The Vertex Linear Arboricity of a Special Class of Integer Distance Graphs

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Abstract

The vertex linear arboricity $vla(G)$ of a nonempty graph G is the minimum number of subsets into which the vertex set $V(G)$ can be partitioned so that each subset induces a subgraph whose connected components are paths. An integer distance graph is a graph $G(D)$ with the set of all integers as vertex set and two vertices $u, v \in Z$ are adjacent if and only if $|u - v| \in D$ where the distance set D is a subset of the positive integers set. Let $D_{m,k,3} = [1, m] \setminus \{k, 2k, 3k\}$ for $m \geq 4k \geq 4$. In this paper, it is obtained that some upper and lower bounds of the vertex linear arboricity of the integer distance graph $G(D_{m,k,3})$ and the exact value of it for some special cases.

Keywords Distance graph; Vertex linear arboricity; Path coloring; Proper coloring; Chromatic number
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1 Introduction

In this paper, $\mathbb{R}$ and $\mathbb{Z}$ denote the sets of all real numbers and all integers, respectively. For $x \in \mathbb{R}$, $\lfloor x \rfloor$ denotes the greatest integer not exceeding x; $\lceil x \rceil$ denotes the least integer not less than x; we use $[m, n]$ for the set of the integers from m to n $(m \leq n)$ and $[m, n] = \phi$ if $m > n$. $|S|$ denotes the cardinality of a set S ($|S| = +\infty$ means that S is an infinite set).

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Coloring of graphs has been one of the most fascinating and well-studied topics in graph theory. Its root goes back to the Four Color Conjecture and more recently, it was motivated by such application problems as the frequency assignment problem (i.e., $L(2,1)$–labeling), the control of traffic signals (i.e., circular coloring) and other problems from wide range of industrial areas. A vertex-coloring can be viewed as a function from V to $\mathbb{Z}$. More precisely, a *k-coloring* of a graph G is a mapping f from $V(G)$ to $[1,k]$. Given a k-coloring, let V_i denote the set of all vertices of G colored with i, and $\langle V_i \rangle$ denote the subgraph induced by V_i in G. If V_i is an independent set for every $1 \leq i \leq k$, then f is called a *proper k-coloring*. The *chromatic number* $\chi(G)$ of a graph G is the minimum integer k for which G has a proper k-coloring. If V_i induces a subgraph whose connected components are paths, then f is called a *path k-coloring*. The *vertex linear arboricity* of a graph G, denoted by $vla(G)$, is the minimum number k for which G has a path k-coloring. Clearly, $\chi(G) \geq vla(G)$ for any graph G.

Matsumoto [8] proved that for a finite graph G, $vla(G) \leq \lceil \frac{\Delta(G)+1}{2} \rceil$; moreover, if $\Delta(G)$ is even, then $vla(G) = \lceil \frac{\Delta(G)+1}{2} \rceil$ if and only if G is a complete graph of order $\Delta(G)+1$ or a cycle. Goddard [5] and Poh [9] proved that $vla(G) \leq 3$ for a planar graph G. Akiyama *et al.* [1] proved that $vla(G) \leq 2$ if G is an outerplanar graph.

Let S be a subset of all real numbers and D a subset of all positive real numbers. Then *distance graph* $G(S, D)$ has the vertex set S and two real numbers x and y are adjacent if and only if $|x-y| \in D$, where the set D is called the *distance set*. In particular, if all elements of D are positive integers and $S = \mathbb{Z}$, the graph $G(\mathbb{Z}, D)$, or $G(D)$ in short, is called a *integer distance graph*. The distance graphs were introduced by Eggleton *et al.* [3] in 1985 to study the chromatic number. They proved that $\chi(G(\mathbb{R}, D)) = n+2$, where D is an interval between 1 and δ, and n satisfies $1 \leq n <$

$\delta \leq n+1$. They also partially determined the values of $\chi(G(D_{m,k}))$, where $D_{m,k} = [1,m] \setminus \{k\}$. The complete solution to $\chi(G(D_{m,k}))$ is provided by Chang, Liu and Zhu in [2]. Many people discussed the chromatic number of the integer distance graph $G(D)$. More results on the chromatic number of integer distance graphs, see [3], [4], [6], [7], [10] and [11]. In [13] and [14], it is discussed that vertex linear arboricity of the real distance graphs. In [12], it is studied that the vertex linear arboricity of $G(D_{m,k})$ where $D_{m,k} = [1,m] \setminus \{k\}$. In [15], it is studied that the fractional version of the vertex linear arboricity of some graphs.

Now the integer distance graph is applied widely to gene sequence, sequential series, on-line computing and so on.

Let $D_{m,k,3} = [1,m] \setminus \{k, 2k, 3k\}$. In Section 3 we shall determine the vertex linear arboricity of distance graph $G(D_{m,1,3})$; in Section 4, we will discuss the distance graph $G(D_{m,k,3})$ for $k \geq 2$.

2 Preliminary

We summarize the basic tactics used in the proof of the main results as several lemmas. Suppose that $m \geq 6k$.

Lemma 2.1. *Suppose that there are three vertices $b_1 < b_2 < b_3$ receiving the same color in the path-coloring f of $G(D_{m,k,3})$.*

(1) *If there is a (b_1, b_2)-path in $G(D_{m,k,3})$, then $b_3 \in \{b_1 + ik, b_2 + ik | i \in [1,3]\}$ or $b_3 \geq b_1 + (m+1)$;*

(2) *if there is a (b_1, b_3)-path in $G(D_{m,k,3})$ and $b_3 - b_1 \leq m$, then $b_2 \in \{b_1 + ik, b_3 - ik | i \in [1,3]\}$;*

(3) *if there is a (b_2, b_3)-path in $G(D_{m,k,3})$, then $b_1 \in \{b_2 - ik, b_3 - ik | i \in [1,3]\}$ or $b_1 \leq b_3 - (m+1)$.*

Proof. (1) Otherwise, if $b_3 \notin \{b_1 + ik, b_2 + ik | i \in [1,3]\}$ and $b_3 \leq b_1 + m$, then $b_1b_3, b_2b_3 \in E(H)$ and thus the (b_1, b_2)-path and two edges b_1b_3, b_2b_3 form a cycle, a contradiction.

(2) and (3) can be proved similarly. □

Lemma 2.2. *Suppose that there are four vertices b_1, b_2, b_3, b_4 receiving the same color in the path-coloring f of $G(D_{m,k,3})$ and $b_1b_2, b_2b_3 \in E(G(D_{m,k,3}))$, then $b_4 \in \{b_2 \pm ik | i \in [1,3]\}$ or $|b_4 - b_2| \geq m+1$.*

Proof. Otherwise, four vertices $b_i (i \in [1,4])$ will induce a $K_{1,3}$. □

For the convenience of arguments, we introduce a new term. If six vertices $v+ik (i \in [0,5])$ receive the same color β, then such a set $\{v+ik | i \in [0,5]\}$ is called an *F-type set associated with β and v* and denoted by V_{β_v}. If there is no confusion arisen, we often call it an F-type set, for short.

Lemma 2.3. *If V_{β_v} is an F-type set associated with β and v, then $f(v+i) \neq \beta$ in the path-coloring f of $G(D_{m,k,3})$ for any $i \in [5k+1, m+5k]$.*

Proof. Assume, to the contrary, that $f(v+i) = \beta$ for some $i \in [5k+1, m+5k]$. Since v is adjacent to $v+4k$ and $v+5k$, by taking $b_1 = v+4k, b_2 = v+5k$ and $b_3 = v+i$ in Lemma 2.1 (1), we have $v+i \in \{v+jk | j \in [6,8]\}$ or $b_3 \geq b_1 + (m+1)$ by the hypothesis.

Since $m \geq 6k$, if $i = 6k$ then $v, v+4k, v+5k, v+i$ induce a $K_{1,3}$; if $i = 7k$ then $v+k, v+2k, v+3k, v+i$ induce a $K_{1,3}$; if $i = 8k$ then $v+2k, v+3k, v+4k, v+i$ induce a $K_{1,3}$; and if $b_3 \geq b_1 + (m+1)$ then $v, v+k, v+5k, v+i$ induce a $K_{1,3}$. We come to a contradiction in any case. Hence the lemma. □

Lemma 2.4. *In the path-coloring f of $G(D_{m,k,3})$, if vertices $a_0 < a_1 < \cdots < a_5$ receive the same color with $a_5 - a_0 \leq m$, then $\{a_i | i \in [0,5]\}$ is an F-type set.*

Proof. We show that the lemma for $k = 1$ only, and the other cases can be proved similarly. If there is an i with $0 \le i < 5$, such that $a_{i+1} - a_i > 1$, then $a_0a_5, a_0a_4, a_0a_3 \in E(H)$ or $a_0a_5, a_1a_5, a_2a_5 \in E(H)$, i.e., the same color vertices induce a $K_{1,3}$, a contradiction. □

Lemma 2.5. *In the path-coloring f of $G(D_{m,k,3})$, if vertices $0 \le a_0 < a_1 < \cdots < a_6 \le m + 5k$ receive the same color, then $a_{i+5} - a_i > m$ for $i = 0, 1$.*

Proof. Assume that $a_5 - a_0 \le m$, then $\{a_i | i \in [0, 5]\}$ is an F-type set by Lemma 2.4. Therefore $a_0a_4, a_0a_5, a_1a_5 \in E(H)$, and $a_6 - a_5 > m$ by Lemma 2.3. Then $a_6 \ge m+1+a_5 > m+5k$ which is contrary to $a_6 \in V(H)$. Therefore $a_5 - a_0 > m$. Similarly $a_6 - a_1 > m$ and hence the lemma. □

3 The vertex linear arboricity of integer distance graphs $G(D_{m,1,3})$

In this section, we discuss the case of $k = 1$, i.e., $D_{m,1,3} = [1, m] \setminus [1, 3] = [4, m]$. For $m = 4$, it is obvious that $vla(G(D_{m,1,3})) = vla(G(\{4\})) = 1$.

Theorem 3.1. *For any integer $m > 4$, $vla(G(D_{m,1,3})) = \lceil \frac{m}{6} \rceil + 1$.*

Proof. Let $n = \lceil \frac{m}{6} \rceil$. Firstly, we show that $vla(G(D_{m,1,3})) \le \lceil \frac{m}{6} \rceil + 1$ for any integer $m > 4$. Let $f(6t + i) = t$ for $0 \le t \le n = \lceil \frac{m}{6} \rceil$ and $0 \le i \le 5$, and $f(6(n + 1)s + h) = f(h)$ for all $h, s \in Z$. Since $6(n + 1) \ge m + 6$,

$$V_t = \cup_{k \in Z}[6k(n + 1) + 6t, 6k(n + 1) + 6t + 5]$$

induces a linear forest for each $0 \le t \le n$. Then f is a path $(n+1)-$coloring of $G(D_{m,1,3})$, and $va(G(D_{m,1,3})) \le \lceil \frac{m}{6} \rceil + 1$.

Now we show that the lower bound. For $5 \le m \le 6$, $vla(G(D_{m,1,3})) \le 2$ by the upper bound. Since $G(D_{m,1,3})$ has a $K_{1,3}$ induced by vertices

$1, 5, 9, 10$, $vla(G(D_{m,1,3})) = 2$. We will show that the lower bound, i.e., $vla(G(D_{m,1,3})) \geq n + 1$ for $m > 6$.

Assume, on the contrary, that $vla(G(D_{m,1,3})) \leq n$, then $G(D_{m,1,3})$ has a path $n-$coloring f. Let H be the subgraph of $G(D_{m,1,3})$ induced by vertex subset $[0, 6n]$, then f is also a path coloring of H. Note that $|V(H)| = 6n+1$. There are at least seven vertices in H, say $0 \leq a_0 < a_1 < \cdots < a_6 \leq 6n$, receiving the color α.

By Lemmas 2.4- 2.5, the following two claims are obvious.

Claim 1. If $a_5 - a_0 \leq m$, then $a_5 = a_4 + 1 = \cdots = a_0 + 5$.

By Claim 1, if $1 \leq i \leq a \leq i + 5$ then it is impossible that there are $n - 1$ colors for the remaining $m - 1 = 6(n - 1)$ vertices $[i, m + i]$ in H such that each color has exactly an F-type set V_{β_v} with $v \in [i, m + i]$.

Claim 2. $\min\{a_5 - a_0, a_6 - a_1\} > m$ and then $a_0 \leq 3$ and $a_1 \leq 4$.

Claim 3. $m = 6(n - 1) + 1$.

Assume that $m \geq 6(n - 1) + 2 \geq 8$, by Claim 2, $a_5 - a_0 > m$ and $a_i \leq m + i - 2$ for $i \in [4, 6]$. There is a integer k with $1 \leq k < 5$ such that $a_k - a_0 \leq m$ and $a_{k+1} - a_0 > m$. If $k = 1$, then $a_2 - a_0 > m$ and $a_6 - a_2 \leq 6n - a_2 \leq 6n - (m+1) \leq 3$ which is impossible. Hence, $k \in [2, 4]$. In the following, we will come to a contradiction in any case.

Case 3.1. $k = 2$.

Since $a_3 - a_0 > m$, $a_6 - a_3 \leq 6n - (m + 1) \leq 3$, so $a_6 = 6n = a_5 + 1 = a_4 + 2 = a_3 + 3$ and $m = 6(n-1)+2$. By Claim 2, $a_1 \leq a_6 - (m+1) \leq 3$, and $a_0 \leq 2$. If $a_1 = 3$, then a_1 is adjacent to a_3, a_4, a_5, a contradiction. If $a_1 = 2$, then $a_1a_3, a_1a_4 \in E(H)$ and $a_2 - a_1, a_3 - a_2 \in [1, 3]$ by Lemmas 2.1-2.2, so $m = 7, a_2 = 5$, and vertices a_2, a_4, a_5, a_6 induce a $K_{1,3}$, a contradiction. Hence $a_1 = 1$, $a_0 = 0$ and $2 \leq a_2 \leq m$, then $a_2 = 2$ (otherwise, a_2 is adjacent to vertices a_3, a_4, a_5 for $2 < a_2 < m - 2$, and a_2 is adjacent to vertices a_0, a_1, a_6 for $a_2 \geq m-2$). Therefore $a_1a_3, a_2a_3, a_2a_4 \in E(H)$ and there are $n-1$ colors for the remaining $m-2 = 6(n-1)$ vertices $3, 4, \cdots, m$ in H such that each color has an F-type set V_{β_h} with $3 \leq h \leq m-5$ by Claims $1-2$. By Lemma 2.3, the vertex $6n+1$ receives the color α and induces a $K_{1,3}$ with vertices a_1, a_2, a_3, a contradiction, too.

Case 3.2. $k = 3$.

Then $a_3 - a_0 \leq m$, $a_4 - a_0 \geq m+1$ and $2 \leq a_6 - a_4 \leq 6n-(m+1) \leq 3$.

If $a_3 - a_0 = 3$, then $a_2a_4, a_3a_4 \in E(H)$ because $m+1 \leq a_4 - a_0 \leq m+2$, and $|a_1 - a_4| > m$ by Lemma 2.2. Thus $a_4 = m+2$, $a_0 = 0$, $a_{i+1} = a_i + 1$ for $i \in [4, 5]$, and $m = 6(n-1)+2$. There are $n-1$ colors for the remaining $6(n-1)$ vertices $4, \cdots, m+1$ in H such that each color has exactly an F-type set V_{β_h} with ($4 \leq h \leq m-4$) by Claim 1. By Lemma 2.3, the vertex $6n+2 = m+6$ receives the color α and induces a $K_{1,3}$ with vertices a_2, a_3, a_4, a contradiction.

Therefore $a_3 - a_0 \geq 4$, $a_0a_3 \in E(H)$, then at most one of vertices a_4, a_5, a_6 is adjacent to a_3. Thus, $a_5 - a_3 \leq 3$. Since $a_5 - a_0 \geq m+2$, $m \geq a_3 - a_0 = a_5 - a_0 - (a_5 - a_3) \geq m-1$. By Claim 2, $a_1 \leq 3$, so that $a_3 - a_1 \geq m - 1 - 3 \geq 4$, then $a_1a_3 \in E(H)$. By Lemma 2.2, $a_6 - a_3 = 3$, and $a_6 - a_0 \leq m+2$ if $a_3 - a_0 = m-1$, which is contrary to $a_4 - a_0 \geq m+1$. Hence $a_3 - a_0 = m$, and $a_6 - a_0 = m+3$.

Clearly, $a_3 - a_2 \leq 3$ (otherwise, vertices a_0, a_1, a_2, a_3 induce a $K_{1,3}$), and $a_2 - a_0 = a_3 - a_0 - (a_3 - a_2) \geq m - 3 > 3$, then $a_0a_2, a_2a_6 \in E(H)$. By Lemma 2.2, $a_2 - a_1, a_5 - a_2 \in [1, 3]$, then $a_5 - a_2 = 3$, and $a_6 - a_1 \leq 7$, which is contrary to Claim 2.

Case 3.3. $k = 4$.

Then $a_4 - a_0 \leq m$ and $a_0a_4 \in E(H)$. If there is an $i \in [0, 3]$ such that $a_{i+1} - a_i > 2$ or an $i \in [0, 1]$ such that $a_{i+j+1} - a_{i+j} = 2$ for $j \in [0, 2]$, then

a_0 is adjacent to vertices a_2, a_3, a_4, or a_4 is adjacent to vertices a_0, a_1, a_2, a contradiction. If there are exactly a_{i_1}, a_{i_2} with $0 \le i_1 < i_2 \le 3$ such that $a_{i_j+1} - a_{i_j} = 2$ for $j = 1, 2$ and $a_{i+1} - a_i = 1$ for all $i \in [0, 3] \setminus \{i_1, i_2\}$, then $a_0a_3, a_0a_4, a_1a_4 \in E(H)$, and $a_4 - a_0 = 6$, so that $a_4a_6 \notin E(H)$, i.e., $a_6 - a_4 \le 3$ and $a_6 - a_1 \le 8$ which is contrary to Claim 2. If there is a unique a_{i_0} $(0 \le i_0 < 4)$ such that $a_{i_0+1} - a_{i_0} = 2$, and $a_{i+1} - a_i = 1$ for $i \in [0, 3] \setminus \{i_0\}$, then $a_4 - a_0 = 5$, and $a_6 - a_4 \le m - 1$, by Claim 2, $a_5 - a_4 \ge m - 4 \ge 4$, so that vertices a_0, a_4, a_5, a_6 induce a $K_{1,3}$, a contradiction.

Therefore $a_4 - a_0 = 4$, i.e., $a_{i+1} = a_i + 1$ for $i \in [0, 3]$, and $a_4a_6 \in E(H)$ since $a_6 - a_4 \le 6(n-1) + 2 \le m$, by Lemma 2.2, $a_5 - a_4 = t \le 3$. By Claim 2, $m \le a_5 - a_0 - 1 = a_5 - a_4 + a_4 - a_0 - 1 = 3 + t \le 6$ which is contrary to $m \ge 6(n-1) + 2 \ge 8$.

By all above arguments, we have $m = 6(n-1)+1$, hence Claim 3 holds.

Claim 4. $a_2 \le 2$ and $a_4 \ge 6(n-1) + 4$, i.e., $a_0 = 0, a_6 = m + 5 = 6n$ and $a_{i+1} = a_i + 1$ for $i \in [0, 1] \cup [4, 5]$.

First, we have the following subclaim.

Subclaim 4.1. $a_2 - a_0 \le 3$ and $a_6 - a_4 \le 3$.

It is clear that the relation of a_2, a_0 is symmetric with the relation of a_4, a_6, so we need only to show that $a_2 - a_0 \le 3$.

Otherwise, assume that $a_2 - a_0 \ge 4$, we prove that $a_2 - a_0 \le m$ in the following. Suppose that $a_2 - a_0 \ge m+1$, then $a_6 = m+5 = 6n, a_{i+1} = a_i + 1$ for $i \in [2, 5]$, $a_0 = 0$, $a_6a_2 \in E(H)$ and $a_1a_2 \in E(H)$ by Claim 2. By Lemmas 2.3-2.5, there are $n - 1$ colors for the remaining $6(n-1) = m - 1$ vertices $[1, m] \setminus \{a_1\}$ in H such that each color has exactly an F-type set V_{β_v} with $1 \le v \le m - 5$, and the vertex $6n + 1$ in $G(D_{m,1,3})$ receives the color α and induces a $K_{1,3}$ with a_1, a_6, a_2, a contradiction. Hence $a_2 - a_0 \le m$, so $a_0a_2 \in E(H)$, and then at most one of a_1, a_3, a_4, a_5, a_6 is adjacent to a_2.

By Lemmas 2.1-2.2, if $a_1a_2 \in E(H)$, then $a_2 \geq 5$, $a_5 - a_2 = 3$, and $a_6 \geq a_2 + m + 1 > 6n$, which is impossible. Similarly, if $a_2a_i \in E(H)$ for some $i \in [3,4]$, then $a_{i+1} \geq a_2 + m + 1 \geq m + 5 = 6n$; if $a_2a_5 \in E(H)$, then $a_2 - a_1 \leq 3$, $a_4 - a_2 \leq 3$, and $a_6 \geq a_2 + m + 1$, so that $a_6 = m + 5 = 6n$, $a_2 = 4$, $a_0 = 0$, $a_4 \leq 7$, and $a_0a_2, a_0a_3, a_0a_4 \in E(H)$, a contradiction.

Finally, if $a_2a_6 \in E(H)$ then $a_2 - a_1 = t \leq 3$, $a_5 - a_2 = 3$ and $a_1a_5 \in E(H)$, so $a_2 = a_5 - (a_5 - a_2) \geq m - 2$ and $4 \geq a_1 - a_0 = (a_5 - a_0) - (a_5 - a_2 + a_2 - a_1) \geq m + 1 - 3 - t \geq m - 5$ by Claim 2, i.e., $m = 6(n-1) + 1 = 7$, $a_1 - a_0 \geq 2$ and $a_2 - a_0 \leq 7$. If $a_2 - a_1 = 3$, then a_1 induces a $K_{1,3}$ with vertices a_3, a_4, a_5; if $a_2 - a_1 = 2$, then a_1 is adjacent to vertices a_4, a_5, by Lemma 2.2 and Claim 2, $a_1 - a_0 \leq 3$ and $a_5 - a_0 = a_5 - a_2 + a_2 - a_1 + a_1 - a_0 = 8$, hence $a_1 - a_0 = 3$, $a_2 - a_0 = 5$, and a_0, a_2, a_3, a_4 induce a $K_{1,3}$; if $a_2 - a_1 = 1$, then $a_5 - a_1 = 4$, $a_1 - a_0 > m - 4 = 3$, $a_0a_1 \in E(H)$, and $a_2 - a_0 = m$ (otherwise, $a_2 - a_0 < m$, then $a_0a_3 \in E(H)$, and a_0, a_1, a_2, a_3 induce a $K_{1,3}$), so that $a_1 - a_0 = m - 1 = 6$ and $a_6 - a_1 \leq 7 = m$ which is contrary to Claim 2.

Therefore the subclaim holds.

Subclaim 4.2. $a_2 - a_0 = a_6 - a_4 = 2$.

By contradiction, assume that $a_2 - a_0 = 3$, then $m - 2 \leq a_5 - a_2 \leq m + 1$ by Claim 2 and $a_5 \leq m + 4$. In the following, we will come to a contradiction in any case.

Case 4.2.1. $a_5 - a_2 = m + 1$.

Then $a_2 = 3$, $a_5 = m + 4$, $a_0 = 0$ and $a_6 = m + 5 = 6n$. If $6 < a_3 \leq m$, then vertices a_0, a_1, a_2 are all adjacent to a_3, a contradiction. So that $a_3 \leq 6$ or $a_3 \geq m + 1$. For $a_3 \leq 6$, $a_0a_3, a_3a_5 \in E(H)$, then $a_4 - a_3, a_3 - a_1 \in [1,3]$ and $a_6 - a_3 \geq m + 1$ by Lemma 2.2, then $a_3 = 4$ and $a_4 \leq 7$, by Subclaim

4.1, $a_6 = m+5 \leq 10$ which is impossible. For $a_3 \geq m+1$, $a_2a_3 \in E(H)$ since $a_3 \leq m+2$, then $a_1a_3 \notin E(H)$ or $a_3a_6 \notin E(H)$, i.e., $a_3 - a_1 > m$ or $a_6 - a_3 = 3$. In the former case, $a_3 = m+2$ and $a_1 = 1$, then there are $n-1$ colors for the remaining $m-1 = 6(n-1)$ vertices $[2, m+1]$ in H such that each color has exactly an F-type set V_{β_v} with $2 \leq v \leq m-4$, then vertex $m+6$ receives the color α and induces a cycle with vertices a_2, a_3, a_4, a contradiction. In the latter case, similarly, we have $a_1 = 2$ and $a_3 = m+2$, then there are $n-1$ colors for the remaining $m-1 = 6(n-1)$ vertices $[1, m+1] \setminus \{2\}$ in H such that each color has exactly an F-type set, which is impossible.

Case 4.2.2. $m-1 \leq a_5 - a_2 \leq m$.

Then $a_2a_5, a_2a_4 \in E(H)$ by Subclaim 4.1, $a_3 - a_2 \leq 3$ by Lemma 2.2, and $a_0a_3 \in E(H)$, so that $a_4 - a_3 \leq 3$ (otherwise, a_2, a_3, a_4, a_5 induce a cycle), and $a_5 - a_3 \leq 3$ or $a_6 - a_3 \geq m+1$. In the former case, $m = 7$, $a_5 - a_2 = 6$ and $a_5 - a_3 = a_3 - a_2 = 3$, then a_0, a_1, a_3, a_6 induce a $K_{1,3}$. In the latter case, $a_3 = 4$, $a_6 = m+5$, $a_2 = 3$ and $a_0 = 0$, then $m-1 \leq a_5 - a_2 = a_5 - a_4 + a_4 - a_3 + a_3 - a_2 \leq 6$ by Subclaim 4.1, i.e., $m = 7$, $a_6 = 12$, $a_5 = 9$ and $a_4 = 7$, so that vertices a_0, a_1, a_2, a_4 induce a $K_{1,3}$, a contradiction.

Case 4.2.3. $a_5 - a_2 = m-2$.

In this case, $a_5 - a_0 = m+1$, and $a_2a_5, a_1a_5 \in E(H)$, then $a_5 - a_3 \leq 3$ by Lemma 2.2, and $a_3 - a_0 \geq m-2$. Thus $a_0a_3, a_0a_4 \in E(H)$, then $a_1a_3 \notin E(H)$ (otherwise, $a_1a_3, a_1a_4 \in E(H)$, so vertices a_0, a_3, a_4, a_1 induce a 4−cycle, a contradiction), i.e., $a_3 - a_1 \leq 3$, so that $a_5 - a_1 \leq 6$ and $a_5 - a_0 \leq 8$, then $a_5 - a_0 = 8, m = 7$ and $a_5 - a_2 = 5$. By Subclaim 4.1, $a_6 - a_5 \leq 2$, then $a_2a_6 \in E(H)$ and $a_4 - a_2 \leq 3$. Since $a_4 - a_1 = a_6 - a_1 - (a_6 - a_4) \geq 5$, $a_1a_4 \in E(H)$, then $a_6 - a_3 = 3$ (otherwise, vertices

$a_0, a_3, a_6, a_2, a_5, a_1, a_4$ induce a 7−cycle), and $a_6 - a_1 \leq 6 < m$ which is contrary to Claim 2.

In a word, we have obtained that $a_2 - a_0 = 2$.

Similarly, we can show that $a_6 - a_4 = 2$, hence the subclaim.

Subclaim 4.3. $a_2 = 2$ and $a_4 = m + 3$.

If $a_0 > 0$, then $a_2 \geq 3$ and $a_1 \geq 2$. Since $a_6 - a_1 \geq m + 1$, $a_1 \leq a_6 - (m+1) \leq 6n - (m+1) = 4$, i.e., $2 \leq a_1 \leq 4$.

Case 4.3.1 $a_1 = 4$.

Hence $a_6 = m + 5$, $a_2 = 5$, $a_0 = 3$, $a_5 = m + 4$, $a_3 \geq 6$ and $a_4 = m + 3$ by Subclaim 4.2, then a_0, a_1, a_2, a_4, induce a $K_{1,3}$, a contradiction.

Case 4.3.2. $a_1 = 3$.

Then $a_0 = 2$, $a_2 = 4$ and $a_6 \geq m + 4$. For $a_6 = m + 4$, $a_5 = m + 3$ and $a_4 = m + 2$, then a_1, a_2, a_4, a_5, induce a cycle, a contradiction. For $a_6 = m + 5$, $a_5 = m + 4$ and $a_4 = m + 3$ by Subclaim 4.2, then a_0, a_1, a_2, a_4 induce a $K_{1,3}$, a contradiction, too.

Case 4.3.3. $a_1 = 2$.

In this case, $a_0 = 1, a_2 = 3, a_5 \geq m + 2$, and $a_2a_4 \in E(H)$ by Subclaim 4.2, then $a_3 \leq 6$ or $a_3 \geq a_4 - 3$. In the former case, if $m > 7$ or $m = 7$ and $a_4 - a_3 > 3$, then $a_3a_4, a_3a_5 \in E(H)$, and $a_3 = 4$, $a_6 = m + 5$ and $a_4 = m + 3$ by Lemma 2.2, so there are $n - 1$ colors for the remaining $m - 1 = 6(n-1)$ vertices $\{0\} \cup [5, m+2]$ in H such that each color has exactly an F-type set V_{β_v} with $5 \leq v \leq m - 3$, except one color γ has six vertices $0 < h_1 < \cdots < h_5$. By Lemmas 2.1-2.3, the vertice $m + 6$ receives the color γ and induces a $K_{1,3}$ with vertices h_3, h_4, h_5, a contradiction. Hence $m = 7$ and $a_4 - a_3 \leq 3$, $a_1a_4 \in E(H)$, i.e., $a_4 - a_1 \leq 7$ and $a_4 - a_3 = a_3 - a_2 = 3$, so a_0, a_3, a_5, a_6 induce a $K_{1,3}$, a contradiction. In the latter case, $m + 1 \geq a_3 \geq a_4 - 3 \geq m - 2$, then $a_0a_3 \in E(H)$, and

$a_3 - a_1 \leq 3$ or $a_6 - a_3 = 3$ by Lemma 2.1, so $4 \leq a_3 \leq 5$ or $a_3 \geq m$, and thus vertices a_0, a_3, a_4, a_5 induce a $K_{1,3}$ or vertices a_0, a_1, a_2, a_3 induce a $K_{1,3}$, a contradiction, too.

In a word, $a_0 = 0, a_1 = 1$ and $a_2 = 2$. Similarly, we have $a_6 = m + 5$, $a_5 = m + 4$ and $a_4 = m + 3$. Hence the claim.

Claim 5. $a_3 = 3$ or $a_3 = m + 2$.

Assume that $4 \leq a_3 \leq m$, then $a_0a_3, a_3a_5 \in E(H)$, and $a_3 - a_1 \leq 3$ and $a_6 - a_3 \geq m + 1$ by Lemma 2.2, so $a_3 = 4$ and vertices a_0, a_3, a_4, a_5 induce a $K_{1,3}$ by claim 4, a contradiction. If $a_3 = m+1$, then vertices a_1, a_2, a_3, a_6 induce a $K_{1,3}$, a contradiction.

Therefore $a_3 = 3$ or $a_3 = m + 2$. So Claim 5 holds.

Without loss of generality, suppose that $a_3 = 3$, then there are $n - 1$ colors for $6(n - 1)$ vertices $[4, m + 2]$ in H such that each color has exactly an F-type set V_{β_v} with $v \geq 4$ by Lemma 2.4. Therefore, the vertex $m + 9$ receives the color α and induces a $K_{1,3}$ with a_4, a_5 and a_6, a contradiction, too.

In a word, we have proved that $vla(G(D_{m,1,3})) \geq \lceil \frac{m}{6} \rceil + 1$.

Therefore, $vla(G(D_{m,1,3})) = \lceil \frac{m}{6} \rceil + 1$. □

4 The vertex linear arboricity of integer distance graphs $G(D_{m,k,3})$ for $k \geq 2$

Secondly, we study the integer distance graph $G(D_{m,2,3})$ with the distance set $D_{m,2,3} = [1, m] \setminus \{2, 4, 6\}$. Clearly, for $m \leq 9$, there is at most one even number in $D_{m,2,3}$, then $vla(G(D_{m,2,3})) \leq 2$ since $f(n) = n (mod\, 2)$ is a path coloring. But vertices $0, 1, 4, 5$ induce a cycle in $G(D_{m,2,3})$, so $vla(G(D_{m,2,3})) = 2$. Suppose that $m \geq 10$ in the following.

Theorem 4.1. *For each integer $m = 12q + j \geq 10$ with $0 < j \leq 12$,*

$$\lceil \frac{m+1}{6} \rceil + 1 \leq vla(G(D_{m,2,3})) \leq \begin{cases} 2\lceil \frac{m}{12} \rceil, & \text{for } j = 1, \\ 2\lceil \frac{m}{12} \rceil + 1, & \text{for } 2 \leq j \leq 4, \\ 2(\lceil \frac{m}{12} \rceil + 1), & \text{else.} \end{cases}$$

Proof. Firstly, we show that the upper bound. Let $\lceil \frac{m}{12} \rceil = n$ and $m = 12q + j$ with $0 < j \leq 12$. If $j = 1$, let $f(12l + i + 2t) = i + 2l$ for $t \in [0, 5]$, where $i = 0, 1$, and $0 \leq l \leq \lceil \frac{m}{12} \rceil - 1$, and the other vertices be colored periodically, then f is a path coloring, so that $vla(G(D_{m,2,3})) \leq 2\lceil \frac{m}{12} \rceil$.

If $2 \leq j \leq 4$, let $f(12l + i + 2t) = i + 2l$ for $t \in [0, 5]$, where $i = 0, 1$, and $0 \leq l \leq \lceil \frac{m}{12} \rceil - 1$, $f(12n) = f(12n + 1) = f(12n + 2) = 2n$, and the other vertices be colored periodically, then f is a path coloring, so that $vla(G(D_{m,1,3})) \leq 2n + 1 = 2\lceil \frac{m}{12} \rceil + 1$.

If $5 \leq j \leq 12$, let $f(12l + i + 2t) = i + 2l$ for $t \in [0, 5]$, where $i = 0, 1$, and $0 \leq l \leq \lceil \frac{m}{12} \rceil = n$, and the other vertices be colored periodically, i.e., $f(12(n + 1)t + u) = f(u)$ for all $t, u \in Z$, then f is a path coloring, and $vla(G(D_{m,2,3})) \leq 2(\lceil \frac{m}{12} \rceil + 1)$.

Secondly, we show that $vla(G(D_{m,2,3})) \geq \lceil \frac{m+1}{6} \rceil + 1$. We discuss the case of $m = 6q$ and $q \geq 2$ at the first.

By contradiction. Assume that $vla(G(D_{m,2,3})) \leq \lceil \frac{m+1}{6} \rceil = q + 1 = n$, then $G(D_{m,2,3})$ has a path $n-$coloring f. Clearly, f is also a path coloring of the subgraph H induced by vertex subset $[0, 6n]$. Note that $|V(H)| = 6n+1$. There are at least seven vertices in H, say $0 \leq a_0 < a_1 < \cdots < a_6 \leq 6n = m + 6$, receiving the color α.

By Lemmas 2.4- 2.5, we have the following two claims.

Claim 1. Suppose that $a_5 - a_0 \leq m$, then $a_5 = a_4 + 2 = a_3 + 4 = a_2 + 6 = a_1 + 8 = a_0 + 10$.

Claim 2. $\min\{a_5 - a_0, a_6 - a_1\} > m$.

It is straightforward to prove the following claim.

Claim 3. If $a_{i+3} - a_i \leq 5$, or $a_{i+3} - a_i = 6$ and $\{a_{i+j+1} - a_{i+j} | j \in [0,2]\} \setminus \{2\} \neq \phi$, then vertices a_{i+j} ($j \in [0,3]$) induce a $K_{1,3}$ or a 4−cycle; Hence this is impossible in the path coloring.

In the following, there is a contradiction regardless the relative positions of a_0 and a_2.

Case 1. $a_2 - a_0 = 2$.

Now a_0a_1, $a_1a_2 \in E(H)$, so $a_3 - a_1 \in \{2,4,6\}$ or $a_3 - a_1 \geq m+1$ by Lemma 2.2.

If $a_3 - a_1 \in \{2,4,6\}$, then vertices a_0, a_1, a_2, a_3 induce a 4−cycle, a contradiction. If $a_3 - a_1 \geq m+1$, then $a_3 \geq m+2$ and $a_6 - a_3 \leq 4$ which is impossible by Claim 3.

Case 2. $a_2 - a_0 = 3$.

In this case, $a_0a_2 \in E(H)$, and $a_1 - a_0 = 1$ or $a_2 - a_1 = 1$.

If $a_1 - a_0 = 1$, then $a_0a_1 \in E(H)$, and $a_3 - a_0 \in \{4,6\}$ or $a_3 - a_0 \geq m+1$ by Lemma 2.2. In the former case, vertices a_0, a_1, a_3, a_2 induce a 4−cycle; in the latter case, $a_6 - a_3 \leq 5$ which is impossible by Claim 3.

If $a_2 - a_1 = 1$, then $a_0a_2, a_1a_2 \in E(H)$, and $a_3 - a_2 \in \{2,4,6\}$ (otherwise, $a_3 - a_2 \geq m+1$, then $a_3 \geq m+4$, and $a_6 \geq m+7$ which is impossible) by Lemma 2.2, so that $a_0a_3, a_1a_3 \in E(H)$, and vertices a_0, a_2, a_1, a_3 induce a 4−cycle, a contradiction, too.

Case 3. $a_2 - a_0 = 4$.

If $a_2 - a_1 \neq 2$, then $a_0a_1, a_1a_2 \in E(H)$. Hence $a_3 - a_1 \in \{2,4,6\}$ or $a_3 - a_1 \geq m+1$ by Lemma 2.2. In the former case, vertices a_0, a_1, a_2, a_3 induce a 4−cycle; in the latter case, $a_3 \geq m+2$ and $a_6 - a_3 \leq 4$ which is impossible by Claim 3. Therefore, $a_2 - a_1 = a_1 - a_0 = 2$.

Then $a_3 - a_2 \in \{2,4,6\}$ (for otherwise, vertices a_0, a_1, a_2, a_3 induce a $K_{1,3}$, or $a_6 - a_3 \leq 5$ which is impossible by Claim 3). Similarly, $a_4 - a_2 \in$

$\{4,6\}$ (if $a_4-a_2>m$, $a_4\geq m+5$ and $a_6\geq m+7$, a contradiction), and then $a_3-a_2\in\{2,4\}$. It is clear that $a_0a_4\in E(H)$ and $m+1\leq a_5-a_0\leq m+5$. We shall show that it educes a contradiction in any subcase.

(3.1) For $a_5-a_0\in\{m+1,m+3\}$, the difference of a_5 and each of a_2,a_3,a_4 is odd and $<m$, so that a_5, a_2,a_3,a_4 induce a $K_{1,3}$.

(3.2) For $a_5-a_0=m+2$, $a_1a_5,a_2a_5\in E(H)$, then $m=12, a_5-a_3=6, a_3-a_2=4$ and $a_5-a_4=4$ by Lemma 2.2, so that $a_0a_3,a_0a_4,a_1a_4,a_1a_5$, $a_2a_5\in E(H)$, and $a_6-a_5\in\{2,4\}$. If $a_6-a_5=4$, then $a_4a_6\in E(H)$, and vertices a_0,a_1,a_4,a_6 induce a $K_{1,3}$. If $a_6-a_5=2$, then $a_0a_4,a_4a_1,a_1a_5,a_5a_2$, $a_2a_6,a_6a_3,a_3a_0\in E(H)$, i.e., vertices a_0,a_4,a_1,a_5, a_2,a_6,a_3 induce a 7-cycle, a contradiction.

(3.3) For $a_5-a_0=m+4$, then a_5,a_2,a_3,a_4 induce a $K_{1,3}$ when $m>12$. If $m=12$, then $a_5-a_4=6$ (otherwise, $a_5-a_4=4$, vertices a_4,a_0,a_1,a_2 induce a $K_{1,3}$). Hence, $a_4-a_2=6$ and $a_6-a_5=2$, then vertices a_0,a_1,a_4,a_6 induce a $K_{1,3}$.

(3.4) For $a_5-a_0=m+5$, $a_0=0,a_1=2$, $a_2=4$, $a_5=m+5$, $a_6=m+6$, $a_3\in\{6,8\}$ and $a_4\in\{8,10\}$, so a_5,a_3,a_4,a_6 induce a $K_{1,3}$ since $m=6q\geq 12$, a contradiction.

Case 4. $a_2-a_0=5$.

Now $a_0a_2\in E(H)$, and $a_0a_1\in E(H)$ or $a_1a_2\in E(H)$. If $a_0a_1\in E(H)$, then $a_3-a_0=6$ or $a_3-a_0>m$ by Lemma 2.2, so that vertices a_0,a_1,a_3,a_2 induce a 4$-$cycle, or $a_6-a_3\leq 5$ which is impossible by Claim 3. If $a_1a_2\in E(H)$, then $a_3-a_2\in\{2,4,6\}$ by Lemma 2.2, and $a_1a_3,a_0a_3\in E(H)$, i.e., vertices a_0,a_1,a_3,a_2 induce a 4$-$cycle, a contradiction, too.

Case 5. $a_2-a_0=6$.

It is not difficult to educe a contradiction as the proof of Case 3 similarly.

Case 6. $a_2-a_0\geq 7$.

(6.1) If $a_2 - a_0 \leq m$, then $a_0a_2 \in E(H)$. Hence, $a_0a_1 \notin E(H)$ or $a_1a_2 \notin E(H)$.

(6.1.1) Assume that $a_0a_1 \notin E(H)$ and $a_1a_2 \in E(H)$, then $a_1 - a_0 \in \{2,4,6\}$, and $\{a_i - a_2 | i \in [3,6]\} = \{2,4,6\}$ which is impossible.

(6.1.2) Assume that $a_0a_1, a_1a_2 \notin E(H)$, then $a_6 - a_2 > 6$ and $a_2a_6 \in E(H)$ by Claim 3. By Lemma 2.2, $a_5 = a_4 + 2 = a_3 + 4 = a_2 + 6$, then $a_0a_3 \in E(H)$, $a_6 - a_5 \in \{2,4,6\}$ (otherwise, a_6, a_3, a_4, a_5 induce a $K_{1,3}$) and $a_2 - a_1 \in \{2,4\}$ (otherwise, $a_2 - a_1 = 6$ and then a_1, a_3, a_4, a_5 induce a $K_{1,3}$), so $a_6 - a_3 = 6$ by Lemma 2.1, and $a_6 - a_1 = 12$ which is contrary to Claim 2.

(6.1.3) Assume that $a_0a_1 \in E(H)$ and $a_1a_2 \notin E(H)$, then $a_2 - a_1 \in \{2,4,6\}$, and $a_3 - a_0 \geq m+1$ by Lemma 2.2, so $a_6 - a_3 \leq 5$ which is impossible by Claim 3.

(6.2) If $a_2 - a_0 \geq m+1$, then $a_6 - a_2 \leq 5$ and $a_6 - a_3 \leq 4$ which is impossible by Claim 3.

In a word, for $m = 6q$ with $q \geq 2$, $vla(G(D_{m,2,3})) \geq \lceil \frac{m+1}{6} \rceil + 1$.

For $m = 6q + j > 12$ with $1 \leq j \leq 5$, since $D_{6q,2,3} \subseteq D_{6q+j,2,3}$, $vla(G(D_{m,2,3})) \geq vla(G(D_{6q,2,3})) \geq \lceil \frac{6q+1}{6} \rceil + 1 = \lceil \frac{m+1}{6} \rceil + 1$.

For $m = 10$, $vla(G(D_{m,2,3})) \geq 3$: otherwise, $vla(G(D_{m,2,3})) = 2$, then the subgraph H_1 induced by vertices $0, 1, 2, \cdots, 12$ has a path 2−coloring, so that there are at least seven vertices in H_1, say $a_0 < a_1 < \cdots < a_6$, receiving the same color, then Claims 1 − 3 hold, too. So $a_5 = 11$, $a_6 = 12$, $a_0 = 0$, $a_1 = 1$, and $a_0a_1, a_1a_5, a_5a_6 \in E(H)$, then $a_{i+1} = a_i + 2$ for $i \in [1,3]$ by Lemma 2.2, and vertices a_1, a_2, a_5, a_6 induce a $K_{1,3}$, a contradiction. Hence $vla(G(D_{11,2,3})) \geq vla(G(D_{10,2,3})) \geq 3$. Therefore, the lower bound is obtained. □

By Theorem 4.1, we have below result easily.

Corollary 4.2. *For $m \geq 10$, we have $vla(G(D_{m,2,3})) = \lceil \frac{m}{6} \rceil + 2$ if $m \equiv 0 (mod\, 12)$, and $vla(G(D_{m,2,3})) = \lceil \frac{m}{6} \rceil + 1$ if $m \equiv 1 (mod\, 12)$.*

Lastly, we consider the case of $k \geq 3$.

For $3k < m < 5k$, there is at most one multiple of k in $D_{m,k,3}$, $vla(G(D_{m,k,3})) \leq k$ since $f(n) = n (mod\, k)$ is a path coloring. Let $X_0 = \{0, k, 2k\}, X_1 = \{1, k+1, 2k+1\}, \cdots, X_{k-1} = \{k-1, 2k-1, 3k-1\}$, then vertices $X_0 \cup X_1 \cup \cdots \cup X_{k-1}$ induce a complete $k-$partite graph $K(3, 3, \cdots, 3)$, so that $vla(G(D_{m,k,3})) \geq k$ since any four vertices induce a cycle or a $K_{1,3}$. Hence $vla(G(D_{m,k,3})) = k$.

Similarly, $k \leq vla(G(D_{m,k,3})) \leq 2k$ for $5k \leq m < 7k$.

For $m \geq 7k$, the following conclusion can be obtained as Theorem 4.1 similarly.

Theorem 4.3. *For $m \geq 7k$, $\lceil \frac{m+k+3}{6} \rceil \leq vla(G(D_{m,k,3})) \leq k \lceil \frac{m+5k+1}{6k} \rceil$.*

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