Smallest regular and almost regular triangle-free graphs without perfect matchings

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Abstract

A graph G is regular if the degree of each vertex of G is d and almost regular or more precisely a $(d, d+1)$-graph, if the degree of each vertex of G is either d or $d+1$. If $d \geq 2$ is an integer, G a triangle-free $(d, d+1)$-graph of order n without an odd component and $n \leq 4d$, then we show in this paper that G contains a perfect matching. Using a new Turán type result, we present an analogue for triangle-free regular graphs. With respect to these results, we construct smallest connected, regular and almost regular triangle-free even order graphs without perfect matchings.

Keywords: Perfect matching, Regular graph, Almost regular graph, Triangle-free graph, Turán type result.

In this paper, all graphs are finite and simple. The vertex set and edge set of a graph G are denoted by $V(G)$ and $E(G)$, respectively. The number $n = n(G) = |V(G)|$ is called the *order* of G. The *neighborhood* $N_G(x) = N(x)$ of a vertex x is the set of vertices adjacent with x, and the number $d_G(x) = d(x) = |N_G(x)|$ is the *degree* of x in the graph G. A *d-regular graph* G is a graph such that $d_G(x) = d$ for every vertex x in G. If $d \leq d_G(x) \leq d+1$ for each vertex x in a graph G, then we speak of an *almost regular graph* or more precisely of a $(d, d+1)$*-graph*. If M is a matching in a graph G with the property that every vertex is incident with an edge of M, then M is a *perfect matching.* The *clique number* $\omega(G)$ of a graph G is the maximum order among the complete subgraphs of G. We denote by $K_{r,s}$ the complete bipartite graph with partite sets X and Y, where $|X| = r$ and $|Y| = s$. If G is a graph and $A \subseteq V(G)$, then we denote by $G[A]$ the subgraph induced by A and by $q(G - A)$ the number of odd components in the subgraph $G - A$.

In his important classical work, König [5] proved in 1916 that each d-regular bipartite graph contains a perfect matching when $d \geq 1$. Clearly, this is not valid for d-regular graphs in general. However, if the order of the graph G is even and at most $3d+2$, then Wallis [10] has shown that G contains a perfect matching as well. More precisely, Wallis [10] proved the following.

Theorem 1 (Wallis [10] 1981) Let $d \geq 3$ be an integer, and let G be a d-regular graph of order n without an odd component. If

(i) $n \leq 3d + 2$ when $d \geq 4$ is even or

(ii) $n \leq 3d + 5$ when $d \geq 3$ is odd or

(iii) $n \leq 20$ when $d = 4$,

then G has a perfect matching.

In the case that d is even, Zhao [11] has proved in 1991 the following more general result.

Theorem 2 (Zhao [11] 1991) Let $d \geq 2$ be an integer, and let G be a $(d, d+1)$-graph without an odd component. If $|V(G)| \leq 3d + 3$, then G has a perfect matching.

For supplements, extensions or generalizations of Theorems 1 and 2, see the articles by Caccetta and Mardiyono [1], Volkmann [9] and Klinkenberg and Volkmann [2, 3, 4].

In this paper, we will prove similar results for triangle-free graphs. The proofs of our main theorems are based on the well-known theorem of Turán [6] and Tutte's famous 1-factor theorem [7] (for proofs of these theorems, see e.g., [8] pp. 137-139 and 211-213).

Theorem 3 (Tutte [7] 1947) A nontrivial graph G has a perfect matching (or a 1-factor) if and only if $q(G - S) \leq |S|$ for every proper subset S of $V(G)$.

Theorem 4 (**Turán [6] 1941**) Let $p \geq 1$ be an integer. If G is a graph of order n with clique number $\omega(G) \leq p$, then

$$2|E(G)| \leq \frac{(p-1)n^2}{p}.$$

Theorem 5 Let $d \geq 2$ be an integer, and let G be a triangle-free $(d, d+1)$-graph of order n without an odd component. If $n \leq 4d$, then G contains a perfect matching.

Proof. Suppose on the contrary that G does not contain a perfect matching. Then Theorem 3 implies that there exists a non-empty set $A \subset V(G)$ such that $q(G-A) \geq |A|+1$. Since n is even, the numbers $q(G-A)$ and $|A|$ are of the same parity, and we deduce that

$$q(G-A) \geq |A| + 2. \tag{1}$$

We call an odd component of $G-A$ large if it has at least $2d+1$ vertices and small otherwise. If we denote by α and β the number of large and small components, respectively, then we deduce from (1) that

$$\alpha + \beta = q(G-A) \geq |A| + 2. \tag{2}$$

In addition, we observe that

$$n \geq |A| + \alpha(2d+1) + \beta. \tag{3}$$

First we will show that there are at least d edges of G joining each small component of $G-A$ with A. Let Q be a small component of $G-A$ of order t with $1 \leq t \leq 2d-1$. Since G is triangle-free, Theorem 4 implies that $2|E(Q)| \leq t^2/2$. In addition, we deduce from the hypothesis that G is a $(d, d+1)$-graph that $2|E(Q)| = \sum_{v \in V(Q)} d_Q(v) \geq dt$ and consequently there are at least $\lceil dt - t^2/2 \rceil$ edges of G joining Q with A. If we define $t = x$ and $g(x) = dx - x^2/2$, then, because of $1 \leq t \leq 2d-1$, we like to determine the minimum of the function g in the interval $I : 1 \leq x \leq 2d-1$. It is straightforward to verify that

$$\min_{x \in I}\{g(x)\} = g(1) = g(2d-1) = d - \frac{1}{2}.$$

Thus there are at least d edges of G joining Q with A.

Using the hypothesis that G is $(d, d+1)$-graph without an odd component, we deduce that

$$\alpha + d\beta \leq |A|(d+1). \tag{4}$$

Next we distinguish three cases.

Case 1: Assume that $\alpha \geq 2$. The hypothesis $n \leq 4d$ and (3) lead to the contradiction

$$\begin{aligned} 4d \geq n &\geq |A| + \alpha(2d+1) + \beta \\ &\geq 2(2d+1) = 4d+2. \end{aligned}$$

Case 2: Assume that $\alpha = 1$. Inequality (2) yields $\beta \geq |A| + 1$, and thus we obtain by (4)

$$|A| \geq d+1.$$

Applying (3) and the hypothesis $n \leq 4d$, we arrive at

$$\begin{aligned} 4d \geq n &\geq |A| + \alpha(2d+1) + \beta \\ &\geq d+1+2d+1+\beta \\ &= 3d+2+\beta \end{aligned}$$

and so $\beta \leq d-2$. Combining this with $\beta \geq |A| + 1$ and $|A| \geq d+1$, we obtain the contradiction

$$d+2 \leq |A|+1 \leq \beta \leq d-2.$$

Case 3: Assume that $\alpha = 0$. Inequality (2) yields $\beta \geq |A| + 2$, and thus (4) leads to

$$|A| \geq 2d. \qquad (5)$$

Applying the bound $\beta \geq |A| + 2$, we obtain

$$\beta \geq |A| + 2 \geq 2d + 2. \qquad (6)$$

According to (3), (5) and (6), we finally arrive at the contradiction

$$4d \geq n \geq |A| + \alpha(2d+1) + \beta \geq |A| + \beta \geq 4d + 2. \quad \|$$

In view of Theorem 5, the following examples are smallest connected, triangle-free almost regular even order graphs without perfect matchings.

Example 6 Let $d \geq 2$ be an integer and, let $K_{d,d+1}$ be the complete bipartite graph with the partite sets $\{x_1, x_2, \ldots, x_{d+1}\}$ and $\{y_1, y_2, \ldots, y_d\}$. If we delete in the graph $K_{d,d+1}$ the edge x_1y_1, then we denote the resulting graph by H_1. In addition, let $K_{d,d+1}$ be the complete bipartite graph with the partite sets $\{u_1, u_2, \ldots, u_{d+1}\}$ and $\{v_1, v_2, \ldots, v_d\}$. If we delete the edge u_1v_1, then we denote the resulting graph by H_2. Now let H be the disjoint union of H_1 and H_2 together with the two edges u_1y_1 and v_1x_1. It is straightforward to verify that H is a connected bipartite (and thus triangle-free) $(d, d+1)$-graph of order $|V(H)| = 4d + 2$ without a perfect matching. This example shows that the bound on n in Theorem 5 is sharp.

As Theorem 4 is not strong enough for the proof of our next main theorem (cf., Theorem 8 below), we need the following Turán type result for triangle-free graphs.

Theorem 7 Let $d \geq 5$ be an integer, and let G be a triangle-free graph of order $n = 2d + 1$ such that $\Delta(G) = d$ and $\delta(G) = d - 1$. Then G contains at least d vertices of degree $d - 1$.

Proof. Suppose on the contrary that G contains at most $d - 1$ and thus at most $d - 2$ vertices of degree $d - 1$. Let w be a vertex of degree d, and let $x_1, x_2, \dots, x_d$ be the neighbors of w. Assume, without loss of generality, that $d(x_1) = d$. Since G is triangle-free, the vertex x_1 has $d - 1$ further neighbors $y_1, y_2, \dots, y_{d-1}$. Let u be the remaining vertex of G. Assume, without loss of generality, that $d(x_2) = d$ and $\{y_1, y_2, \dots, y_{d-2}\} \subset N(x_2)$.

If $u \in N(x_2)$, then $N(u) \subseteq \{x_2, x_3, \dots, x_d\} \cup \{y_{d-1}\}$. In the case that $y_{d-1} \in N(u)$, we obtain the contradiction $d(y_{d-1}) \leq 3 < d - 1 = \delta(G)$. In the remaining case we have $N(u) = \{x_2, x_3, \dots, x_d\}$ and so $d(u) = d - 1$. This implies that there are at most $(d-2)(d-2)$ edges joining the set $\{x_3, x_4, \dots, x_d\}$ with the set $\{y_1, y_2, \dots, y_{d-1}\}$. Applying the assumption that there are at most $d - 2$ vertices of degree $d - 1$, we arrive at the contradiction

$$
\begin{aligned}
d^2 - 2d + 3 &= 2d + (d-3)(d-1) \leq \sum_{i=1}^{d-1} d(y_i) \\
&\leq (d-2)(d-2) + 2(d-1) - 1 \\
&= d^2 - 2d + 1.
\end{aligned}
$$

Thus $u \notin N(x_2)$ and $N(x_2) = \{y_1, y_2, \dots, y_{d-1}\} \cup \{w\}$.

Next assume, without loss of generality, that $\{y_1, y_2, \dots, y_{d-3}\} \subset N(x_i)$ for an index $3 \leq i \leq d$, say $\{y_1, y_2, \dots, y_{d-3}\} \subset N(x_3)$.

If $u \in N(x_3)$, then $y_{d-1} \in N(u)$ or $y_{d-2} \in N(u)$. If $d(x_3) = d$, then we obtain a contradiction as above. So assume that $d(x_3) = d - 1$.

Assume first that $y_{d-2}, y_{d-1} \in N(u)$. Then u has $d-4$ further neighbors in $N(w)$, say $x_4, x_5, \dots, x_{d-1}$. If $d(u) = d$, then $x_d \in N(u)$, and we obtain the contradiction $d(y_{d-2}) = d(y_{d-1}) = 3 < d - 1 = \delta(G)$. Thus assume that $d(u) = d - 1$. This implies $d(y_{d-2}), d(y_{d-1}) \leq 4$, a contradiction when $d \geq 6$. In the case $d = 5$, we observe that $d(x_3) = d(u) = d - 1 = 4$ and $d(y_3), d(y_4) \leq 4$, a contradiction to the assumption that there are most $d - 2 = 3$ vertices of degree $d - 1 = 4$.

Assume second that, without loss of generality, $y_{d-1} \in N(u)$ and $y_{d-2} \notin N(u)$. This yields to $N(u) = \{x_3, x_4, \ldots, x_d\} \cup \{y_{d-1}\}$, and we arrive at the contradiction $d(y_{d-1}) = 3$.

This shows that we obtain a contradition when $u \in N(x_i)$ for an index $2 \leq i \leq d$. Consequently, $N(u) = \{y_1, y_2, \ldots, y_{d-1}\}$ and thus $d(u) = d - 1$. Since there are at most $(d-1)(d-3)$ edges joining $\{y_1, y_2, \ldots, y_{d-1}\}$ with $\{x_3, x_4, \ldots, x_d\}$, we finally arrive at the contradiction

$$\begin{aligned} d^2 - 3d + 3 &= d + (d-3)(d-1) \leq \sum_{i=3}^{d} d(x_i) \\ &\leq (d-1)(d-3) + d - 2 = d^2 - 3d + 1. \quad \| \end{aligned}$$

Theorem 8 Let $d \geq 3$ be an integer, and let G be a triangle-free d-regular graph of order n without an odd component. If

(i) $n \leq 6d + 2 = 20$ when $d = 3$ or

(ii) $n \leq 9d = 36$ when $d = 4$ or

(iii) $n \leq 6d + 8$ when $d \geq 5$,

then G has a perfect matching.

Proof. Suppose on the contrary that G does not contain a perfect matching. Then Theorem 3 implies that there exists a non-empty set $A \subset V(G)$ such that

$$q(G - A) \geq |A| + 2. \tag{7}$$

We call an odd component of $G - A$ large if it has at least $2d+1$ vertices and small otherwise. If we denote by α and β the number of large and small components, respectively, then we deduce from (7) that

$$\alpha + \beta = q(G - A) \geq |A| + 2. \tag{8}$$

As we have seen in the proof of Theorem 5, there are at least d edges of G joining each small component of $G - A$ with A. Using the hypothesis that G is a d-regular graph without an odd component, we deduce that

$$\alpha + d\beta \leq |A|d. \tag{9}$$

This implies that $\beta \leq |A|$, and thus (8) yields to $\alpha \geq 2$. Applying (9) once more, we obtain $\beta \leq |A| - 1$, and therefore (8) leads to $\alpha \geq 3$. Since $A \neq \emptyset$, we arrive at

$$n \geq |A| + \alpha(2d + 1) + \beta \geq 1 + 3(2d + 1) = 6d + 4. \tag{10}$$

In the case that $d = 3$, this is contradiction to our hypothesis, and (i) is proved.

Assume that $d = 4$. Suppose that $|A| = 1$. As we have seen above, $G-A$ has at least three odd components of order greater or equal $2d + 1 = 9$. Since G is 4-regular without an odd component, there are at least 2 edges of G joining each such large component of $G - A$ with A, a contradiction to $d = 4$.

It remains tha case that $|A| \geq 2$. If $\alpha = 3$, then it follows from (8) that $\beta \geq |A| - 1$. Using the fact that $\beta \leq |A| - 1$, we obtain $\beta = |A| - 1$. Since there are at least d edges of G joining each small component of $G - A$ with A, and at least 6 edges of G joining the three large component of $G - A$ with A, we arrive at the contradiction

$$2\alpha + 4\beta = 6 + 4(|A| - 1) \leq 4|A|.$$

Thus $\alpha \geq 4$, and we conclude that

$$n \geq |A| + \alpha(2d + 1) + \beta \geq 2 + 4(2d + 1) = 8d + 6 = 38.$$

However, this is a contradiction to our hypothesis, and (ii) is proved.

Assume now that $d \geq 5$. Assume first that $|A| = 1$. Combining the fact that $\alpha \geq 3$ with Theorem 7, we find that each large component of $G - A$ is of order at least $2d + 3$, and therefore it follows that

$$n \geq |A| + \alpha(2d + 3) + \beta \geq 1 + 3(2d + 3) = 6d + 10. \qquad (11)$$

Assume next that $|A| \geq 2$. If $\alpha = 3$, then it follows from (8) that $\beta \geq |A| - 1$. Using the fact that $\beta \leq |A| - 1$, we obtain $\beta = |A| - 1$. If $|A| \geq 4$, then

$$n \geq |A| + \alpha(2d + 1) + \beta \geq 4 + 3(2d + 1) + 3 = 6d + 10. \qquad (12)$$

If $|A| = 3$, then then each small component of $G - A$ has order at least $d - 2$. If U is a small component of minimum order, then we observe that

$$n \geq |A| + \alpha(2d + 1) + \beta|V(U)| \geq 3 + 3(2d + 1) + 2(d - 2) = 8d + 2. \qquad (13)$$

If $|A| = 2$, then the hypothesis that G is triangle-free implies that the small component of $G - A$ has order at least d, and it follows that

$$n \geq |A| + \alpha(2d + 1) + d \geq 2 + 3(2d + 1) + d = 7d + 5. \qquad (14)$$

If $\alpha \geq 4$, then we conclude that

$$n \geq |A| + \alpha(2d + 1) + \beta \geq 2 + 4(2d + 1) = 8d + 6. \qquad (15)$$

Combining (11), (12), (13), (14) and (15), we deduce that

$$n \geq \min\{6d+10, 8d+2, 7d+5, 8d+6\} = 6d+10.$$

This is a contradiction to our hypothesis, and (iii) is also proved. ||

In view of Theorem 8, the following examples are smallest connected, triangle-free regular even order graphs without perfect matchings.

Example 9 a) Let H consists of a cycle $x_1x_2\ldots x_7x_1$ of length 7 together with the cords x_2x_5, x_3x_6 and x_4x_7. Furthermore, let H_1, H_2 and H_3 be three copies of H such that $d_{H_1}(u) = d_{H_2}(v) = d_{H_3}(w) = 2$. Now let G be the disjoint union of H_1, H_2, H_3 and a further vertex z together with the 3 edges uz, vz and wz. Then G is a connected, 3-regular and triangle-free graph of order 22 without a perfect matching, and therefore Theorem 8 (i) is best possible.

b) Let H consists of a cycle $x_1x_2\ldots x_9x_1$ of length 9 together with the cords x_1x_5, x_1x_7, x_2x_6, x_2x_8, x_3x_7, x_4x_9, x_5x_8 and x_6x_9. Furthermore, let H_1, H_2, H_3 and H_4 be four copies of H such that

$$\begin{aligned} d_{H_1}(u) &= d_{H_1}(u') = d_{H_2}(v) = d_{H_2}(v') \\ &= d_{H_3}(w) = d_{H_3}(w') = d_{H_4}(x) = d_{H_4}(x') = 3. \end{aligned}$$

Now let G be the disjoint union of H_1, H_2, H_3, H_4 and two further vertices z and z' together with the 8 edges uz, $u'z'$, vz, $v'z'$, wz, $w'z'$, xz and $x'z'$. The resulting graph G is connected, 4-regular and triangle-free graph of order 38 without a perfect matching, and therefore Theorem 8 (ii) is best possible.

c) Let $d \geq 5$ an integer, and let $K_{d+1,d+1}$ be the complete bipartite graph with the partite sets $\{x_1, x_2, \ldots, x_{d+1}\}$ and $\{y_1, y_2, \ldots, y_{d+1}\}$. If M is the perfect matching $M = \{y_1x_2, y_2x_3, \ldots, y_dx_{d+1}, y_{d+1}x_1\}$, then $H' = K_{d+1,d+1} - M$ is a d-regular graph.

Case 1: Assume that d is odd. Then

$$M' = \{x_1y_1, x_2y_2, \ldots, x_{\frac{d-1}{2}}y_{\frac{d-1}{2}}\}$$

is a matching of H'. Let H be the disjoint union of $H' - M'$ and a further vertex w together with the edges $wx_1, wy_1, wx_2, wy_2, \ldots, wx_{\frac{d-1}{2}}, wy_{\frac{d-1}{2}}$. Then H is a triangle-free graph such that $d_H(w) = d-1$ and $d_H(x) = d$ for all vertices different from w. In addition,

$$M'' = \left\{x_{\frac{d+1}{2}}y_{\frac{d+1}{2}}, x_{\frac{d+3}{2}}y_{\frac{d+3}{2}}, \ldots, x_{\frac{2d-4}{2}}y_{\frac{2d-4}{2}}\right\}$$

is a matching of H. Let $H_1 = H - M''$, and let H_2, H_3 be two copies of H with $d_{H_2}(u) = d - 1$ and $d_{H_3}(v) = d - 1$. Now let G be the disjoint union of H_1, H_2, H_3 and a further vertex z together with the d edges $zx_{\frac{d+1}{2}}, zy_{\frac{d+1}{2}}, zx_{\frac{d+3}{2}}, zy_{\frac{d+3}{2}}, \ldots, zx_{\frac{2d-4}{2}}, zy_{\frac{2d-4}{2}}$, zu, zv and zw. The resulting graph G is connected, d-regular, triangle-free and of order $6d + 10$ without a perfect matching, and therefore Theorem 8 (iii) is best possible when $d \geq 5$ is odd.

Case 2: Assume that $d \geq 6$ is even. Then

$$M' = \{x_1y_1, x_2y_2, \ldots, x_{\frac{d+2}{2}}y_{\frac{d+2}{2}}\}$$

is a matching of H'. Now let H be the disjoint union of $H' - M'$ and a further vertex w together with the edges $wx_1, wy_1, wx_2, wy_2, \ldots, wx_{\frac{d}{2}}, wy_{\frac{d}{2}}$. Then H is a triangle-free graph such that $d_H(x_{\frac{d+2}{2}}) = d_H(y_{\frac{d+2}{2}}) = d - 1$ and $d_H(x) = d$ for all other vertices of H. In addition,

$$M'' = \left\{x_{\frac{d+4}{2}}y_{\frac{d+4}{2}}, x_{\frac{d+6}{2}}y_{\frac{d+6}{2}}, \ldots, x_{\frac{2d-4}{2}}y_{\frac{2d-4}{2}}\right\}$$

is a matching of H for $d \geq 8$. Define $M'' = \emptyset$ when $d = 6$. Let $H_1 = H - M''$, and let H_2, H_3 be two copies of H such that $d_{H_2}(u_1) = d_{H_2}(u_2) = d - 1$ and $d_{H_3}(v_1) = d_{H_3}(v_2) = d - 1$. Now let G be the disjoint union of H_1, H_2, H_3 and a further vertex z together with the d edges $zx_{\frac{d+2}{2}}, zy_{\frac{d+2}{2}}, zx_{\frac{d+4}{2}}, zy_{\frac{d+4}{2}}, \ldots, zx_{\frac{2d-4}{2}}, zy_{\frac{2d-4}{2}}$, zu_1, zu_2, zv_1 and zv_2. The resulting graph G is connected, d-regular, triangle-free and of order $6d+10$ without a perfect matching. Therefore G shows that Theorem 8 (iii) is best possible when $d \geq 6$ is even.

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