# FURTHER APPLICATIONS OF HSU'S SUMMATION FORMULA

Yidong Sun†, Shuang Wang and Xiao Guan
Department of Mathematics,
Dalian Maritime University, 116026 Dalian, P.R. China
†*Email: sydmath@yahoo.com.cn*

**Abstract.** In this paper, we first survey the connections between Bell polynomials (numbers) and the derangement polynomials (numbers). Their close relations are mainly based on Hsu' summation formula. According to this formula, we present some new identities involving harmonic numbers, Bell polynomials (numbers) and the derangement polynomials (numbers). Moreover, we find that the series $\sum_{m\geq 0}(\frac{D_m}{m!} - \frac{1}{e})$ is (absolutely) convergent and their sums are also determined, where $D_m$ is the $m$th derangement number.

**Keywords**: Bell polynomial, Derangement polynomial, Harmonic number, Stirling number.

2000 MATHEMATICS SUBJECT CLASSIFICATION: 05A10; 05A19; 11A07; 11B73.

## 1. INTRODUCTION

It is well known [11] that the unsigned Stirling number of the first kind $(-1)^{m-j}s(m,j)$ counts the number of permutations on $[m] = \{1, 2, \ldots, m\}$ with exactly $j$ cycles, and the Stirling number of the second kind $S(m,j)$ enumerates the number of set partitions on $[m]$ with exactly $j$ blocks. They also obey the following relations

$$(x)_m = \sum_{j=0}^{m} s(m,j)x^j, \tag{1.1}$$

$$x^m = \sum_{j=0}^{m} S(m,j)(x)_j, \tag{1.2}$$

where $(x)_m = x(x-1)\cdots(x-m+1)$ for $m \geq 1$ with $(x)_0 = 1$ is the Pochhammer symbol.

The Bell polynomials $\{\mathcal{B}_m(x)\}_{m\geq 0}$ are defined by

$$\mathcal{B}_m(x) = \sum_{j=0}^{m} S(m,j)x^j. \tag{1.3}$$

It is clear that $\mathcal{B}_m(1)$ is the $m$-th Bell number, denoted by $B_m$, counting the total number of partitions of $[m]$ (with $B_0 = 1$). The Bell polynomials $\mathcal{B}_m(x)$ satisfy the recurrence

$$\mathcal{B}_{m+1}(x) = x\sum_{j=0}^{m} \binom{m}{j} \mathcal{B}_j(x). \tag{1.4}$$

The derangement polynomials $\{\mathcal{D}_m(x)\}_{m\geq 0}$ are defined by

$$\mathcal{D}_m(x) = \sum_{j=0}^{m} \binom{m}{j} j!(x-1)^{m-j}. \tag{1.5}$$

Clearly, $\mathcal{D}_m(1) = m!$ and $\mathcal{D}_m(0)$ is the $m$-th derangement number, denoted by $D_m$, counting the number of fixed-point-free permutations on $[m]$ (with $D_0 = 1$). The derangement polynomials $\mathcal{D}_m(x)$, also called $x$-factorials of $m$, have been considerably investigated by Eriksen, Freij and Wästlund [5], Sun and Zhuang [13]. The derangement polynomials also satisfy the recursive relation

$$\mathcal{D}_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} \mathcal{D}_k(x) y^{n-k}. \tag{1.6}$$

It seems that the Bell numbers (polynomials) and the derangement numbers (polynomials) have no direct connections. In fact, there exists many implications between them. As early as 1933, Broggi [1] presented a nice identity

$$\sum_{j=0}^{n} \binom{n}{j} j^n D_{n-j} = n! B_n. \tag{1.7}$$

Using the explicit formula for Stirling number of the second kind

$$S(n,j) = \frac{1}{j!} \sum_{k=0}^{j} (-1)^{j-k} \binom{j}{k} k^n,$$

Riordan [8, P193] generalized the Broggi identity to the polynomial case

$$n!\mathcal{B}_n(x) = \sum_{j=0}^{n} \binom{n}{j} j^n x^j \mathcal{D}_{n-j}(1-x). \tag{1.8}$$

In 1993, by the Inclusion-Exclusion principle, Clarke and Sved [2] proved that for any integers $n \geq m \geq 0$, there holds

$$\sum_{j=0}^{n} \binom{n}{j} j^m D_j = n! \sum_{i=0}^{m} (-1)^i \binom{m}{i} n^{m-i} B_i, \tag{1.9}$$

which is a generalized version of Tao's identity [15]. In 1990, Rousseau [10] also obtained an equivalent form of (1.9) by a somewhat different algebraic route.

In 2005, Vinh [16] provided another general identity for any integers $n \geq m \geq 0$,

$$\sum_{j=0}^{n} \binom{n}{j} j^m D_{n-j} = n! B_m, \tag{1.10}$$

and deduced a common generalization of (1.9) and (1.10),

$$\sum_{j=0}^{n} \binom{n}{j} g_m(j) D_{n-j} = n! \sum_{i=0}^{m} a_i B_i, \tag{1.11}$$

where $g_m(x) = \sum_{i=0}^{m} a_i x^i$ is a polynomial of degree $m$.

It should be noticed that (1.10) has abundant and nontrivial applications. The Broggi identity, as observed by Riordan [8], is a truncated version of the famous Dobinski formula for Bell numbers [4, 9]. Actually, multiplying both sides by $\frac{1}{n!}$ in (1.10) and then taking $n \to \infty$, we have

$$B_m = \frac{1}{e} \sum_{j=0}^{\infty} \frac{j^m}{j!},$$

where we use the well-known result

$$\lim_{n \to \infty} \frac{D_n}{n!} = \frac{1}{e}. \tag{1.12}$$

The second example indicates that the Broggi-Vinh identity implies the dual form of Sun-Zagier's congruence [14]. Precisely, according to $\binom{p-1}{j} \equiv (-1)^j \pmod{p}$, and Wilson's congruence $(p-1)! \equiv -1 \pmod{p}$, after shifting $j = n - j$, the case $n = p - 1$ in (1.10) for a prime $p$ $(> m)$ generates

$$(-1)^{m+1} B_m \equiv \sum_{j=0}^{p-1} (-1)^j (j+1)^m D_j \pmod{p}. \tag{1.13}$$

Note that another new equivalent form of Sun-Zagier's congruence can also be obtained by setting $n = p - 1$ in (1.9),

$$(1.14) \qquad (-1)^{m+1} B_{m+1} \equiv \sum_{j=0}^{p-1} (-1)^j j^m D_j \pmod{p}.$$

In 1993, Hsu [6] proposed the following summation rule. Let $F(n,k)$ be a bivariate function defined for integers $n, k \geq 0$. If there can be found a summation formula for $n \geq j \geq 0$

$$(1.15) \qquad \sum_{k=j}^{n} \binom{k}{j} F(n,k) = \phi(n,j),$$

then for $m \geq 0$ one has the summation formula

$$(1.16) \qquad \sum_{k=0}^{n} F(n,k) k^m = \sum_{j=0}^{m} S(m,j) j! \phi(n,j).$$

Hsu [6] also gave an extension of the above summation rule, and presented many special summation identities. Hsu and Shiue [7] gave applications of the extended rule to generalized Eulerian polynomials and remarked that this simple summation rule stated can still be used to find various special sums. Note that (1.7)-(1.10) are brought into this general framework.

In this paper, we will give further applications of Hsu's summation rule.

## 2. Identities involving Bell and derangement numbers or polynomials

**Lemma 2.1.** *For any integers $n \geq m \geq 0$, there holds*

$$(2.1) \qquad \sum_{k=0}^{n} \binom{n}{k} k^m x^{n-k} = n! \sum_{j=0}^{m} S(m,j) \frac{(x+1)^{n-j}}{(n-j)!},$$

*or equivalently*

$$(2.2) \qquad x \sum_{k=0}^{n} \binom{n}{k} k^m \mathcal{B}_{n-k}(x) = n! \sum_{j=0}^{m} S(m,j) \frac{\mathcal{B}_{n-j+1}(x)}{(n-j)!},$$

$$(2.3) \qquad \sum_{k=0}^{n} \binom{n}{k} k^m \mathcal{D}_{n-k}(x) = n! \sum_{j=0}^{m} S(m,j) \frac{\mathcal{D}_{n-j}(x+1)}{(n-j)!}.$$

*Proof.* Let $F(n,k) = \binom{n}{k} x^{n-k}$ in (1.15). By the identity

$$\binom{n}{k}\binom{k}{j} = \binom{n}{j}\binom{n-j}{n-k},$$

we have $\phi(n,j) = \binom{n}{j}(1+x)^{n-j}$. Then (1.16) in this case produces (2.1).

Define a linear (invertible) transformation

$$L_1(x^k) = \mathcal{B}_k(x), \quad (k = 0, 1, 2, \dots).$$

Then by (1.4) we have

$$xL_1((x+1)^n) = x\sum_{k=0}^{n}\binom{n}{k}L_1(x^k) = x\sum_{k=0}^{n}\binom{n}{k}\mathcal{B}_k(x) = \mathcal{B}_{n+1}(x).$$

Hence (2.2) follows by acting $xL_1$ on the two sides of (2.1).

Similarly, define another linear transformation

$$L_2(x^k) = \mathcal{D}_k(x), \quad (k = 0, 1, 2, \dots).$$

Then by (1.6) in the case $y = 1$ we have

$$L_2((x+1)^n) = \sum_{k=0}^{n}\binom{n}{k}L_2(x^k) = \sum_{k=0}^{n}\binom{n}{k}\mathcal{D}_k(x) = \mathcal{D}_n(x+1).$$

Hence (2.3) follows by acting $L_2$ on the two sides of (2.2). □

Clearly, the case $x = 0$ in (2.3) reduces to the Vihn identity (1.10).

**Lemma 2.2.** *For any integers $m \geq j \geq 0$, let $g_m(x) = \sum_{i=0}^{m} a_i x^i$ be a polynomial of degree $m$. Then there holds*

$$\sum_{i=j}^{m} a_i S(i,j) = \frac{1}{j!}\sum_{k=0}^{j}(-1)^{j-k}\binom{j}{k}g_m(k). \tag{2.4}$$

Proof. Shifting $m$ by $i$, multiplying by $a_i$ on the two sides of (2.1), and then summing for $i$ from 0 to $m$, after exchanging the summation, one can derive

$$\sum_{k=0}^{n}\binom{n}{k}g_m(k)x^{n-k} = n!\sum_{j=0}^{m}\frac{(x+1)^{n-j}}{(n-j)!}\sum_{i=j}^{m}a_i S(i,j),$$

which, by comparing the coefficient of $\frac{(x+1)^{n-j}}{(n-j)!}$, yields (2.4). □

Next we consider the two special cases when $g_m(x) = (x-1)_m$ or $g_m(x) = \frac{d}{dx}(x)_{m+1}$.

**Corollary 2.3.** *For any integers $m \geq j \geq 0$, there hold*

$$\sum_{i=0}^{m} s(m+1, i+1)S(i,j) = (-1)^{m-j}\frac{m!}{j!}, \tag{2.5}$$

$$\sum_{i=0}^{m} s(m+1, i+1)\mathcal{B}_i(x) = (-1)^m\mathcal{D}_m(1-x). \tag{2.6}$$

*Proof.* For $g_m(x) = (x-1)_m$, by (1.1), one has $a_i = s(m+1, i+1)$. Note that $g_m(k) = 0$ when $1 \le k \le m$ and $g_m(0) = (-1)^m m!$. Then (2.5) follows from (2.4) by setting $a_i = s(m+1, i+1)$.

Multiplying by $x^j$ the two sides of (2.5), and then summing for $j$ from 0 to $m$, after exchanging the order of summation, by (1.3) and (1.5), (2.6) is followed. □

Note that (2.5) and (2.6) have been established recently by Sun, Wu and Zhuang [12]. They also utilized (2.6) to generalize Sun-Zagier's congruence to polynomial cases.

**Corollary 2.4.** *For any integers $m \ge j \ge 0$, there hold*

$$\sum_{i=0}^{m}(i+1)s(m+1,i+1)S(i,j) = (-1)^{m-j}\frac{(m+1)!}{j!(m-j+1)}, \tag{2.7}$$

$$\sum_{i=0}^{m}(i+1)s(m+1,i+1)x^i = (m+1)!\sum_{j=0}^{m}(-1)^{m-j}\frac{1}{m-j+1}\binom{x}{j}, \tag{2.8}$$

$$\sum_{i=0}^{m}(i+1)s(m+1,i+1)\mathcal{B}_i(x) = \sum_{j=0}^{m}(-1)^{m-j}\binom{m+1}{j}(m-j)!x^j, \tag{2.9}$$

$$\begin{aligned}&\sum_{i=1}^{m}(i+1)s(m+1,i+1)\mathcal{B}_{i-1}(x)\\ &= (-1)^{m+1}\sum_{j=1}^{m}\binom{m+1}{j}(m-j)!\mathcal{D}_{j-1}(1-x).\end{aligned} \tag{2.10}$$

*Proof.* For $g_m(x) = \frac{d}{dx}(x)_{m+1}$, by (1.1), one has $a_i = (i+1)s(m+1, i+1)$. Note that $g_m(k) = (-1)^{m-k}k!(m-k)!$ for $0 \le k \le m$. Then

$$\begin{aligned}&\sum_{i=0}^{m}(i+1)s(m+1,i+1)S(i,j)\\
&= \frac{1}{j!}\sum_{k=0}^{j}(-1)^{j-k}\binom{j}{k}(-1)^{m-k}k!(m-k)!\\
&= (-1)^{m-j}\sum_{k=0}^{j}\frac{(m-k)!}{(j-k)!}\\
&= (-1)^{m-j}(m-j)!\sum_{k=0}^{j}\binom{m-k}{m-j}\\
&= (-1)^{m-j}(m-j)!\binom{m+1}{j},\end{aligned}$$

which proves (2.7), where we use the binomial summation

$$\sum_{k=0}^{j}\binom{a+k}{n}=\binom{a+j+1}{n+1}.$$

Multiplying by $(x)_j$ (resp. $x^j$) the two side of (2.7), and then summing for $j$ from 0 to $m$, after exchanging the order of summation, by (1.2) (resp. (1.3)), (2.8) (resp. (2.9)) follows.

Removing first the constant term in (2.8), and multiplying by $\frac{1}{x}$, then acting by $L_1$ on the two sides, by (2.7), and using

$$\begin{aligned}L_1((x-1)_{j-1}) &= L_1\Big(\sum_{i=0}^{j-1}s(j,i+1)x^i\Big)\\ &= \sum_{i=0}^{j}s(j,i+1)\mathcal{B}_i(x)=(-1)^{j-1}\mathcal{D}_{j-1}(1-x),\end{aligned}$$

we can get (2.10), another simple connection between $\mathcal{B}_n(x)$ and $\mathcal{D}_n(x)$. □

**Theorem 2.5.** *For any integers $n\geq m\geq 0$, there holds*

$$\sum_{k=0}^{n}\binom{n}{k}\binom{k-1}{m}x^{n-k}=\sum_{j=0}^{m}(-1)^{m-j}\binom{n}{j}(x+1)^{n-j}, \tag{2.11}$$

*or equivalently*

$$\sum_{k=0}^{n}\binom{n}{k}\binom{k-1}{m}\mathcal{D}_{n-k}(x)=\sum_{j=0}^{m}(-1)^{m-j}\binom{n}{j}\mathcal{D}_{n-j}(x+1), \tag{2.12}$$

$$x\sum_{k=0}^{n}\binom{n}{k}\binom{k-1}{m}\mathcal{B}_{n-k}(x)=\sum_{j=0}^{m}(-1)^{m-j}\binom{n}{j}\mathcal{B}_{n-j+1}(x). \tag{2.13}$$

*Proof.* Setting $g_m(x)=\binom{x-1}{m}=\frac{(x-1)_m}{m!}$ in Lemma 2.2, by (2.5), we have (2.11). Then (2.12) and (2.13) can been obtained by acting $L_2$ and $xL_1$ on the two sides of (2.11) respectively. □

**Corollary 2.6.** *For any integers $n\geq m\geq 0$, there hold*

$$\sum_{k=m+1}^{n}\binom{n}{k}\binom{k-1}{m}D_{n-k}=(-1)^m n!\Big(\frac{D_m}{m!}-\frac{D_n}{n!}\Big), \tag{2.14}$$

$$\sum_{m=0}^{\infty}\Big(\frac{D_m}{m!}-\frac{1}{e}\Big)=\frac{1}{e}, \tag{2.15}$$

$$\sum_{m=0}^{\infty}(-1)^m\Big(\frac{D_m}{m!}-\frac{1}{e}\Big)=\frac{e}{2}-\frac{1}{2e}. \tag{2.16}$$

*Proof.* Setting $x = 0$ in (2.12), by $\mathcal{D}_{n-k}(0) = D_{n-k}$ and $\mathcal{D}_{n-k}(1) = (n-k)!$, using (1.5) in the case $x = 0$, after routine simplification, we get (2.14).

Divided by $n!$ on the two sides of (2.14), and let $n \to \infty$, by (1.12), we have

$$\begin{aligned}(-1)^m\Big(\frac{D_m}{m!} - \frac{1}{e}\Big) &= \lim_{n\to\infty}(-1)^m\Big(\frac{D_m}{m!} - \frac{D_n}{n!}\Big)\\ &= \sum_{k=m+1}^{\infty}\frac{1}{k!}\binom{k-1}{m}\lim_{n\to\infty}\frac{D_{n-k}}{(n-k)!}\\ &= \frac{1}{e\, m!}\sum_{k=0}^{\infty}\frac{1}{(k+m+1)k!}\\ &= \frac{1}{e\, m!}\int_0^1\sum_{k=0}^{\infty}\frac{x^{m+k}}{k!}dx\\ &= \int_0^1\frac{x^m}{m!}e^{x-1}dx. \qquad (2.17)\end{aligned}$$

Summing (2.17) for $m$ from 0 to $\infty$ yields

$$\begin{aligned}\sum_{m=0}^{\infty}(-1)^m\Big(\frac{D_m}{m!} - \frac{1}{e}\Big) &= \int_0^1\sum_{m=0}^{\infty}\frac{x^m}{m!}e^{x-1}dx\\ &= \int_0^1 e^{2x-1}dx = \frac{e^{2x-1}}{2}\Big|_0^1 = \frac{e}{2} - \frac{1}{2e},\end{aligned}$$

which proves (2.16).

Similarly, divided by $(-1)^m$ on the two sides of (2.17) and then summing for $m$ from 0 to $\infty$, after routine computation, (2.15) follows. □

**Remark 2.7.** *It should be noticed that (2.15) states the sum of the errors* $\frac{D_m}{m!} - \frac{1}{e}$ *for* $m \geq 0$ *converges to* $\frac{1}{e}$ *and that (2.16) states the sum of the absolute errors* $(-1)^m\Big(\frac{D_m}{m!} - \frac{1}{e}\Big)$ *for* $m \geq 0$ *still converges.*

## 3. Identities involving harmonic numbers and derangement polynomials

**Theorem 3.1.** *For any integers* $n \geq m \geq 0$, *there holds*

$$\begin{aligned}&\sum_{k=m+1}^{n}\binom{n}{k}\binom{k}{m+1}(H_k - H_{k-m-1})x^{n-k} \qquad (3.1)\\ &= \sum_{j=0}^{m}(-1)^{m-j}\binom{n}{j}\frac{1}{m-j+1}\Big\{(x+1)^{n-j} - \frac{x^{n-j}}{\binom{m+1}{j}}\Big\},\end{aligned}$$

*or equivalently*

$$(3.2)\quad \sum_{k=m+1}^{n}\binom{n}{k}\binom{k}{m+1}(H_k-H_{k-m-1})\mathcal{D}_{n-k}(x)$$
$$=\sum_{j=0}^{m}(-1)^{m-j}\binom{n}{j}\frac{1}{m-j+1}\Big\{\mathcal{D}_{n-j}(x+1)-\frac{\mathcal{D}_{n-j}(x)}{\binom{m+1}{j}}\Big\},$$

*where $H_k = 1+\frac{1}{2}+\cdots+\frac{1}{k}$ is the k-th harmonic number with $H_0 = 0$.*

*Proof.* Setting $g_m(x) = \frac{d}{dx}\binom{x}{m+1} = \frac{1}{(m+1)!}\frac{d}{dx}(x)_{m+1} = \binom{x}{m+1}\sum_{j=0}^{m}\frac{1}{x-j}$ in Lemma 2.2, using $g_m(k) = (-1)^{m-k}k!(m-k)!$ for $0 \le k \le m$ and $g_m(k) = \binom{k}{m+1}(H_k - H_{k-m-1})$ for $m+1 \le k \le n$, by (2.7), after routine simplification, we have (3.1). Then (3.2) follows by acting $L_2$ on the two sides of (3.1). □

**Corollary 3.2.** *For any integers $n, m \ge 0$, there hold*

$$(3.3)\qquad H_{m+1}=\sum_{j=0}^{m}\frac{(x+1)^{j+1}}{j+1}-\sum_{j=0}^{m}\binom{m+1}{j+1}\frac{x^{j+1}}{j+1},$$

$$(3.4)\qquad \frac{H_{m+1}}{m+1}=\sum_{j=0}^{m}(-1)^j\binom{m}{j}\frac{1}{(j+1)^2},$$

$$(3.5)\qquad H_{n+m+1}-H_n=\sum_{j=0}^{m}\frac{(-1)^j\binom{m+1}{j+1}}{(n+j+1)\binom{n+j}{j}},$$

$$(3.6)\qquad \frac{1}{e}\sum_{k=0}^{\infty}\frac{H_{k+m+1}-H_k}{k!}=\sum_{j=0}^{m}(-1)^j j!\Big\{\binom{m+1}{j+1}-\frac{1}{e}\Big\},$$

$$(3.7)\qquad \frac{1}{e}\sum_{k=0}^{\infty}\frac{H_{k+m+1}-H_k}{k!}=H_{m+1}+\sum_{j=0}^{m}(-1)^j j!\Big\{\frac{D_{j+1}}{(j+1)!}-\frac{1}{e}\Big\}.$$

*Proof.* Setting $n = m+1$ in (3.1), its left side collapses to $H_{m+1}$ and its right side reduces, after replacing $j$ by $m-j$, to that of (3.3). So (3.3) follows.

Setting $x = -1$ in (3.3), after simplification, (3.4) follows. Setting $x = 0$ in (3.1), then replacing $n$ by $n+m+1$ and $j$ by $m-j$, after routine computation, we get (3.5). Note that the case $n = 0$ in (3.5) can also lead to (3.4).

Divided by $n!$ on both sides of (3.2), and replacing $k$ by $m+k+1$ and $j$ by $m-j$, we have

$$\sum_{k=0}^{n-m-1} \frac{H_{k+m+1}-H_k}{k!(m+1)!} \frac{\mathcal{D}_{n-m-k-1}(x)}{(n-m-k-1)!} = \sum_{j=0}^{m}(-1)^j$$

$$\frac{1}{(m-j)!(j+1)} \frac{1}{(n-m+j)!}\left\{\mathcal{D}_{n-m+j}(x+1) - \frac{\mathcal{D}_{n-m+j}(x)}{\binom{m+1}{j+1}}\right\}. \tag{3.8}$$

Noting $\mathcal{D}_{n-j}(0) = D_{n-j}$ and $\mathcal{D}_{n-j}(1) = (n-j)!$, the case $x=0$ in (3.8) produces

$$\sum_{k=0}^{n-m-1} \frac{H_{k+m+1}-H_k}{k!} \frac{D_{n-m-k-1}}{(n-m-k-1)!}$$
$$= \sum_{j=0}^{m}(-1)^j j! \binom{m+1}{j+1}\left\{1 - \frac{D_{n-m+j}}{(n-m+j)!} \frac{1}{\binom{m+1}{j+1}}\right\},$$

which, when $n \to \infty$, by (1.12), generates (3.6).

Replacing $x$ by $-x$ in (3.3), and acting $L_2$ on the two sides, then taking value at $x=1$, we can obtain

$$\sum_{j=0}^{m}(-1)^j j! \binom{m+1}{j+1} = H_{m+1} + \sum_{j=0}^{m}(-1)^j \frac{D_{j+1}}{j+1},$$

which, together with (3.6), produces (3.7). □

## Acknowledgements

The authors are grateful to the anonymous referees for the helpful suggestions and comments. The work was partially supported by the Fundamental Research Funds for the Central Universities (Grant No. 2011QN056 and No. 2012TD032), by the Dr Start-up Fund of Liaoning (Grant No. 20091005) and by the Science and Technology Fund of Dalian (Grant No. 2009J22DW010).

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