Super connectivity and super arc connectivity of the Mycielskian of a digraph

Xianglan Cao[1], Litao Guo[2,3], * Xiaofeng Guo[3], Jixiang Meng[1]

1 College of Mathematics and System Sciences, Xinjiang University, Urumqi 830046, P.R.China

2 Department of Mathematics and Physics, Xiamen University of Technology, Xiamen Fujian 361024, P.R.China

3 School of Mathematical Sciences, Xiamen University, Xiamen Fujian 361005, P.R. China

Abstract: In a search for triangle-free graphs with arbitrarily large chromatic numbers, Mycielski developed a graph transformation that transforms a graph G into a new graph $\mu(G)$, which is called the Mycielskian of G. This paper shows that: for a strongly connected digraph D with $|V(D)| \geq 2$, $\mu(D)$ is super-κ if and only if $\delta(D) < 2\kappa(D)$; $\mu(D)$ is super-λ if and only if $D \not\cong \overrightarrow{K_2}$.

Key words: Mycielskian; super connectivity; super arc connectivity

1 Introduction

In this paper, $D = (V, A)$ is a digraph with no loops and parallel arcs. For a vertex $v \in V$, we denote the indegree, the outdegree, the minimum indegree and the minimum outdegree in D by $d_D^-(v), d_D^+(v)$ (simply $d^-(v), d^+(v)$), $\delta^-(D), \delta^+(D)$, respectively. The minimum degree of D is $\delta(D) = \min\{\delta^-(D), \delta^+(D)\}$. Moreover, we denote by $N_D^+(v)$ the set of out-neighbors of v, $N_D^-(v)$ the set of in-neighbors of v, $E_D^+(v)$ the set of out-arcs of v, $E_D^-(v)$ the set of in-arcs of v (simply $N^+(v), N^-(v), E^+(v), E^-(v)$). More generally for $S \subset V$, the sets $N_D^+(S) = \bigcup_{x\in S} N_D^+(x) - S$ and $N_D^-(S) = \bigcup_{x\in S} N_D^-(x) - S$ are called out-neighbors and in-neighbors of S, and $D - S$

*Corresponding author. E-mail address: ltguo@yahoo.cn (L.Guo).

denotes the subdigraph of D induced by the vertex set of $V - S$. $[X,Y]$ denotes the set of all arcs with tail in X and head in Y and let $\|[X,Y]\|$ denote the cardinality of $[X,Y]$. $\overrightarrow{K_n}$ is the complete digraph of order n.

A vertex cut of a strongly connected digraph D is a set of vertices whose removal makes D no longer strongly connected or becomes trivial. The vertex connectivity $\kappa(D)$ of a strongly connected digraph D is the minimum cardinality of a vertex cut over all vertex cuts of D. A strongly connected digraph D is *super connected* or *super-κ*, if every minimum vertex cut is either $N_D^+(v)$ or $N_D^-(v)$ for some vertex v. The arc connectivity $\lambda(D)$, *super arc connected* or *super-λ* of a strongly connected digraph D is similarly defined.

It is convenient to denote a digraph by D and let two distinguished vertices $x, y \in D$. An *(x,y)-vertex-cut* is a subset S of $V \setminus \{x,y\}$ whose deletion destroys all directed (x,y)-paths. For notation and terminology not defined here we refer to Bondy and Murty [2].

In a search for triangle-free graphs with arbitrarily large chromatic numbers, Mycielski [9] developed an interesting graph transformation the Mycielskian $\mu(G)$ of a graph G. For a graph $G = (V, E)$, the Mycielskian of G is the graph $\mu(G)$ with the vertex set $V \cup V' \cup \{u\}$, where $V' = \{x' : x \in V\}$ and edge set $E \cup \{xy' : xy \in E\} \cup \{y'u : y' \in V'\}$. The vertex x' is called the twin of the vertex x (and x the twin of x') and the vertex u is called the root of $\mu(G)$. For $n \geq 2$, $\mu^n(G)$ is defined iteratively $\mu^n(G) = \mu(\mu^{n-1}(G))$. From the definition, if G is connected then $\mu(G)$ is connected and $\delta(\mu^n(G)) = \delta(G) + n$.

We define the Mycielskian $\mu(D)$ of a digraph D as follows [4]. For a digraph $D = (V, A)$, the Mycielskian of D is the digraph with the vertex set $V \cup V' \cup \{u\}$, where $V' = \{x' : x \in V\}$ and arc set $A \cup \{(x,y') : (x,y) \in A\} \cup \{(x',y) : (x,y) \in A\} \cup \{(x',u) : x' \in V'\} \cup \{(u,x') : x' \in V'\}$. The vertex x' is called the twin of the vertex x (and x the twin of x') and the vertex u is called the root of $\mu(D)$. For $n \geq 2$, $\mu^n(D)$ is defined iteratively $\mu^n(D) = \mu(\mu^{n-1}(D))$. An obvious inference from the definition of $\mu(D)$ is that $d^-_{\mu(D)}(x') = d^-_D(x) + 1, d^+_{\mu(D)}(x') = d^+_D(x) + 1$ for all $x \in V$. Consequently, $\delta(\mu(D)) = \delta(D) + 1$. If D is a strongly connected digraph, then $\mu(D)$ is strongly connected.

Chang et al. studied the circular chromatic numbers of the Mycielskian $\mu(G)$ of a graph G [3, 5 - 8], Balakrishnan and Francis Raj [1] investigated the vertex connectivity and edge connectivity of the Mycielskian $\mu(G)$ of a graph G. We investigated connectivity and arc connectivity of the Mycielskian of a digraph [4]. In this paper, we study the super connectivity and super arc connectivity of the Mycielskian $\mu(D)$ of a digraph D.

2 Super connectivity of the Mycielskian

Firstly, we need the following results.

Lemma 2.1. [4] *For a strongly connected digraph D with $|V(D)| \geq 2$, $\kappa(\mu(D)) = \kappa(D) + i + 1$ if and only if $\delta(D) = \kappa(D) + i$ for each $i, 0 \leq i < \kappa(D)$.*

Remark. [4] If S is a minimum vertex cut of D with $|S| = \kappa(D)$ and S' is the twin of S in V', then $S \cup S' \cup \{u\}$ is a vertex cut of $\mu(D)$. Hence $\kappa(D) + 1 \leq \kappa(\mu(D)) \leq 2\kappa(D) + 1$.

Lemma 2.2. *For a strongly connected digraph D with $|V(D)| \geq 2$, $\kappa(\mu(D)) = 2\kappa(D) + 1$ if and only if $\delta(D) \geq 2\kappa(D)$.*

Proof. Let $\delta(D) \geq 2\kappa(D)$. By Remark there is a vertex cut of $\mu(D)$ with the cardinality $2\kappa(D) + 1$, hence $\kappa(\mu(D)) \leq 2\kappa(D) + 1$. If $\kappa(\mu(D)) < 2\kappa(D) + 1$, then by Lemma 2.1, we have $\delta(D) < 2\kappa(D)$, which is not true. Thus $\kappa(\mu(D)) = 2\kappa(D) + 1$.

Conversely, let $\kappa(\mu(D)) = 2\kappa(D) + 1$. Since $2\kappa(D) + 1 = \kappa(\mu(D)) \leq \delta(\mu(D)) = \delta(D) + 1$, we have $\delta(D) \geq 2\kappa(D)$. □

As a consequence of Lemma 2.1 and Lemma 2.2, we have the following corollary.

Corollary 2.3. *If D is a strongly connected digraph D with $|V(D)| \geq 2$, then $\kappa(\mu(D)) = min\{\delta(D) + 1, 2\kappa(D) + 1\}$.*

Lemma 2.4. *(Menger's Theorem [2]) In any digraph $D(x, y)$, where $(x, y) \notin A(D)$, the maximum number of pairwise internally disjoint directed (x, y)-paths is equal to the minimum number of vertices in an (x, y)-vertex-cut.*

Theorem 2.5. *For a strongly connected digraph D with $|V(D)| \geq 2$, $\mu(D)$ is super-κ if and only if $\delta(D) < 2\kappa(D)$.*

Proof. Suppose $\mu(D)$ is super-κ, but $\delta(D) \geq 2\kappa(D)$. By Lemma 2.2, $\kappa(\mu(D)) = 2\kappa(D) + 1$. Let F be a minimum vertex cut of D with $|F| = \kappa(D)$. Then $F \cup F' \cup \{u\}$ is a minimum vertex cut of $\mu(D)$, where F' is the twin of F in V'. But $F \cup F' \cup \{u\}$ is neither the out-neighbor set nor the in-neighbor set of a vertex of $\mu(D)$, which contradicts the fact that $\mu(D)$ is *super-κ*.

Now suppose $\delta(D) < 2\kappa(D)$ but $\mu(D)$ is not super-κ. Then, by Lemma 2.1, $\kappa(\mu(D)) = \delta(D) + 1$. Hence there is a minimum vertex cut S of $\mu(D)$ with $|S| = \kappa(\mu(D)) = \delta(D) + 1 \leq 2\kappa(D)$ such that $\mu(D) - S$ is not strongly connected and S is not the out-neighbor set or in-neighbor set of a vertex in $\mu(D)$.

Case 1. $|V \cap S| < \kappa(D)$.
Then $D - (V \cap S)$ is strongly connected, and each vertex of $V' - S$ has at least $\kappa(D)$ in-neighbors and $\kappa(D)$ out-neighbors in V and so has at least one in-neighbor and one out-neighbor in $V - S$. Thus $\mu(D) - S$ is strongly connected, which is impossible.

Case 2. $|V \cap S| \geq \kappa(D)$.

Subcase 2.1 $u \notin S$. Then $(V' - S) \cup \{u\}$ induces a strongly connected star in $\mu(D) - S$, say S^*. In addition, since $|S| = \kappa(\mu(D)) = \delta(D) + 1 \leq 2\kappa(D)$, we have $|V' \cap S| \leq \kappa(D)$.
If $|V' \cap S| < \kappa(D)$, then each vertex in $V - S$ has at least one in-neighbor and one out-neighbor in $V' - S$ (that is, in S^*), and so $\mu(D) - S$ is strongly connected, a contradiction.
Hence, $|V' \cap S| = \kappa(D)$, and so $|V \cap S| = \kappa(D)$. Then $|S| = 2\kappa(D) = \delta(D) + 1$. If $\kappa(D) > 1$, then $\kappa(D) < \delta(D)$. So any vertex in $V - S$ has at least one in-neighbor and one out-neighbor in $V' - S$, and so $\mu(D) - S$ is strongly connected, a contradiction. In the other case, $\kappa(D) = \delta(D) = 1$, and $|S| = 2$. Let $S = \{x, y'\}$, $x \in V$, and $y' \in V'$. Then, for any vertex $z \in V - x$, either z has at least one in-neighbor and one out-neighbor in $V' - y'$ or y' is the only out-neighbor or the only in-neighbor of z in V'. For the latter, the twin y of y' must be not equal to x (otherwise S is the out-neighbor set or the in-neighbor set of a vertex, contradicting our assumption). Without loss of generality, y' is the only out-neighbor of z,

thus $d^+_D(z) = 1$. y has an out-neighbor $y'_1(\neq y')$ in V', then zyy'_1 is a directed path from z to S^*. And z has an in-neighbor z'_1 in V'. If $z'_1 \neq y'$, then there is a directed path from S^* to z. If $z'_1 = y'$, then $z_1 = y$ and z also is an in-neighbor of y. $z'yz$ is the directed path from S^* to z, that is $\mu(D) - S$ is strongly connected, a contradiction.

Subcase 2.2 $u \in S$. Then $|V' \cap S| \leq 2\kappa(D) - |V \cap S| - 1 < \kappa(D)$, and every vertex z' in $V' - S$ has at least one out-neighbor and one in-neighbor in $V - S$ (otherwise, S would be the out-neighbor set or the in-neighbor set of z', a contradiction).

If $D - (V \cap S)$ is strongly connected, then $\mu(D) - S$ is strongly connected, a contradiction. We assume that $D - (V \cap S)$ is not strongly connected. Let $D_1, D_2, \cdots, D_k$ be all strongly connected components of $D - S$.

Claim For any two strongly connected components D_s and D_t $(s \neq t)$, there is a directed path from D_s to D_t in $\mu(D) - S$.

Proof of Claim. If the component D_s has out-neighbor in D_t, then there exists a directed path from D_s to D_t in $\mu(D) - S$.

If D_s has no out-neighbor in D_t. For any vertex $x \in V(D_s)$ and any vertex $y \in V(D_t)$, by Lemma 2.4, there are $t \geq \kappa(D)$ internally vertex disjoint directed paths, say $P_1, P_2, \cdots, P_t$, from x to y in D. For any $i = \{1, 2, \cdots, t\}$, if the length of P_i is even, set

$$P_i = xx_1 \cdots x_k \cdots x_{2k-1}y \, (k \geq 1).$$

Then there are two internally vertex disjoint directed paths from x to y in $\mu(D)$,

$$P_{i1} = xx'_1 \cdots x_k \cdots x'_{2k-1}y, \text{ and } P_{i2} = xx_1x'_2 \cdots x'_k \cdots x'_{2k-2}x_{2k-1}y.$$

If the length of P_i is odd, set

$$P_i = xx_1x_2 \cdots x_{2k-1}x_{2k}y.$$

Then there are two internally vertex disjoint directed paths from x to y in $\mu(D)$,

$$P_{i1} = xx'_1x_2 \cdots x'_{2k-1}x_{2k}y, \text{ and } P_{i2} = xx_1x'_2 \cdots x_{2k-2}x'_{2k}y.$$

Thus there are $2t \geq 2\kappa(D)$ directed paths from x to y in $\mu(D)$ which are internally vertex disjoint each other, since any two directed paths from x to y in D are internally vertex disjoint. Therefore there is at least one directed path from x to y after deleting $|S| \leq 2\kappa(D) - 1$ vertices. The proof of Claim is completed.

By Claim, we have that any two strongly connected components of D are in the same strongly connected component in $\mu(D) - S$. Also any vertex of $V' - S$ has at least one out-neighbor and one in-neighbor in $V - S$. Thus $\mu(D) - S$ is strongly connected, a final contradiction. □

Lemma 2.6. *[4] If D is a strongly connected digraph with $|V(D)| \geq 2$, then $\kappa(\mu^n(D)) = \kappa(D) + n$ if and only if $\delta(D) = \kappa(D)$.*

Corollary 2.7. *If G is a strongly connected digraph with $|V(D)| \geq 2$ and $\delta(D) = \kappa(D)$, then $\mu^n(D)$ is super-κ for any integer $n \geq 1$.*

Proof. Since $\delta(D) = \kappa(D) < 2\kappa(D)$, by Lemma 2.6, $\delta(\mu^n(D)) = \delta(D) + n < 2(\delta(D) + n) = 2(\kappa(D) + n) = 2\kappa(\mu^n(D))$ for any $n \geq 1$. By Theorem 2.5, we have $\mu^n(D)$ is super-κ for $n \geq 1$. □

3 Super arc connectivity of the Mycielskian

Lemma 3.1. *[4] For a strongly connected digraph D with $|V(D)| \geq 2$, $\lambda(\mu(D)) = \delta(D) + 1$.*

Theorem 3.2. *For a strongly connected digraph D with $|V(D)| \geq 2$, $\mu(D)$ is super-λ if and only if $D \not\cong \overrightarrow{K_2}$.*

Proof. If $D \cong \overrightarrow{K_2}$, then $\mu(D) = C_5$, where C_5 is a cycle of order 5 and each edge represents two opposite arcs. Obviously, $\mu(D)$ is not super-λ.

Conversely, suppose $\mu(D)$ is not super-λ. By Lemma 3.1, $\lambda(\mu(D)) = \delta(D)+1 = \delta(\mu(D))$. There exists a minimum arc cut F of $\mu(D)$ with $|F| = \delta(D) + 1$ such that $\mu(D) - F$ is not strongly connected but $F \neq E^-_{\mu(D)}(v)$ and $F \neq E^+_{\mu(D)}(v)$ for any $v \in V(\mu(D))$.

Claim Let D_i be a strongly connected component of $D-(A\cap F)$. Then there are a directed path from u to D_i and a directed path from D_i to u in $\mu(D)-F$.

Proof of Claim. Firstly, we show that there is a directed path from u to D_i in $\mu(D)-F$.

Suppose to the contrary that there is no directed path from u to D_i in $\mu(D)-F$.

Let $X' = N^-_{V'}(D_i)$. Then $|[u,X']| = |X'| \geq \delta(D)$, and $1 \leq |V(D_i)| \leq |V(D)|$ (equation in the last inequality holds only if $D-(A\cap F)$ is strongly connected).

Case 1. $D-(A\cap F)$ is strongly connected. Then $|V(D_i)| = |V(D)| = \delta(D)+1$ (otherwise, if $|V(D)| > \delta(D)+1$, then $|[u,V']| > \delta(D)+1$ and $|[V',V]| > \delta(D)+1$. There are at least $\delta(D)+2$ arc disjoint directed paths from u to D_i and so has at least one directed path in $\mu(D)-F$, a contradiction). Hence D is a complete digraph, and $\delta(D) \geq 2$ since $D \not\cong \overrightarrow{K_2}$. Thus $|[u,V']| = |V(D)| = \delta(D)+1$ and $|[V',V]| \geq |V(D)|\delta(D) \geq \delta(D)+2$. Since $F \neq [u,V']$, there exists at least one directed path from u to D_i in $\mu(D)-F$, a contradiction.

Case 2. $D-(A\cap F)$ is not strongly connected. Then $|A\cap F| \geq 1$ and $|([u,X'] \cup [X', V(D_i)]) \cap F| \leq \delta(D)$. If $|([u,X'] \cup [X', V(D_i)]) \cap F| < \delta(D)$, then there exists a directed path from u to D_i in $\mu(D)-F$ since $|X'| \geq \delta(D)$, which is not true. Thus we have $|([u,X'] \cup [X', V(D_i)]) \cap F| = \delta(D)$ and so $|A\cap F| = 1$. Furthermore, we have $|X'| = \delta(D)$ (otherwise, if $|X'| > \delta(D)$, then there are at least $\delta(D)+1$ arc disjoint directed paths from u to D_i in $\mu(D)$ and so there is at least one directed path from u to D_i in $\mu(D)-F$, which is impossible).

Subcase 2.1. $|V(D_i)| \geq 2$. Let $x, y \in V(D_i)$ and $y \in N^-_D(x)$. Then y' is an in-neighbor of x in V' and so $N^-_{V'}(y) \cup \{y'\} \subseteq N^-_{V'}(y) \cup N^-_{V'}(x) \subseteq X'$. Thus we have $|X'| \geq |N^-_{V'}(y)| + |\{y'\}| \geq \delta(D)+1$, which contradicts $|X'| = \delta(D)$.

Subcase 2.2. $|V(D_i)| = |\{x\}| = 1$. Then $\delta(D) = 1$ since $|A\cap F| = 1$. Moreover, we have $|X'| = \delta(D) = 1$ and so $|F| = 2$. Thus $d^-_D(x) = |X'| = 1$

(if not, $d_D^-(x) \geq 2$, it is not difficult to find a directed path from u to D_i in $\mu(D) - F$, a contradiction). Let y be the in-neighbor of x in D. Since $F \neq E_{\mu(D)}^-(x)$ and there is no directed path from u to D_i, $F = \{(y, x), (u, y')\}$. Note that $D \not\cong \overrightarrow{K_2}$, we have $|V(D)| \geq 3$. Thus $N_{D-x}^-(y) \neq \emptyset$. If there is a vertex $z \in N_{D-x}^-(y)$ which has an in-neighbor $v \neq y$ in D, then $uv'zy'x$ is a directed path from u to x in $\mu(D) - F$, which is impossible. We assume that for any $z \in N_{D-x}^-(y)$, y is the only in-neighbor of z in D. Thus y is both an in-neighbor and out-neighbor of z. In this case, there exists a directed path $uz'yzy'x$ from u to x in $\mu(D) - F$, contradicting to our assumption.

Similarly, we can show that there is at least one directed path from D_i to u in $\mu(D) - F$.

The proof of Claim is completed.

By Claim, in $\mu(D) - F$, $V(D_i) \cup \{u\}$ are contained in the same strongly connected component of $\mu(D) - F$. In particular, $V \cup \{u\}$ are contained in the same strongly connected component of $\mu(D) - F$. For any vertex in $V' - S$ has at least one out-neighbor and one in-neighbor in $V \cup \{u\}$. Thus $\mu(D) - F$ is strongly connected, a final contradiction. □

Corollary 3.3. *If D is a strongly connected digraph with $|V| \geq 3$, then $\mu^n(D)$ is super-λ for any integer $n \geq 1$.*

Proof is by induction on n.

Acknowledgements

The project is supported by NSFC (No.11171283, 11171279,11126321), Xiamen University of Technology(YKJ12030R), NSFXJ (2010211A06) and Fund of Xinjiang University (07.02.0432.01). We would like to thank the referees for kind help and valuable suggestions.

References

[1] R. Balakrishnan, S. Francis Raj, Connectivity of the Mycielskian of a graph, Discrete Math. 308 (2008) 2607-2610.

[2] J. A. Bondy, U. S. R. Murty, Graph Theory, Springer, 2008.

[3] G. J. Chang, L. Huang, X. Zhu, Circular chromatic number of Mycielski's graph, Discrete Math. 205 (1999) 23-37.

[4] L. Guo, X. Guo, Connectivity of the Mycielskian of a digraph, Appl. Math. Lett. 22 (2009) 1622-1625.

[5] H. Hajibolhassan, X. Zhu, The circular chromatic number and Mycielski construction, J.Graph Theory 44 (2003) 106-115.

[6] P. C. B. Lam, W. Lin, G. Gu, Z. Song, Circular chromatic number and a generalization of the construction of Mycielski, J.Combin.Theory Ser.B 89 (2003) 195-205.

[7] D. D. F. Liu, Circular chromatic number for iterated Mycielski graphs, Discrete Math. 285 (2004) 335-340.

[8] H. Liu, Circular chromatic number and Mycielski graphs, Acta Math.Sci.26B (2)(2006) 314-320.

[9] J. Mycielski, Sur le colouriage des graphes, Colloq.Math. 3 (1955) 23-29.