# Characterization of $M_3(2)$-graphs *†

S. AKBARI, D. KIANI, F. MOHAMMADI, S. MORADI

**Abstract**

A graph $G$ is called an $M_r(k)$*-graph* if $G$ has no $k$-list assignment to its vertices with exactly $r$ vertex colorings. We characterize all $M_3(2)$-graphs. More precisely, it is shown that a connected graph $G$ is an $M_3(2)$-graph if and only if each block of $G$ is a complete graph with at least three vertices.

# 1 Introduction

Throughout this paper $G$ denotes a simple graph. For any graph $G$, $V(G)$ and $E(G)$ denote the vertex set and the edge set of $G$, respectively. Let $G$ be a graph. For every $v \in V(G)$, we mean $N_G(v)$, the set of neighbors of $v$ in $G$. A *list assignment* $L$ is a function that assigns to each vertex $v$ of $G$ a set $L_v$ of colors. An $L$*-coloring* of $G$ is a function $c$ that assigns a color to each vertex of $G$ such that $c(v) \in L_v$ for all $v \in V(G)$ and $c(u) \neq c(v)$, whenever $u$ and $v$ are adjacent in $G$. A graph $G$ is called an $M_r(k)$*-graph* (M for Marshal Hall) if $G$ has no $k$-list assignment with exactly $r$ colorings. Uniquely 2-list colorable graphs are completely characterized. It was shown that a graph $G$ is not uniquely 2-list colorable(i.e., it is an $M_1(2)$-graph) if and only if each block of $G$ is a complete graph, a complete bipartite graph or a cycle, see Theorem A of [3]. $M_1(3)$-graphs are studied in [2]

* *Key Words*: List coloring, $M_r(k)$-graphs.

† 2000 *Mathematics Subject Classification*: 05C15, 05C38.

and [3]. In [1] it was proved that every graph with complete blocks such that each block has at least three vertices is an $M_3(2)$-graph. In this paper we characterize all $M_3(2)$-graphs. We show that a connected graph $G$ is an $M_3(2)$-graph if and only if each block of $G$ is a complete graph with at least three vertices.

## 2 Results

First we state the following simple remark without proof.

**Remark.** Every connected bipartite graph $G$ with the 2-list assignment $L_v = \{1, 2\}$, for each $v \in V(G)$, has exactly two $L$-colorings.

Now, we have an immediate consequence.

**Theorem 1.** *Every connected bipartite graph is not an $M_3(2)$-graph.*

**Proof.** Let $G$ be a connected bipartite graph, $T$ be a spanning tree of $G$ and $v \in V(T)$ be a pendant vertex. The graph $G \setminus \{v\}$ is connected. Let $V_1$ and $V_2$ be two parts of $G$ and $v \in V_1$. Set $L_u = \{1, 2\}$ for each $u \in V(G) \setminus \{v\}$ and $L_v = \{1, 3\}$. By the previous remark, $G \setminus \{v\}$ has two $L$-colorings $c_1$ and $c_2$ in which the colors of parts $V_1$ and $V_2$ in the coloring $c_1$ are $1, 2$, respectively. Also, the colors of parts $V_1$ and $V_2$ in the coloring $c_2$ are 2 and 1, respectively. Thus, with these lists $G$ has three $L$-colorings. □

If $p, q$ and $r$ $(p \leq q \leq r)$ are positive integers and at most one of them equals 1, then by $\theta_{p,q,r}$ we mean a graph which consists of three internally disjoint paths of lengths $p, q$ and $r$, which have the same endpoints.

**Theorem 2.** *Every $\theta_{p,q,r}$ is not an $M_3(2)$-graph.*

**Proof.** If $\theta_{p,q,r}$ has no odd cycle, then by Theorem 1 we obtain the result. So without loss of generality we assume that $p + q$ is odd and $r \geqslant 2$. Let

$v, w$ be the endpoints of three paths $P, Q$ and $R$ in $\theta_{p,q,r}$ such that their lengths are $p, q$ and $r$, respectively. Set $L_w = \{1,3\}$, $L_x = \{1,2\}$ for any $x \in V(P) \bigcup V(Q) \setminus \{w\}$ and $L_y = \{1,4\}$ for any $y \in V(R) \setminus \{v, w\}$. It is easy to see that with this list assignment $\theta_{p,q,r}$ has three $L$-colorings. $\square$

**Lemma 1.** *Let $G$ be a graph and $u, v \in V(G)$ be two adjacent vertices. If $G \setminus \{u, v\}$ is a uniquely 2-list colorable graph, then $G$ is not an $M_3(2)$-graph.*

**Proof.** Let $L$ be a 2-list assignment for $G \setminus \{u, v\}$ such that it has a unique $L$-coloring. Set $L_u = \{a, b\}$ and $L_v = \{a, c\}$, where $a, b$ and $c$ are new colors. With this list assignment $G$ has three $L$-colorings. $\square$

The following lemma is useful in the proof of the next theorem.

**Lemma 2.** *Suppose that $G$ is a 2-connected non-bipartite graph. If $G$ is not a complete graph or a cycle, then it has an induced subgraph $\theta_{p,q,r}$, for some natural numbers $p, q$ and $r$, where $p + q$ is odd.*

**Proof.** Let $C$ be an odd cycle of minimum length in $G$. Then $C$ is an induced (chordless) cycle. Since $G$ is 2-connected and not a cycle, it contains a path $R$ of length at least 2 whose endpoints are in $C$ and whose other vertices are not in $C$. Choose such a path $R$ of minimum length. If $C \cup R$ is an induced subgraph of $G$, then it is the required subgraph $\theta_{p,q,r}$. So assume that $C \cup R$ is not an induced subgraph of $G$. Since $C$ is the smallest odd cycle and $R$ has minimum length, there exists an edge with one end point in $V(R) \setminus V(C)$ and other one in $V(C) \setminus V(R)$. If the length of $R$ is at least three, then we obtain a path whose length is less than the length of $R$ and its end points are on $C$, a contradiction. On the other hand since $C$ is the smallest odd cycle, we conclude that $R$ has length 2 and its middle vertex $x$ is adjacent to at least three vertices of $C$. Assume that $x_1, \ldots, x_k$ are all vertices in $N_G(x) \cap V(C)$ (in clockwise order). Since the length of $C$ is odd, there exists an index $j$ such that the arc $x_j x_{j+1}$ has odd length. Now, if for every $i$, the length of arc $x_i x_{i+1}$ is at least two, then $x$ and arc $x_jx_{j+1}$ form an odd cycle of length smaller than length of $C$, a contradiction. Thus there exists some $t$, $1 \leq t \leq k$, such that the arc $x_tx_{t+1}$ has length 1. Thus $G$ contains a triangle, and so $C$ is a triangle and $C \cup \{x\}$ induces a $K_4$.

Let $K$ be a largest complete subgraph of $G$. Since $G$ is 2-connected and not complete, it contains a path $R$ of length at least 2 whose endpoints $u, v$ are in $K$ and whose other vertices are not in $K$. Choose such a path $R$ of minimum length, choose $w \in V(K)\backslash\{u, v\}$, and let $C$ be the triangle $uvw$. By the same argument as in the previous paragraph, if the result does not hold then $R$ has length 2 and $C \cup \{x\}$ induces a $K_4$, where $x$ is the middle vertex of $R$. By the maximality of $K$, there is a vertex $y$ of $K$ such that $x$ is not adjacent to $y$, and then $\{u, v, x, y\}$ induces the required subgraph $\theta_{1,2,2}$ in $G$. □

**Theorem 3.** *Every non-complete 2-connected graph is not an $M_3(2)$-graph.*

**Proof.** If $G$ is bipartite, then by Theorem 1, $G$ is not an $M_3(2)$-graph. Thus assume that $G$ has an odd cycle. First assume that $G$ is an odd cycle with vertices $v_1, \ldots, v_n$. Let $L_{v_i} = \{1, 2\}$, for $i = 1, \ldots, n-3$, $L_{v_{n-2}} = L_{v_{n-1}} = \{1, 3\}$ and $L_{v_n} = \{1, 4\}$. Clearly, $G$ has exactly three coloring with these lists and we are done. Thus assume that $G$ is not an odd cycle. By Lemma 3, $G$ has an induced subgraph $\theta_{p,q,r}$ with three paths $P, Q$ and $R$ of sizes $p, q$ and $r$, where $p + q$ is odd. If $G$ is $\theta_{p,q,r}$, then by Theorem 2, $G$ is not an $M_3(2)$-graph. We can assume that $V(G) \setminus V(\theta_{p,q,r}) \neq \emptyset$. Suppose that there exists a vertex $v \in V(G) \setminus V(\theta_{p,q,r})$ which is adjacent to none of the vertices of $\theta_{p,q,r}$. Since $G$ is 2-connected, by Exercise 4.2.8 of [4], there exists a vertex $u$ adjacent to $v$ such that $G \setminus \{u, v\}$ is connected. The graph $G \setminus \{u, v\}$ has $\theta_{p,q,r}$ as an induced subgraph. The graph $\theta_{p,q,r}$ is uniquely 2-list colorable(see Theorem A of [3]). Now, we show that $G \setminus \{u, v\}$ is uniquely 2-list colorable. Let $w$ be a vertex of $G \setminus \{u, v\}$ such that $w' \in N_{G\setminus\{u,v\}}(w) \cap V(\theta_{p,q,r}) \neq \varnothing$. The induced subgraph on $V(\theta_{p,q,r}) \cup \{w\}$ is a uniquely 2-list colorable graph, because if one can assign the list $\{t, s\}$ to $w$, where $t$ is the color of $w'$ in the unique coloring of $\theta_{p,q,r}$ and $s$ is a new color not appeared in the list of vertices of $\theta_{p,q,r}$, then $\theta_{p,q,r} \cup \{w\}$ is uniquely 2-list colorable with these lists. Now, inductively we conclude that $G \setminus \{u, v\}$ is uniquely 2-list colorable. Therefore, by Lemma 2, $G$ is not $M_3(2)$. Hence we may assume that each vertex $v \in V(G) \setminus V(\theta_{p,q,r})$ is adjacent to some vertices of $\theta_{p,q,r}$. If there exist two adjacent vertices $u, v \in V(G) \setminus V(\theta_{p,q,r})$, then the graph $G \setminus \{u, v\}$ is connected and uniquely 2-list colorable, since it contains $\theta_{p,q,r}$ as an induced subgraph. By Lemma 2, $G$ is not an $M_3(2)$-graph. Assume that no two vertices of $V(G) \setminus V(\theta_{p,q,r})$ are adjacent. Since any vertex $v \in V(G) \setminus V(\theta_{p,q,r})$ is adjacent to some vertex in $\theta_{p,q,r}$, the graph $G \setminus \{v\}$ is 2-connected. Therefore, $G \setminus \{u, v\}$ is connected for any $v \in V(G) \setminus V(\theta_{p,q,r})$ and $u \in V(G)$. Let $u', v' \in V(\theta_{p,q,r})$ be two adjacent vertices. If $G \setminus \{u', v'\}$ is not connected, then there exists a vertex $w \in V(G) \setminus V(\theta_{p,q,r})$ adjacent to $u', v'$ such that $G \setminus \{u', w\}$ is connected. Thus, for any two adjacent vertices $u, v \in V(\theta_{p,q,r})$ either $G \setminus \{u, v\}$ is connected or for some vertex $w \in V(G) \setminus V(\theta_{p,q,r})$ adjacent to $u$, $G \setminus \{u, w\}$ is connected. Let $x, y$ be the endpoints of the paths $P : x, u_1, \ldots, u_{p-1}, y$, $Q : x, v_1, \ldots, v_{q-1}, y$ and $R : x, w_1, \ldots, w_{r-1}, y$, where $q = min\{p, q, r\}$. Let $C_1$ and $C_2$ be the cycles formed by $(P, Q)$ and $(Q, R)$, respectively. The following cases may be considered:

Case (i). Assume that the cycles $C_1$ and $C_2$ have lengths at least 5 or $p \geqslant 4$, $q = 1$, $r = 3$. If there exists a vertex $z \in V(G) \setminus V(\theta_{p,q,r})$ such that $z$ is adjacent to at least two vertices in $V(C_1)$ or $V(C_2) \setminus V(C_1)$, then there exists an edge $uv \in E(G)$ such that either $G \setminus \{u, v\}$ is connected or for some vertex $w \in V(G) \setminus V(\theta_{p,q,r})$, $G \setminus \{u, w\}$ is connected and uniquely 2-list colorable. So $G \setminus \{u, v\}$ or $G \setminus \{u, w\}$ is uniquely 2-list colorable and by Lemma 2 the assertion is true. Assume that any vertex $v \in V(G) \setminus V(\theta_{p,q,r})$ is adjacent to exactly one vertex in $C_1$ and exactly one vertex in $V(C_2) \setminus V(C_1)$. Let $v \in V(G) \setminus V(\theta_{p,q,r})$. So $v$ is not adjacent to both $x$ and $y$. Assume that $v$ is not adjacent to $x$. Consider the list assignment $L_{u_1} = L_{u_2} = \cdots = L_{u_{p-1}} = L_y = L_{v_1} = L_{v_2} = \cdots = L_{v_{q-1}} = \{1, 2\}$, $L_x = \{1, 3\}$, $L_{w_1} = \{3, 4\}$, $L_{w_2} = \{4, 5\}, \ldots, L_{w_{r-1}} = \{r + 1, r + 2\}$

and $L_v = \{1, a\}$, where $a$ is a new color. The graph $\theta_{p,q,r} \bigcup \{v\}$ has three $L$-colorings and the colors of $x, w_1, \dots, w_{r-1}$ in three $L$-colorings are the same. Let $c(x), c(w_1), \dots, c(w_{r-1})$ denote the colors of these vertices. For any $z \in V(G) \setminus V(\theta_{p,q,r}) \bigcup \{v\}$ which is adjacent to $w_j \in V(C_2)$ for some $1 \leqslant j \leqslant r-1$, set $L_z = \{c(w_j), b\}$, where $b$ is a new color. With this list assignment $G$ has exactly three $L$-colorings.

Case (ii). Let $p \geqslant 3$, $q = r = 2$. Consider the list assignment $L_{u_1} = \{1,4\}$, $L_{w_1} = \{2,3\}$ , $L_x = L_{v_1} = \{1,3\}$ and $L_y = L_{u_2} = \dots = L_{u_{p-1}} = \{1,2\}$. It is easy to see that $\theta_{p,q,r}$ has three $L$-colorings $c_1, c_2$ and $c_3$, where the colors of the vertices $u_1, x$ and $v_1$ in three colorings are the same. For any vertex $v \in V(G) \setminus V(\theta_{p,q,r})$ and any $1 \leqslant i \leqslant 3$ let $c_i(N_G(v)) = \{c_i(u); u \in N_G(v)\}$. If there exists $a \in c_1(N_G(v)) \bigcap c_2(N_G(v)) \bigcap c_3(N_G(v))$, then set $L_v = \{a, b\}$, where $b$ is a new color. If $v$ is adjacent to $u_1$ or $x$ or $v_1$, then one can assign an appropriate list to $v$. So we can assume that $u_1, x$ and $v_1$ are not adjacent to $v$. If $w_1 \notin N_G(v)$ and $c_1(N_G(v)) \bigcap c_2(N_G(v)) \bigcap c_3(N_G(v)) = \emptyset$, then set $L_v = \{1,2\}$. Assume that $w_1 \in N_G(v)$. If $y \in N_G(v)$, then the graph $G \setminus \{u_i, u_{i+1}\}$ or $G \setminus \{u_i, z\}$ is connected and uniquely 2-list colorable for some $1 \leqslant i \leqslant p-1$ and $z \in V(G) \setminus V(\theta_{p,q,r})$, since it has a uniquely 2-list colorable subgraph induced by the vertices $x, y, v_1, w_1, v$. So by Lemma 2, $G$ is not an $M_3(2)$-graph. Let $y \notin N_G(v)$, therefore $u_i \in N_G(v)$ for some $2 \leqslant i \leqslant p-1$. Assume that there exist at least two vertices between $x$ and $u_i$ or between $u_i$ and $y$ on $P$, say $u_j, u_{j+1}$. If $j < i-1$, then the subgraph induced by the vertices $v, x, v_1, w_1, u_i, \dots, u_{p-1}, y$ is uniquely 2-list colorable. If $j > i$, then the subgraph induced by the vertices $v, x, v_1, w_1, u_1, \dots, u_i, y$ is uniquely 2-list colorable. Thus $G \setminus \{u_j, u_{j+1}\}$ or $G \setminus \{u_j, z\}$ is connected and uniquely 2-list colorable, for some $z \in V(G) \setminus V(\theta_{p,q,r})$. Therefore, by Lemma 2, $G$ is not an $M_3(2)$-graph. Assume that there exists at most one vertex between $x$ and $u_i$ and at most one vertex between $u_i$ and $y$, so $p = 3$. Thus, the path $P$ is $x, u_1, u_2, y$ and so $N_G(v) = \{w_1, u_2\}$. Set $L_v = \{1,3\}$. The graph $G$ has three $L$-colorings.

Case (iii). Assume that $p = 2$, $q = 1$ and $r \geqslant 2$. If there exists no vertex $v$ in $V(G) \setminus V(\theta_{p,q,r})$ such that $N_G(v) = \{x, w_1\}$, then set $L_x = L_{u_1} = \{1,2\}$, $L_y = L_{w_2} = \cdots = L_{w_{r-1}} = \{1,3\}$ and $L_{w_1} = \{2,4\}$. The graph $\theta_{p,q,r}$ has three $L$-colorings and the colors of $y, w_2, \ldots, w_{r-1}$ are the same in three $L$-colorings. Let $c(y), c(w_2), \ldots, c(w_{r-1})$ denote the colors of these vertices.

For any vertex $z \in V(G) \setminus V(\theta_{p,q,r})$, if there is $u \in N_G(z) \bigcap \{y, w_2, \ldots, w_{r-1}\}$, then set $L_z = \{c(u), a\}$, where $a$ is a new color. So $z$ gets the color $a$ in three colorings. If $N_G(z) = \{x, u_1\}$ or $N_G(z) = \{x, u_1, w_1\}$, then set $L_z = \{1, a\}$, where $a$ is a new color. If $N_G(z) = \{u_1, w_1\}$, then set $L_z = \{1,2\}$. With this list assignment $G$ has three $L$-colorings $c_1, c_2$ and $c_3$. Assume that there exists a vertex $v \in V(G) \setminus V(\theta_{p,q,r})$ such that $N_G(v) = \{x, w_1\}$. Set $L_x = \{1,3\}$, $L_{u_1} = L_y = L_{w_1} = \cdots = L_{w_{r-1}} = \{1,2\}$ and $L_v = \{1,4\}$. So $\theta_{p,q,r} \bigcup \{v\}$ has three $L$-colorings. For any $u \in V(G) \setminus V(\theta_{p,q,r}) \bigcup \{v\}$, if there exists $a \in c_1(N_G(u)) \bigcap c_2(N_G(u)) \bigcap c_3(N_G(u))$, then set $L_u = \{a, b\}$, where $b$ is a new color, otherwise, set $L_u = \{1,2\}$. With this list assignment $G$ has three $L$-colorings. □

The following results were proved in [1].

**Theorem A** *Let $G$ be a graph with complete blocks. If each block of $G$ has at least three vertices, then $G$ is an $M_3(2)$-graph.*

**Theorem B** *Let $G$ be a connected graph formed by the union of two graphs $G_1, G_2$ which are joined in exactly one vertex $v$. If $G_1$ is not an $M_3(2)$-graph, then $G$ is not an $M_3(2)$-graph.*

Now, we are in a position to state our main theorem.

**Theorem 4.** *A connected graph $G$ is an $M_3(2)$-graph if and only if each block of $G$ is a complete graph with at least three vertices.*

**Proof.** Let $G$ be a connected $M_3(2)$-graph. By Theorem B each block of $G$ is an $M_3(2)$-graph. Therefore, by Theorem 3, each block of $G$ is a complete graph. Since a complete graph with two vertices is not an $M_3(2)$-graph, each block of $G$ has at least three vertices. The other side is clear by Theorem A. □

**Conjecture.** Suppose that $G$ is not a connected $M_3(2)$-graph. There exists a 2-list assignment $L$ to $G$ such that $1 \in \bigcap_{v \in V(G)} L_v$ and $G$ has three $L$-colorings.

## Acknowledgement

The first author is indebted to the Research Council of Sharif University of Technology for support. The research of the second author was in part supported by a grant from IPM with number (No. 86150116). Also, the authors wish to thank the referee for her/his fruitful comments.

# References

[1] S. Akbari, D. Kiani, F. Mohammadi, S. Moradi, Coloring of graphs with 2-lists, submitted.

[2] D.A. Mojdeh, A. Ahmadi Haji, H. Abdollahzadeh Ahangar and A. Khodkar, Graphs $K_{1*4,5}$, $K_{1*5,4}, K_{1*4,4}, K_{2,3,4}$ have the property $M(3)$, Ars Combinatoria 84 (2007), 171-190.

[3] D.A. Mojdeh, A. Ahmadi Haji, H. Abdollahzadeh Ahangar and A. Khodkar, The graph $K_{2,2,r}$ has the property $M(3)$ for $4 \leq r \leq 8$, Bulletin of the Institute of Combinatorics and its Applications 47 (2006) 63-70.

[4] D.B. West, Introduction to Graph Theory, Second Edition, Prentice Hall, 2001.

SAIEED AKBARI [a] s_akbari@sharif.edu
DARIUSH KIANI [b,c] dkiani@aut.ac.ir
FATEMEH MOHAMMADI [b] f_mohammadi@cic.aut.ac.ir

SOMAYEH MORADI [b] s_moradi@cic.aut.ac.ir
[a]Department of Mathematical Sciences Sharif University of Technology P. O. Box 11365-9415, Tehran, Iran.
[b]Department of Pure Mathematics, Faculty of Mathematics and Computer Science, Amirkabir University of Technology (Tehran Polytechnic), 424, Hafez Ave., Tehran 15914, Iran.
[c]Institute for Studies in Theoretical Physics and Mathematics (IPM).