Signed total k-domination in graphs*

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Abstract

A signed total k-dominating function of a graph $G = (V, E)$ is a function $f : V \rightarrow \{+1, -1\}$ such that for every vertex v, the sum of the values of f over the open neighborhood of v is at least k. A signed total k-dominating function f is minimal if there does not exist a signed total k-dominating function g, $f \neq g$, for which $g(v) \leq f(v)$ for every $v \in V$. The weight of a signed total k-dominating function is $w(f) = \sum_{v \in V(G)} f(v)$. The signed total k-domination number of G, denoted by $\gamma^s_{t,k}(G)$, is the minimum weight of a signed total k-dominating function on G. The upper signed total k-domination number $\Gamma^s_{t,k}(G)$ of G is the maximum weight of a minimal signed total k-dominating function on G. In this paper we present sharp lower bounds on $\gamma^s_{t,k}(G)$ for general graphs and K_{r+1}-free graphs and characterize the extremal graphs attaining some lower bounds. Also, we give a sharp upper bound on $\Gamma^s_{t,k}(G)$ for an arbitrary graph.

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1 Introduction

Let $G = (V, E)$ be a finite simple graph with vertex set V and edge set E. Terminology not defined here will generally conform to that in [1]. For a

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vertex $v \in V$, the *open neighborhood* of v is $N_G(v) = \{u \in V | \ uv \in E\}$ and the *closed neighborhood* of v is $N_G[v] = \{v\} \cup N_G(v)$. The *degree* of v in G, denoted by $d_G(v)$, is the cardinality of $N_G(v)$, and the *minimum degree* and *maximum degree* of G is denoted by $\delta(G)$ and $\Delta(G)$, respectively. If each vertex in G has an odd degree, then we call G an *odd-degree graph.* If no ambiguity, we will omit the subscript G. For $S \subseteq V$, we let $d_S(v)$ denote the number of vertices in S that are adjacent to v. If $d(v) = k$ for all $v \in V$, then we call G a k-regular graph. If $d(v) = k - 1$ or k for all $v \in V$, then we call G a nearly k-regular graph. The subgraph of G induced by S is denoted by $G[S]$. If $X, Y \subseteq V(G), X \cap Y = \emptyset$, we write $e(X, Y)$ for the number of edges between X and Y. A graph $G = (V, E)$ is called *r-partite* if V admits a partition into r classes such that every edge has its ends in different classes: vertices in the same partition class must not be adjacent. An r-partite graph in which every two vertices from different partition classes are adjacent is called *complete.* A *clique* in G is a complete subgraph of G. More precisely, an r-clique is a clique of order r, denoted by K_r. A graph is *K_r-free* if it contains no r-clique as its subgraph.

Let $f : V \to \{+1, -1\}$ be a function which assigns to each vertex of G an element of the set $\{+1, -1\}$. The *weight* of f is $w(f) = \sum_{v \in V} f(v)$, and for $S \subseteq V$ we define $f(S) = \sum_{v \in S} f(v)$, so $w(f) = f(V)$. For a vertex $v \in V$, we denote $f(N(v))$ by $f[v]$ for notational convenience. The function f is said to be a *signed total dominating function* (STDF) of G if $f[v] \geq 1$ for every $v \in V$. The *signed total domination number* of G, denoted by $\gamma_t^s(G)$, is the minimum weight of a STDF on G. We say f is a minimal STDF if there does not exist a STDF $g : V \to \{+1, -1\}$, $f \neq g$, for which $g(v) \leq f(v)$ for every vertex $v \in V$. The *upper signed total domination number* of G, denoted by $\Gamma_t^s(G)$, is the maximum weight of a minimal STDF of G. The study of signed total domination was begun by Zelinka [9], and continued by Henning [2], Kang et al. [3], Shan et al. [5], Shi et al. [6], Xing et al. [8]. A set D of vertices of G is defined in [4] to be a *total k-dominating set* of G if $|N(v) \cap D| \geq k$ for all $v \in V$. The *total k-domination number* $\gamma_t^k(G)$ of G is the minimum cardinality of a total k-dominating set of G.

In this paper, we generalize the concepts of signed total domination and total k-domination to signed total k-domination. If G is a graph with $\delta(G) \geq k$, where $k \in \mathbb{N}$, the function f is called *signed total k-dominating function* (STkDF) of G if $f[v] \geq k$ for every $v \in V$. The *signed total k-domination number* $\gamma_{t,k}^s(G)$ of G is the minimum weight of a STkDF on G. We say f is a minimal STkDF if there does not exist a STkDF $g : V \to \{+1, -1\}$, $f \neq g$, for which $g(v) \leq f(v)$ for every vertex $v \in V$. The *upper signed total k-domination number* of G, denoted by $\Gamma_{t,k}^s(G)$, is the maximum weight of a minimal STkDF of G. It is obvious that $\gamma^s_{t,1}(G) = \gamma^s_t(G)$ and $\Gamma^s_{t,1}(G) = \Gamma^s_t(G)$. Throughout this paper, we always assume that a graph G has minimum degree $\delta(G) \geq k$ and $k \in \mathbf{N}$. A (minimal) STkDF of weight $\gamma^s_{t,k}(G)$ (respectively, $\Gamma^s_{t,k}(G)$) we call a $\gamma^s_{t,k}(G)$-function (respectively, $\Gamma^s_{t,k}(G)$-function).

This paper is organized as follows. In Section 2, we establish several sharp lower bounds on $\gamma^s_{t,k}(G)$ of a general graph G and a K_{r+1}-free graph G. In particular, we characterize the extremal graphs attaining some lower bounds. In Section 3, we present a sharp upper bound on $\Gamma^s_{t,k}(G)$ of an arbitrary graph G. These results improve or imply most of previous results on the signed total domination.

2 Signed k-Total Domination Number

For an arbitrary graph G, $|V(G)|$ clearly is a trivial upper bound on signed total k-domination number. In this section we firstly characterize the graphs attaining this bound.

Theorem 1 *If G is a graph of order n, then $\gamma^s_{t,k}(G) = n$ if and only if there exists a vertex $u \in N(v)$ such that $d(u) \in \{k, k+1\}$ for every vertex $v \in V(G)$.*

Proof. Suppose that $\gamma^s_{t,k}(G) = n$ and there exists a vertex $v \in V(G)$ such that $d(u) \geq k+2$ for every vertex $u \in N(v)$. Consider the function $f : V \to \{+1, -1\}$ for which $f(v) = -1$ and for any other vertex w, $f(w) = +1$. It is easy to verify that f is a signed total k-dominating function of G with the weight $f(V(G)) = n-2$. Therefore, $\gamma^s_{t,k}(G) \leq n-2$, a contradiction.

On the other hand, suppose that there exists a vertex $u \in N(v)$ such that $d(u) \in \{k, k+1\}$ for every vertex $v \in V(G)$. A signed total k-dominating function must assign $+1$ to every vertex of G, and so $\gamma^s_{t,k}(G) = n$. ■

By Theorem 1, we have the following result when $k = 1$.

Corollary 2 *If G is a graph of order n, then $\gamma^s_t(G) = n$ if and only if there exists a vertex $u \in N(v)$ such that $d(u) \in \{1, 2\}$ for every vertex $v \in V(G)$.*

The next result due to Henning [2] is a special case of Corollary 2.

Corollary 3 ([2]) *If T is a tree of order $n \geq 2$, then $\gamma_t^s(G) = n$ if and only if every vertex of T is a support vertex or is adjacent to a vertex of degree 2.*

Next we establish a lower bound on $\gamma_{t,k}^s(G)$ for a general graph G in terms of its order, and we characterize the extremal graphs achieving this bound. For this purpose, we define a family $\mathcal{C}$ of graphs as follows.

Let n_1, n_2 and k be integers satisfying the following conditions:

(i) $k \geq 1$ and $n_1 \geq k+1$;

(ii) $n_1(n_1 - k - 1) = kn_2$.

Let $G_1 = K_{n_1}$ be a complete graph on n_1 vertices and G_2 an empty graph (that is, $E(G_2) = \emptyset$) of order n_2. Let $G(n_1, n_2)$ be the graph obtained from disjoint union of G_1 and G_2 by adding $n_1(n_1 - k - 1)(= kn_2)$ edges between $V(G_1)$ and $V(G_2)$ so that each vertex of G_1 has exactly $n_1 - k - 1$ neighbors in G_2 while each vertex of G_2 has precisely k neighbors in G_1. Let $\mathcal{C} = \{G(n_1, n_2) \mid n_1 \geq k+1, n_1(n_1 - k - 1) = kn_2\}$.

Theorem 4 *If G is a graph of order n with $\delta \geq k$, then*

$$\gamma_{t,k}^s(G) \geq 1 + \sqrt{1 + 4kn} - n$$

with equality if and only if $G \in \mathcal{C}$.

Proof. Let f be a $\gamma_{t,k}^s(G)$-function and let $P = \{v \in V \mid f(v) = +1\}$ and $M = \{v \in V \mid f(v) = -1\}$. Further, we let $|P| = p$ and $|M| = m$. For every vertex $v \in M$, we have $|N(v) \cap P| \geq k$ as $f[v] \geq k$. Hence

$$e(P, M) = \sum_{v \in M} |N(v) \cap P| \geq km = k(n - p) \tag{1}$$

For each vertex $v \in P$, $f[v] \geq k$ implies that $|N(v) \cap M| \leq |N(v) \cap P| - k$, and so

$$e(P, M) = \sum_{v \in P} |N(v) \cap M| \leq \sum_{v \in P} (|N(v) \cap P| - k) = 2|E(G[P])| - kp. \tag{2}$$

Note that $|E(G[P])| \leq \binom{p}{2}$, it follows from the inequalities (1) and (2) that

$$k(n - p) \leq e(P, M) \leq 2|E(G[P])| - kp \leq p(p-1) - kp, \tag{3}$$

or equivalently, $p^2 - p - kn \geq 0$. Thus $p \geq \left(1 + \sqrt{1 + 4kn}\right)/2$ as $p > 0$. Therefore, $\gamma_{t,k}^s(G) = 2p - n \geq 1 + \sqrt{1 + 4kn} - n$.

If $\gamma^s_{t,k}(G) = 1 + \sqrt{1+4kn} - n$, then all equalities hold in (1), (2) and (3). We denote $G_1 = G[P]$ and $G_2 = G[M]$. The equality in (1) implies that each vertex of G_2 has precisely k neighbors in G_1, and so G_2 is an empty graph of order m. The equalities in (2) and (3) imply that G_1 is a complete graph of order $p \geq k+1$, each vertex of G_1 has exactly $p-k-1$ neighbors in G_2 and $p(p-k-1) = km$. Hence $G \in \mathcal{C}$.

Conversely, suppose that $G \in \mathcal{C}$. Then there exist integers n_1, n_2 and k satisfying conditions (i) and (ii) such that $G = G(n_1, n_2)$. Since $n_1(n_1-k-1) = kn_2$, G has order $n = n_1+n_2 = n_1(n_1-1)/k$. Let $f : V(G) \to \{+1, -1\}$ be a function on G assigning $+1$ to all vertices of G_1 and -1 to all vertices of G_2. It is easily seen that f is a signed total k-domination function of G with weight $w(f) = n_1 - n_2 = 2n_1 - n = 1 + \sqrt{1+4kn} - n$. Consequently, $\gamma^s_{t,k}(G) = 1 + \sqrt{1+4kn} - n$. It completes the proof. ■

Let $k = 1$ in Theorem 4, we obtain a special case of Theorem 4.

Corollary 5 ([2]) *If G is a graph of order n, then*

$$\gamma^s_t(G) \geq 1 + \sqrt{1+4n} - n$$

with equality if and only if $G \in \mathcal{F}$.

Our next aim is to present a sharp lower bound on $\gamma^s_{t,k}(G)$ of a K_{r+1}-free graph G in terms of its order and its minimum degree. The unique complete r-partite graphs on $n \geq r$ vertices whose partition sets differ in size by at most 1 are called *Turán graphs*; we denote them by $T^r(n)$ and their number of edges by $t_r(n)$. Clearly, $T^r(n) = K_n$ for all $n \leq r$. The following Turán theorem from extremal graph theory is well-known and useful for our purpose.

Lemma 6 ([7]) (Turán theorem) *For any integer $r \geq 1$, if $G = (V, E)$ is a K_{r+1}-free graph of order n with maximum number of edges, then G is a $T^r(n)$ and*

$$|E| = t_r(n) \leq \frac{r-1}{2r}n^2$$

with equality if and only if r divides n.

By applying Turán theorem, we shall present a sharp lower bound on the signed total k-domination number for K_{r+1}-free graphs where $r \geq 2$.

Theorem 7 *If G is a K_{r+1}-free graph of order n with $\delta \geq k$ and $c = \lceil (\delta + k)/2 \rceil$, then*

$$\gamma^s_{t,k}(G) \geq \frac{r}{r-1}\left(-(c-k) + \sqrt{(c-k)^2 + 4\frac{r-1}{r}cn}\right) - n$$

and this bound is sharp.

Proof. Let f, P, M, p and m be defined as in the proof of Theorem 4. For each vertex $v \in M$, v is adjacent to at least $\lceil (d(v) + k)/2 \rceil$ vertices of P since $f[v] \geq k$, hence $|N(v) \cap P| \geq \lceil (d(v)+k)/2 \rceil \geq \lceil (\delta + k)/2 \rceil = c$. So we have

$$e(P, M) = \sum_{v \in M} |N(v) \cap P| \geq c|M| = c(n-p). \tag{4}$$

According to the inequalities (2) and (4), we obtain

$$c(n-p) \leq e(P, M) \leq 2|E(G[P])| - kp. \tag{5}$$

Since G is K_{r+1}-free, it follows from Lemma 6 that $|E(G[P])| \leq (r-1)p^2/2r$. By the inequality (5), we have

$$c(n-p) \leq e(P, M) \leq \frac{r-1}{r}p^2 - kp, \tag{6}$$

or equivalently,

$$\frac{r-1}{r}p^2 + (c-k)p - cn \geq 0.$$

Thus

$$p \geq \left(-(c-k) + \sqrt{(c-k)^2 + 4\frac{r-1}{r}cn}\right) \Big/ 2\left(\frac{r-1}{r}\right).$$

Consequently,

$$\gamma^s_{t,k}(G) = 2p - n \geq \frac{r}{r-1}\left(-(c-k) + \sqrt{(c-k)^2 + 4\frac{r-1}{r}cn}\right) - n.$$

That the bound is sharp may be seen as follows. For integers $r \geq 2$, let G_i be a complete bipartite graph with vertex classes X_i and Y_i, where $|X_i| = k$ and $|Y_i| = (r-2)k$, for $i = 1, 2, \ldots, r$. (If $r = 2$, then the graph $K_{k,0}$ is considered as $\overline{K_k}$, i.e., the complement of complete graph K_k.) Now let $G(r)$ be the graph obtained from disjoint union of $G_1, G_2, \ldots, G_r$ by joining each vertex of X_i with all vertices of $\bigcup_{j=1, j\neq i}^{r} X_j$ for all $i = 1, 2, \ldots, r$. Let

$V_i = X_i \cup Y_{i+1}$ where $i+1 \pmod r$. Then $G(r)$ is an r-partite graph of order $n = rk(r-1)$ with vertex classes $V_1, V_2, \ldots, V_r$. So $G(r)$ is K_{r+1}-free. Note that $G(r)$ has minimum degree k and so $c = \lceil (\delta + k)/2 \rceil = k$. Assigning to each vertex of $\bigcup_{i=1}^r X_i$ the value $+1$ and to each vertex of $\bigcup_{i=1}^r Y_i$ the value -1, we produce a signed total k-dominating function g of $G(r)$ with weight

$$\begin{aligned} w(g) &= rk - r(r-2)k \\ &= 3rk - r^2k \\ &= \frac{r}{r-1}\left(-(c-k) + \sqrt{(c-k)^2 + 4\frac{r-1}{r}cn}\right) - n. \end{aligned}$$

Consequently,

$$\gamma_{tk}^s(G(r)) = \frac{r}{r-1}\left(-(c-k) + \sqrt{(c-k)^2 + 4\frac{r-1}{r}cn}\right) - n.$$

This completes the proof. ■

Now we give a sharp lower bound on $\gamma_{t,k}^s(G)$ of a graph G in terms of its order, minimum degree and size.

Theorem 8 *If G is a graph of order n and size q with $\delta \geq k$, then*

$$\gamma_{t,k}^s(G) \geq \frac{\delta + k}{\delta}n - \frac{2q}{\delta}$$

and this bound is sharp.

Proof. Let f,P, M, p, and m defined as in the proof of Theorem 4. By the inequality(5) in the proof of Theorem 7, we obtain

$$|E(G[P])| \geq \frac{c(n-p)}{2} + \frac{kp}{2}. \tag{7}$$

Furthermore, we have

$$2|E(G[M])| = \sum_{v \in M} |N(v) \cap M| = \sum_{v \in M} (d(v) - |N(v) \cap P|) \geq \delta|M| - e(P, M),$$

that is,

$$|E(G[M])| \geq \frac{\delta}{2}|M| - \frac{e(P,M)}{2} = \frac{\delta}{2}(n-p) - \frac{e(P,M)}{2}. \tag{8}$$

Since $c \geq (\delta + k)/2$, combining with the inequalities (4), (7) and (8), we obtain

$$\begin{aligned} q &= |E(G[P])| + |E(G[M])| + e(P, M) \\ &\geq \frac{c(n-p)}{2} + \frac{kp}{2} + \frac{\delta}{2}(n-p) - \frac{e(P, M)}{2} + e(P, M) \\ &\geq \frac{c(n-p)}{2} + \frac{kp}{2} + \frac{\delta}{2}(n-p) + \frac{c(n-p)}{2} \\ &\geq \frac{2\delta + k}{2} n - \delta p. \end{aligned}$$

Thus,

$$p \geq \frac{2\delta + k}{2\delta} n - \frac{q}{\delta}.$$

Consequently,

$$\gamma_{t,k}^{s}(G) = 2p - n \geq \frac{\delta + k}{\delta} n - \frac{2q}{\delta}.$$

To prove the lower bound is sharp, we consider a family $\mathcal{J}$ of graphs defined as follows. Let $G(1) = K_{k+1}$ and for integers $l \geq 2$, let G_i be a complete bipartite graph with vertex classes X_i and Y_i where $|X_i| = k$ and $|Y_i| = (l-1)k - 1$, for $i = 1, 2, \ldots, l$ (the graph $K_{1,0}$ is regarded as K_1 when $k = 1$ and $l = 2$). Further, let $X = \bigcup_{i=1}^{l} X_i$. Now let $G(l)$ be the graph obtained from disjoint union of $G_1, G_2, \ldots, G_l$ by adding $lk(lk-1)/2$ edges joining vertices of X such that $G[X]$ is a complete graph. Let $\mathcal{J} = \{G(l) \mid l \geq 1\}$. Suppose $G \in \mathcal{J}$. If $G = G(1)$, by Theorem 1, then $\gamma_{t,k}^{s}(G) = k + 1 = (\delta + k)n/\delta - 2q/\delta$. So we assume that $G = G(l)$ for some $l \geq 2$. Then G has order $n = l(lk-1)$ and size $q = lk(lk-k-1) + lk(lk-1)/2$. Note that G has the minimum degree k. Let g be a function on G by assigning to each vertex of $X = \bigcup_{i=1}^{l} X_i$ the value $+1$ and to all other vertices the value -1. It is easy to see that g is a signed total k-dominating function of G with weight

$$w(g) = kl - l(kl - k - 1) = -kl^2 + 2kl + l = \frac{\delta + k}{\delta} n - \frac{2q}{\delta}.$$

Therefore,

$$\gamma_{t,k}^{s}(G) = \frac{\delta + k}{\delta} n - \frac{2q}{\delta}.$$

■

For a graph G of order n and size q, Henning [2] presented a lower bound $\gamma_t^s(G) \geq 2(n - q)$. As a special case $k = 1$ of Theorem 8, we improve this previous bound.

Corollary 9 *If G is a graph of order n and size q with $\delta \geq 1$, then $\gamma_t^s(G) \geq \frac{\delta+1}{\delta}n - \frac{2q}{\delta}$ and this bound is sharp.*

3 Upper Signed k-Total Domination Number

In this section we restrict our attention to the upper signed total k-domination of graphs. Next we shall present a sharp upper bound on $\Gamma_{t,k}^s$ of an arbitrary graph in terms of its minimum degree, maximum degree and order. We begin by stating a lemma due to Henning [2].

Lemma 10 ([2]) *If k and n are integers with $k < n$ and n is even, then we can construct a k-regular graph on n vertices.*

Lemma 11 *A signed total k-dominating function of a graph $G = (V, E)$ is minimal if and only if for every vertex $v \in V$ with $f(v) = 1$, there exists a vertex $u \in N(v)$ with $f[u] \leq k+1$, where if $d(u)$ and k are odd, then $f[u] = k$.*

The proof of Lemma 11 is straightforward and therefore omitted.

Theorem 12 *If G is a graph of order n with minimum degree δ and maximum degree Δ, then*

$$\Gamma_{t,k}^s(G) \leq \begin{cases} \dfrac{\Delta(\delta+k+2)-(\delta-k)}{\Delta(\delta+k+2)+(\delta-k)}n & \text{for } \delta-k \text{ even,} \\[2ex] \dfrac{\Delta(\delta+k+1)-(\delta-k-1)}{\Delta(\delta+k+1)+(\delta-k-1)}n & \text{for } \delta-k \text{ odd.} \end{cases}$$

In particular, if G is an odd-degree graph and k is odd, then

$$\Gamma_{t,k}^s(G) \leq \frac{\Delta(\delta+k)-(\delta-k)}{\Delta(\delta+k)+(\delta-k)}n.$$

Furthermore, these bounds are sharp.

Proof. Let f be a $\Gamma_{t,k}^s(G)$-function of G, and let P, M, p, m be defined as in the Section 2. If $\delta = k$ or $k+1$, then the results are trivial. Hence in what follows we assume $\delta \geq k+2$. For natational convenience, we write $\lfloor(\delta-k)/2\rfloor = s_1$, $\lceil(\delta+k)/2\rceil = s_2$, $\lfloor(\Delta-k)/2\rfloor = t_1$ and $\lceil(\Delta+k)/2\rceil = t_2$.

Obviously, $f[v] = d(v) - 2d_M(v)$ for any vertex $v \in V$. Then, by Lemma 11, $M \neq \emptyset$. Let $|P| = p$ and $|M| = m$. Thus, $w(f) = |P| - |M| = n - 2m$.

For any vertex $v \in V$, $d_M(v) \leq \lfloor (d(v) - k)/2 \rfloor$ since $f[v] \geq k$. Hence we can partition P into $t_1 + 1$ sets by defining $P_i = \{v \in P \mid d_M(v) = i\}$ and letting $|P_i| = p_i$ for $i = 0, 1, \ldots, t_1$. Then we have

$$n = m + p = m + \sum_{i=0}^{t_1} p_i. \tag{9}$$

For any vertex $v \in V$, $d_P(v) \geq \lceil (d(v) + k)/2 \rceil$ for otherwise $f[v] < k$. We define

$$M_j = \{v \in M \mid d_P(v) = j\} \text{ for } j = s_2, s_2 + 1, \ldots, t_2$$

and $M' = M - \bigcup_{j=s_2}^{t_2} M_j$. Let $|M_j| = m_j$, and so $|M'| = m - \sum_{j=s_2}^{t_2} m_j$. Clearly, $(M_{s_2}, \ldots, M_{t_2}, M')$ is a partition of M. Since each vertex in M' is adjacent to at most Δ vertices of P, we have

$$\sum_{i=1}^{t_1} ip_i = e(P, M) \leq (s_2 m_{s_2} + \cdots + t_2 m_{t_2}) + \Delta \left(m - (m_{s_2} + \cdots + m_{t_2})\right).$$

Hence,

$$\sum_{i=1}^{t_1} ip_i \leq \Delta m - \left((\Delta - s_2)m_{s_2} + \cdots + (\Delta - t_2)m_{t_2}\right). \tag{10}$$

If $P_0 = \emptyset$, then, by (9) and (10), we have

$$n = m + \sum_{i=1}^{t_1} p_i \leq m + \sum_{i=1}^{t_1} ip_i \leq (\Delta + 1)m.$$

This implies that $m \geq n/(\Delta+1)$, and so $\Gamma_{t,k}^{s}(G) = n - 2m \leq (\Delta - 1)n/(\Delta + 1)$. Let

$$b = \min\left\{ \left(\frac{\Delta(\delta + k + 2) - (\delta - k)}{\Delta(\delta + k + 2) + (\delta - k)}\right) n, \left(\frac{\Delta(\delta + k + 1) - (\delta - k - 1)}{\Delta(\delta + k + 1) + (\delta - k - 1)}\right) n \right\}.$$

Observing that

$$(\Delta - 1)n/(\Delta + 1) \leq \min\left\{b, (\Delta(\delta + k) - (\delta - k))n/(\Delta(\delta + k) + (\delta - k))\right\},$$

then the desired result follows. We therefore may assume that $P_0 \neq \emptyset$.

According to our partition for P and M, we have $f[v] \geq k + 2$ for any $v \in (\bigcup_{i=0}^{s_1 - 1} P_i) \cup M'$, so if $f[v] = k$ or $k + 1$ for $v \in V$, then $v \in$

$(\bigcup_{i=s_1}^{t_1} P_i) \cup (\bigcup_{j=s_2}^{t_2} M_j)$. For any vertex $v \in P_0$, since f is minimal, by Lemma 11, there exists a vertex $u \in N(v)$ such that $f[u] \in \{k, k+1\}$. Let $I = \{u \in N(P_0) \mid f[u] = k \text{ or } k+1\}$. Then $I \subseteq \bigcup_{i=s_1}^{t_1} P_i$. So

$$p_0 = |P_0| \le e(P_0, I) = e(P_0, \bigcup_{i=s_1}^{t_1} (P_i \cap I)) \le e(P_0, \bigcup_{i=s_1}^{t_1} P_i). \tag{11}$$

Furthermore, for every vertex $u \in P_i \cap I$ ($s_1 \le i \le t_1$), there must exist a neighbor u' of u such that $f[u'] = k$ or $k+1$. Note that u' belongs to $(\bigcup_{i=s_1}^{t_1} P_i) \cup (\bigcup_{j=s_2}^{t_2} M_j)$. If $u' \in \bigcup_{i=s_1}^{t_1} P_i$, then u is adjacent to at most $i+k$ vertices of P_0, while if $u' \in \bigcup_{j=s_2}^{t_2} M_j$, then u is adjacent to at most $i+k+1$ vertices of P_0. Hence we can write $P_i \cap I$ ($s_1 \le i \le t_1$) as the disjoint union of two sets IP_i' and IP_i'', where $IP_i' = \{u \in P_i \cap I \mid d_{P_0}(u) = i+k+1\}$ and $IP_i'' = P_i \cap I - IP_i'$. Then $IP_i'' = \{u \in P_i \cap I \mid d_{P_0}(u) \le i+k\}$. Let $|IP_i'| = p_i'$, and so $|IP_i''| = |P_i \cap I| - p_i'$. Since each vertex $u \in \bigcup_{i=s_1}^{t_1} IP_i'$ is adjacent to a vertex in $\bigcup_{j=s_2}^{t_2} M_j$, it follows that

$$\sum_{i=s_1}^{t_1} p_i' \le s_2 m_{s_2} + (s_2+1) m_{(s_2+1)} + \cdots + t_2 m_{t_2}. \tag{12}$$

Thus, by Eqs. (11) and (12), we have

$$\begin{aligned} p_0 &\le \sum_{i=s_1}^{t_1} (i+k+1) p_i' + \sum_{i=s_1}^{t_1} (i+k)(|P_i \cap I| - p_i') \\ &\le \sum_{i=s_1}^{t_1} (i+k+1) p_i' + \sum_{i=s_1}^{t_1} (i+k)(p_i - p_i') \\ &\le \sum_{i=s_1}^{t_1} (i+k) p_i + \sum_{j=s_2}^{t_2} j m_j. \end{aligned} \tag{13}$$

We now distinguish two possibilities depending on the parity of $\delta - k$.

Case 1. $\delta - k$ is even.

Then $s_1 = (\delta-k)/2$. Noting that when $i \ge (\delta-k)/2$, $(\delta+k+2)i/(\delta-k) \ge i+k+1$ holds, then by Eqs. (9), (10) and (13), we obtain

$$\begin{aligned} n &\le m + \Big(\sum_{i=s_1}^{t_1} (i+k) p_i + \sum_{j=s_2}^{t_2} j m_j\Big) + \sum_{i=1}^{t_1} p_i \\ &= m + \sum_{i=s_1}^{t_1} (i+k+1) p_i + \sum_{i=1}^{s_1-1} p_i + \sum_{j=s_2}^{t_2} j m_j \end{aligned}$$

$$
\begin{aligned}
&\le & m+\frac{\delta+k+2}{\delta-k}\sum_{i=s_1}^{t_1} ip_i+\sum_{i=1}^{s_1-1} p_i+\sum_{j=s_2}^{t_2} jm_j \\
&\le & m+\frac{\delta+k+2}{\delta-k}\sum_{i=1}^{t_1} ip_i+\sum_{j=s_2}^{t_2} jm_j \\
&\le & m+\frac{\delta+k+2}{\delta-k}m\Delta-\frac{\delta+k+2}{\delta-k}\sum_{j=s_2}^{t_2}(\Delta-j)m_j+\sum_{j=s_2}^{t_2} jm_j \\
&= & m+\frac{\delta+k+2}{\delta-k}m\Delta-\sum_{j=s_2}^{t_2}\frac{(\delta+k+2)\Delta-2j(\delta+1)}{\delta-k}m_j.
\end{aligned}
$$

Further, observing that

$$
\frac{(\delta+k+2)\Delta-2j(\delta+1)}{\delta-k}\ge\frac{(\delta+k+2)\Delta-2t_2(\delta+1)}{\delta-k}\ge 0,
$$

we have $n \le m+((\delta+k+2)\Delta/(\delta-k))m$, which implies that $m \ge (\delta-k)n/(\Delta(\delta+k+2)+(\delta-k))$. Hence,

$$
\Gamma_{t,k}^{s}(G)=w(f)=n-2m\le\left(\frac{\Delta(\delta+k+2)-(\delta-k)}{\Delta(\delta+k+2)+(\delta-k)}\right)n.
$$

That the bound is sharp may be seen as follows. For any integers $l \ge k$ and $r \ge l+k$, let $F_{l,r}$ be the graph with vertex set $X \cup Y \cup Z$ with $|X| = l$, $|Y| = 2r$ and $|Z| = 2r(l+k)$, where X is an independent set of vertices. The edge set of $F_{l,r}$ is constructed as follows: Add $2rl$ edges between X and Y so that $G[X \cup Y]$ forms a complete bipartite graph with partition sets X and Y. Add $2r(l+k)$ edges between Y and Z so that each vertex of Y is precisely adjacent to $l+k$ vertices of Z and each vertex of Z is precisely adjacent to one vertex of Y. Add edges joining vertices of Y so that Y induces a 1-regular graph. Add edges joining vertices of Z so that Z induces a $(2l+k-1)$-regular graph (since $2l+k-1 < 2r(l+k) = |Z|$ and $|Z|$ is even, it follows from Lemma 10 that such an addition of edges is possible).

By construction, $F_{l,r}$ is a graph of order $n = l+2r+2r(l+k)$ with minimum degree $\delta = 2l+k$ and maximum degree $\Delta = 2r$. The function g that assigns to each vertex of X the value -1 and to each vertex of $Y \cup Z$ the value $+1$ is a minimal signed total k-dominating function of $F_{l,r}$ as $g[u] = k+1$ for any $u \in Y$. It is easy to see that

$$
w(g)=2r+2r(l+k)-l=\left(\frac{\Delta(\delta+k+2)-(\delta-k)}{\Delta(\delta+k+2)+(\delta-k)}\right)n.
$$

Consequently, $\Gamma^s_{t,k}(F_{l,r}) = (\Delta(\delta+k+2)-(\delta-k))n/(\Delta(\delta+k+2)+(\delta-k))$.

Case 2. $\delta - k$ is odd.

Then $s_1 = (\delta-k-1)/2$. Noting that when $i \geq (\delta-k-1)/2$, $(\delta+k+1)i/(\delta-k-1) \geq i+k+1$ holds, then by Eqs. (9), (10) and (13) again, we have

$$\begin{aligned}
n &\leq m + \Big(\sum_{i=s_1}^{t_1}(i+k)p_i + \sum_{j=s_2}^{t_2} jm_j\Big) + \sum_{i=1}^{t_1} p_i \\
&= m + \sum_{i=s_1}^{t_1}(i+k+1)p_i + \sum_{i=1}^{s_1-1} p_i + \sum_{j=s_2}^{t_2} jm_j \\
&\leq m + \frac{\delta+k+1}{\delta-k-1}\sum_{i=s_1}^{t_1} ip_i + \sum_{i=1}^{s_1-1} p_i + \sum_{j=s_2}^{t_2} jm_j \\
&\leq m + \frac{\delta+k+1}{\delta-k-1}\sum_{i=1}^{t_1} ip_i + \sum_{j=s_2}^{t_2} jm_j \\
&\leq m + \frac{\delta+k+1}{\delta-k-1}m\Delta - \frac{\delta+k+1}{\delta-k-1}\sum_{j=s_2}^{t_2}(\Delta-j)m_j + \sum_{j=s_2}^{t_2} jm_j \\
&= m + \frac{\delta+k+1}{\delta-k-1}m\Delta - \sum_{j=s_2}^{t_2}\frac{(\delta+k+1)\Delta-2j\delta}{\delta-k-1}m_j.
\end{aligned}$$

Further, observing that

$$\frac{(\delta+k+1)\Delta-2j\delta}{\delta-k-1} \geq \frac{(\delta+k+1)\Delta-2t_2\delta}{\delta-k-1} \geq 0,$$

we have $n \leq m + ((\delta+k+1)\Delta/(\delta-k-1))m$, which implies that $m \geq (\delta-k-1)n/(\Delta(\delta+k+1)+(\delta-k-1))$. Hence,

$$\Gamma^s_{t,k}(G) = w(f) = n - 2m \leq \left(\frac{\Delta(\delta+k+1)-(\delta-k-1)}{\Delta(\delta+k+1)+(\delta-k-1)}\right)n.$$

That the bound is sharp may be seen as follows. For any integers $l \geq k$, $r \geq l+k$ and q, where $2l+k \leq q \leq 2r-1$, let $G_{l,r}$ be the graph with vertex set $X \cup Y \cup Z$ with $|X| = l$, $|Y| = 2r$ and $|Z| = 2r(l+k)$, where X is an independent set of vertices. The edge set of $G_{l,r}$ is constructed as follows: Add $2rl$ edges between X and Y so that $G[X \cup Y]$ forms a complete bipartite graph with partition sets X and Y. Add $2r(l+k)$ edges between Y and Z so that each vertex of Y is precisely adjacent to $l+k$ vertices of

Z and each vertex of Z is precisely adjacent to one vertex of Y. Add edges joining vertices of Y so that Y induces a 1-regular graph. Add edges joining vertices of Z so that Z induces a q-regular graph (since $q < 2r(l+k) = |Z|$ and $|Z|$ is even, it follows from Lemma 10 that such an addition of edges is possible).

By construction, $G_{l,r}$ is a graph of order $n = l + 2r + 2r(l+k)$ with minimum degree $\delta = 2l+k+1$ and maximum degree $\Delta = 2r$. The function g that assigns to each vertex of X the value -1 and to each vertex of $Y \cup Z$ the value $+1$ is a minimal signed total dominating function of $F_{l,r}$ as $g[u] = k+1$ for any $u \in Y$. It is easy to see that

$$w(g) = 2r + 2r(l+k) - l = \left(\frac{\Delta(\delta+k+1) - (\delta-k-1)}{\Delta(\delta+k+1) + (\delta-k-1)}\right) n.$$

Consequently, $\Gamma^s_{t,k}(F_{l,r}) = (\Delta(\delta+k+1) - (\delta-k-1))n/(\Delta(\delta+k+1) + (\delta-k-1))$.

In particular, if G is an odd-degree graph, then for any vertex $v \in V$, $f[v] = d(v) - 2d_M(v)$ is odd. Further, if k is odd, then for every vertex $u \in P_i \cap I$ $(s_1 \le i \le t_1)$, there exist $u' \in N(u)$ such that $f[u'] = k$ by Lemma 11. If $u' \in \bigcup_{i=s_1}^{t_1} P_i$, then u is adjacent to at most $i+k-1$ vertices of P_0, while if $u' \in \bigcup_{j=s_2}^{t_2} M_j$, then u is adjacent to at most $i+k$ vertices of P_0. Hence we can write $P_i \cap I$ $(s_1 \le i \le t_1)$ as the disjoint union of two sets IP'_i and IP''_i, where $IP'_i = \{u \in P_i \cap I \mid d_{P_0}(u) = i+k\}$ and $IP''_i = P_i \cap I - IP'_i$. Let $|IP'_i| = p'_i$, and so $|IP''_i| = |P_i \cap I| - p'_i$. Therefore, by (11) and (12), we can obtain

$$\begin{aligned} p_0 &\le \sum_{i=s_1}^{t_1} (i+k)p'_i + \sum_{i=s_1}^{t_1} (i+k-1)(|P_i \cap I| - p'_i) \\ &\le \sum_{i=s_1}^{t_1} (i+k-1)p_i + \sum_{j=s_2}^{t_2} jm_j. \end{aligned} \tag{14}$$

Since $(\delta+k)i/(\delta-k)i \ge i+k$ when $i \ge (\delta-k)/2$, it follows from (9), (10) and (14) that

$$n \le m + \Big(\sum_{i=s_1}^{t_1} (i+k-1)p_i + \sum_{j=s_2}^{t_2} jm_j\Big) + \sum_{i=1}^{t_1} p_i \le \left(1 + \frac{\delta+k}{\delta-k}\Delta\right) m.$$

Consequently,

$$\Gamma^s_{t,k}(G) = n - 2m \le \left(\frac{\Delta(\delta+k) - (\delta-k)}{\Delta(\delta+k) + (\delta-k)}\right) n.$$

That the bound is sharp may be seen as follows. For any integers $l \geq k$, $r \geq l + (k-1)/2$ and an even integer q, where $2l + k - 1 \leq q \leq 2r$, let $H_{l,r}$ be the graph with vertex set $X \cup Y \cup Z$ with $|X| = 2l^2$, $|Y| = 2l(2r+1)$ and $|Z| = 2l(2r+1)(l+k-1)$, where X is an independent set of vertices. The edge set of $H_{l,r}$ is constructed as follows: Add $2l^2(2r+1)$ edges between X and Y so that each vertex in X has degree $2r+1$ while each vertex in Y has degree l. Add $2l(2r+1)(l+k-1)$ edges between Y and Z so that each vertex of Y is precisely adjacent to $l+k-1$ vertices of Z and each vertex of Z is precisely adjacent to one vertex of Y. Add $l(2r+1)$ edges joining vertices of Y so that Y induces a 1-regular graph. Add edges joining vertices of Z so that Z induces a q-regular graph (since $q \leq 2r < |Z|$ and $|Z|$ is even, it follows from Lemma 10 that such an addition of edges is possible).

By construction, $H_{l,r}$ is an odd-graph of order $n = 2l^2 + 2l(2r+1) + 2l(2r+1)(l+k-1)$ with minimum degree $\delta = 2l + k$ and maximum degree $\Delta = 2r+1$. The function g that assigns to each vertex of X the value -1 and to each vertex of $Y \cup Z$ the value $+1$ is a minimal signed total dominating function of $H_{l,r}$ as $g[u] = k$ for any $u \in Y$. It is easy to see that

$$w(g) = 2l(2r+1) + 2l(2r+1)(l+k-1) - 2l^2 = \left(\frac{\Delta(\delta+k) - (\delta-k)}{\Delta(\delta+k) + (\delta-k)}\right) n.$$

Consequently, $\Gamma^s_{t,k}(H_{l,r}) = (\Delta(\delta+k) - (\delta-k))n/(\Delta(\delta+k) + (\delta-k))$. ■

As immediate consequences of Theorem 12 when $k = 1$, we have the following results due to Henning [2] and Kang and Shan [3], respectively.

Corollary 13 ([2]) *If $G = (V, E)$ is an r-regular graph with $r \geq 1$ of order n, then*

$$\Gamma^s_t(G) \leq \begin{cases} \left(\dfrac{r^2+1}{r^2+2r-1}\right) n & \text{for } r \text{ odd,} \\[2ex] \left(\dfrac{r^2+r+2}{r^2+3r-2}\right) n & \text{for } r \text{ even.} \end{cases}$$

Furthermore, these bounds are sharp.

Corollary 14 ([3]) *If $G = (V, E)$ is a nearly $(r+1)$-regular graph with $r \geq 1$ of order n, then*

$$\Gamma_t^s(G) \le \begin{cases} \left(\dfrac{r^2+3r+4}{r^2+5r+2}\right)n & \text{for } r \text{ odd,} \\ \left(\dfrac{(r+1)^2+3}{r(r+4)}\right)n & \text{for } r \text{ even.} \end{cases}$$

Furthermore, these bounds are sharp.

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