On the Four Color Ramsey Numbers for Hexagons *

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Abstract

Let G_i be the subgraph of G whose edges are in the i-th color in an r-coloring of the edges of G. If there exists an r-coloring of the edges of G such that $H_i \nsubseteq G_i$ for all $1 \leq i \leq r$, then G is said to be $r-colorable$ to $(H_1, H_2, \ldots, H_r)$. The multicolor Ramsey number $R(H_1, H_2, \ldots, H_r)$ is the smallest integer n such that K_n is not r-colorable to $(H_1, H_2, \ldots, H_r)$. Let C_m be a cycle of length m, the four color Ramsey numbers related to C_6 are studied in this paper. It is well known that $18 \leq R_4(C_6) \leq 21$. We prove that $R(C_6, C_4, C_4, C_4) = 19$ and $18 \leq R(C_6, C_6, H_1, H_2) \leq 20$, where H_i are isomorphic to C_4 or C_6.

Keywords: *Extremal graph; Cycle; Multicolor Ramsey number*

1 Introduction

We consider only finite undirected graphs without loops or multiple edges. Let G be a graph, the vertex set of G is denoted by $V(G)$, the edge set of G by $E(G)$. Let Ψ denote the set of some graphs, then $ex(n; \Psi)$ is the maximum size of a graph with n vertices, which contains no subgraph isomorphic to any graph in Ψ, and $EX(n; \Psi)$ denotes the set of all graphs with $ex(n; \Psi)$ edges.

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Let G_i be the subgraph of G whose edges are in the i-th color in an r-coloring of the edges of G. If there exists an r-coloring of the edges of G such that $H_i \nsubseteq G_i$ for all $1 \leq i \leq r$, then G is said to be $r-$*colorable* to $(H_1, H_2, \ldots, H_r)$. The multicolor Ramsey number $R(H_1, H_2, \ldots, H_r)$ is the smallest integer n such that K_n is not r-colorable to $(H_1, H_2, \ldots, H_r)$. In the case of $H_1 \cong H_2 \cong \ldots \cong H_r \cong H$, we simply write $R(H_1, H_2, \ldots, H_r)$ as $R_r(H)$. Let $\alpha(G)$ denote the independence number of G, $N[v]$ the set of vertices adjacent to a vertex v together with v itself, and $G[W]$ the induced subgraph of G by $W \subseteq V(G)$.

Clapham, Flockhart and Sheehan gave the values of $ex(n;\{C_4\})$ and $EX(n;\{C_4\})$ for $n \leq 21$ in [1]. Yang and Rowlinson[10, 11] studied the values of $ex(n;\Psi)$ for $\Psi = \{C_4\}$ and $\Psi = \{C_6\}$ by a computer. They determined the exact values of $ex(n;\{C_4\})$ for $22 \leq n \leq 31$, $ex(n;\{C_6\})$ for $6 \leq n \leq 21$ and gave the corresponding extremal graphs. Sun et al. further gave the values of $ex(n;\{C_6\})$ for $22 \leq n \leq 26$[5] and obtained the values of $ex(n;\{C_4, C_5\})$ for $n \leq 21$[4]. By the result of $ex(26;\{C_6\}) = 64$, they showed that $R_5(C_6) = 26$. In [6], Sun et al. showed that $R_4(C_4) = 18$ with the help of a substantial amount of computation. Xu and Radziszowski[8, 9] studied the four color Ramsey numbers related to cycles, they proved that $21 \leq R(C_4, C_4, C_4, C_3) \leq 27$ and $28 \leq R(C_4, C_4, C_3, C_3) \leq 36$. Dybizbański and Dzido[2] improved the lower bounds of $R(C_4, C_4, C_4, C_3)$ and $R(C_4, C_4, C_3, C_3)$ to 24 and 30 respectively. For further general reading about the multicolor Ramsey numbers of cycles, see the latest survey provided by Radziszowski[3].

The four color Ramsey numbers related to C_6 are studied in this paper. It is well known that $18 \leq R_4(C_6) \leq 21$[7, 10]. By the lemmas in sections 2 and 3, we prove the following theorem,

Theorem 1.1.

$$R(C_6, C_4, C_4, C_4) = 19,$$
$$18 \leq R(C_6, C_6, C_4, C_4) \leq 20,$$
$$18 \leq R(C_6, C_6, C_6, C_4) \leq 20,$$
$$18 \leq R_4(C_6) \leq 20.$$

2 The proof of the upper bounds

Let $V(G_f) = \{v_1, v_2, \ldots, v_n\}$, $V(G_s) = \{u_1, u_2, \ldots, u_n\}$, and θ is a bijection such that $\theta(v_i) = u_j$ for $1 \leq i, j \leq n$. For a bijection θ, if $v_iv_j \in E(G_f)$ and

$\theta(v_i)\theta(v_j) \in E(G_s)$ for some i,j $(1 \le i < j \le n)$, we say it is bad, otherwise called it a good bijection. Let $V(G_f \uplus G_s) = V(G_f)$ and $E(G_f \uplus G_s) = \{v_iv_j | v_iv_j \in E(G_f) \cup \theta(v_i)\theta(v_j) \in E(G_s), 1 \le i < j \le n\}$. If θ is a good bijection, then $|E(G_f \uplus G_s)| = |E(G_f)| + |E(G_s)|$.

The values of $ex(n;\{C_4\})$ and $ex(n;\{C_6\})$ are obtained in [10, 11], which are shown in Table 1.

Table 1. The values of $ex(n;\{C_4\})$ and $ex(n;\{C_6\})$ for $6 \le n \le 20$

n	6	7	8	9	10	11	12	13	14	15	16
$ex(n;\{C_4\})$	7	9	11	13	16	18	21	24	27	30	33
$ex(n;\{C_6\})$	11	13	16	20	21	23	26	30	31	33	37
n	17	18	19	20							
$ex(n;\{C_4\})$	36	39	42	46							
$ex(n;\{C_6\})$	40	41	44	48							

By the results $ex(19;\{C_6\}) = 44$, $ex(19;\{C_4\}) = 42$, $ex(20;\{C_6\}) = 48$, $ex(20;\{C_4\}) = 46$ and the results in [10], we have Lemma 2.1 and Lemma 2.2 as following.

Lemma 2.1.

$$R(C_6, C_4, C_4, C_4) \le 19,$$

$$R(C_6, C_6, C_4, C_4) \le 20.$$

Lemma 2.2. $|EX(20;\{C_6\})| = 2$.

Let $EX(20;\{C_6\}) = \{H_{20-1}, H_{20-2}\}$, then H_{20-1} and H_{20-2} are shown in Fig. 1, where $V(H_{20-1}) = V(H_{20-2}) = \{v_1, v_2, \ldots, v_{20}\}$.

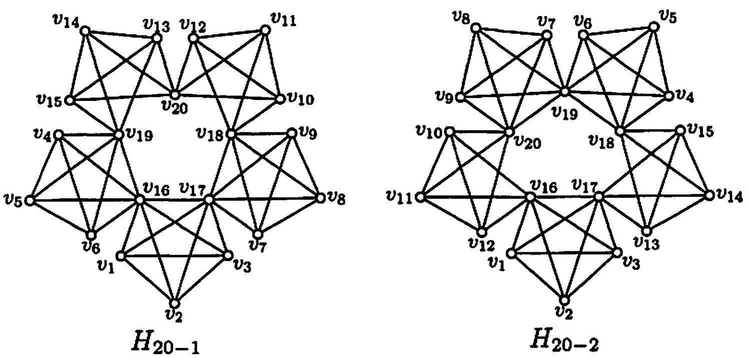

Fig. 1. The graphs H_{20-1} and H_{20-2}.

Lemma 2.3. Let $G_f = H_{20-1}$, $G_s \cong H_{20-1}$ and $V(G_s) = \{u_i | u_i = v_i, 1 \leq i \leq 20\}$. For any bijection θ where $\theta(v_i) = u_j$ for $1 \leq i, j \leq 20$, we have θ is bad.

Proof. Suppose there exists a good bijection θ, then there are two subcases depending on $\theta(v_i)$ for $v_i \in \{v_{16}, v_{17}, v_{18}, v_{19}\}$.

Case 1. Suppose there is one vertex of $\{v_{16}, v_{17}, v_{18}, v_{19}\}$ such that $\theta(v_i) = u_j$ for $1 \leq j \leq 9$ or $16 \leq j \leq 20$. If there exists $\theta(v_i) = u_j$ for $1 \leq j \leq 9$, say $\theta(v_{16}) = u_1$, then since $G_s[N[u_1]] \cong K_5$ and θ is a good bijection, v_{16} together with the four appropriate vertices of $V(G_f)$ have to yield a $5K_1$. Note that $\alpha(G_f[V(G_f) \setminus N[v_{16}]]) = 3$, a contradiction. If there exists $\theta(v_i) = u_j$ for $16 \leq j \leq 20$, say $\theta(v_{16}) = u_{16}$, then since $K_5 \subseteq G_f[N[v_{16}]]$ and θ is a good bijection, u_{16} together with the four appropriate vertices of $V(G_s)$ have to yield a $5K_1$. Note that $\alpha(G_s[V(G_s) \setminus N[u_{16}]]) = 3$, a contradiction too.

Case 2. Suppose each vertex of $\{v_{16}, v_{17}, v_{18}, v_{19}\}$ maps to one vertex of u_j for $10 \leq j \leq 15$ in θ. Without loss of generality, let $\theta(v_{16}) = u_{10}$. Since θ is a good bijection and $v_{16}v_{17} \in E(G_f)$, we have $\theta(v_{17})$ is one vertex of $\{u_{13}, u_{14}, u_{15}\}$, say $\theta(v_{17}) = u_{13}$. Then $\theta(v_{18})$ is one vertex of $\{u_{11}, u_{12}\}$, say $\theta(v_{18}) = u_{11}$. Therefore since $G_s[\{u_{10}, u_{11}, u_{12}, u_{20}\}] \cong K_4$, v_{16} and v_{18} together with the two appropriate vertices of $V(G_f)$ have to yield a $4K_1$. Note that $\alpha(G_f[V(G_f) \setminus (N[v_{16}] \cup N[v_{18}])]) = 1$, a contradiction.

By Case 1 and 2, we have the lemma holds. □

Lemma 2.4. Let $G_f = H_{20-1}$, $G_s \cong H_{20-2}$ and $V(G_s) = \{u_i | u_i = v_i, 1 \leq i \leq 20\}$. For any bijection θ where $\theta(v_i) = u_j$ for $1 \leq i, j \leq 20$, we have θ is bad.

Proof. Suppose there exists a good bijection θ, then there are two subcases depending on $\theta(v_i)$ for $v_i \in \{v_{16}, v_{17}, \ldots, v_{20}\}$.

Case 1. Suppose there is one vertex of $\{v_{16}, v_{17}, \ldots, v_{20}\}$ such that $\theta(v_i) = u_j$ for $1 \leq j \leq 9$ or $16 \leq j \leq 20$, say $\theta(v_{16}) = u_1$(or $\theta(v_{16}) = u_{16}$). Then since $G_s[N[u_1]] \cong K_5$(or $K_5 \subseteq G_s[N[u_{16}]]$) and θ is a good bijection, v_{16} together with the four appropriate vertices of $V(G_f)$ have to yield a $5K_1$. Note that $\alpha(G_f[V(G_f) \setminus N[v_{16}]]) = 3$, a contradiction.

Case 2. Suppose each vertex of $\{v_{16}, v_{17}, \ldots, v_{20}\}$ maps to one vertex of u_j for $10 \leq j \leq 15$ in θ. Without loss of generality, let $\theta(v_{16}) = u_{10}$. Since θ is a good bijection and $v_{16}v_{17} \in E(G_f)$, we have $\theta(v_{17})$ is one vertex of $\{u_{13}, u_{14}, u_{15}\}$, say $\theta(v_{17}) = u_{13}$. Then $\theta(v_{18})$ is one vertex of $\{u_{11}, u_{12}\}$, say $\theta(v_{18}) = u_{11}$. Therefore since $G_s[\{u_{10}, u_{11}, u_{12}, u_{16}\}] \cong K_4$, v_{16} and v_{18} together with the two appropriate vertices of $V(G_f)$ have to yield a $4K_1$. Note that $\alpha(G_f[V(G_f) \setminus (N[v_{16}] \cup N[v_{18}])]) = 1$, a contradiction.

By Case 1 and 2, we have the lemma holds. □

Lemma 2.5. Let $G_f = H_{20-2}$, $G_s \cong H_{20-2}$ and $V(G_s) = \{u_i | u_i = v_i, 1 \leq$

$i \leq 20\}$. For any bijection θ where $\theta(v_i) = u_j$ for $1 \leq i, j \leq 20$, we have θ is bad.

Proof. Suppose there exists a good bijection θ, then there are two subcases depending on $\theta(v_i)$ for $v_i \in \{v_{16}, v_{17}, \dots, v_{20}\}$.

Case 1. Suppose there is one vertex of $\{v_{16}, v_{17}, \dots, v_{20}\}$ such that $\theta(v_i) = u_j$ for $1 \leq j \leq 9$ or $16 \leq j \leq 20$, say $\theta(v_{16}) = u_1$(or $\theta(v_{16}) = u_{16}$). Then since $G_s[N[u_1]] \cong K_5$(or $K_5 \subseteq G_s[N[u_{16}]]$) and θ is a good bijection, v_{16} together with the four appropriate vertices of $V(G_f)$ have to yield a $5K_1$. Note that $\alpha(G_f[V(G_f) \setminus N[v_{16}]]) = 3$, a contradiction.

Case 2. Suppose each vertex of $\{v_{16}, v_{17}, \dots, v_{20}\}$ maps to one vertex of u_j for $10 \leq j \leq 15$ in θ. Without loss of generality, let $\theta(v_{18}) = u_{10}$. Since θ is a good bijection and $v_{18}v_{19} \in E(G_f)$, we have $\theta(v_{19})$ is one vertex of $\{u_{13}, u_{14}, u_{15}\}$, say $\theta(v_{19}) = u_{13}$. Then $\theta(v_{20})$ is one vertex of $\{u_{11}, u_{12}\}$, say $\theta(v_{20}) = u_{11}$. Therefore since $G_s[\{u_{10}, u_{11}, u_{12}, u_{16}\}] \cong K_4$, v_{18} and v_{20} together with the two appropriate vertices of $V(G_f)$ have to yield a $4K_1$. Note that $\alpha(G_f[V(G_f) \setminus (N[v_{18}] \cup N[v_{20}])]) = 1$, a contradiction.

By Case 1 and 2, we have the lemma holds. □

Lemma 2.6. $R_4(C_6) \leq 20$.

Proof. Suppose K_{20} is 4-colorable to C_6. Without loss generality, let $|E(G_1)| \geq |E(G_2)| \geq |E(G_3)| \geq |E(G_4)|$. Since $|E(K_{20})| = 190$, we have $|E(G_1)| = 48$ and $|E(G_2)| = 48$. By Lemma 2.2, both G_1 and G_2 are isomorphic to H_{20-1} or H_{20-2} as shown in Fig. 1. Let $V(G_1) = \{v_i | 1 \leq i \leq 20\}$ and $V(G_2) = \{u_j | 1 \leq j \leq 20\}$, then there exists a good bijection θ such that $\theta(v_i) = u_j$, a contradiction to Lemma 2.3, 2.4 or 2.5. Hence K_{20} is not 4-colorable to C_6, that is, $R_4(C_6) \leq 20$. □

Lemma 2.7. $R(C_6, C_6, C_6, C_4) \leq 20$.

Proof. Suppose K_{20} is 4-colorable to (C_6, C_6, C_6, C_4). Since $|E(K_{20})| = 190$ and $ex(20; \{C_4\}) = 46$ in Table 1, it is forced that $|E(G_1)| = |E(G_2)| = |E(G_3)| = 48$. By Lemma 2.2, we have $G_i (1 \leq i \leq 3)$ are isomorphic to H_{20-1} or H_{20-2} as shown in Fig. 1. It is sufficient to consider G_1 and G_2. Let $V(G_1) = \{v_i | 1 \leq i \leq 20\}$ and $V(G_2) = \{u_j | 1 \leq j \leq 20\}$, then there exists a good bijection θ such that $\theta(v_i) = u_j$, a contradiction to Lemma 2.3, 2.4 or 2.5. Hence K_{20} is not 4-colorable to (C_6, C_6, C_6, C_4), that is, $R(C_6, C_6, C_6, C_4) \leq 20$. □

3 The proof of the lower bounds

Lemma 3.1. $R(C_6, C_4, C_4, C_4) \geq 19$.

Proof. We show a 4-coloring of the edges of K_{18} where $G_i \cong H_{18-i}$ for $1 \leq i \leq 4$ as shown in Fig. 2. We can easily find that $C_6 \nsubseteq H_{18-1}$. H_{18-2} is

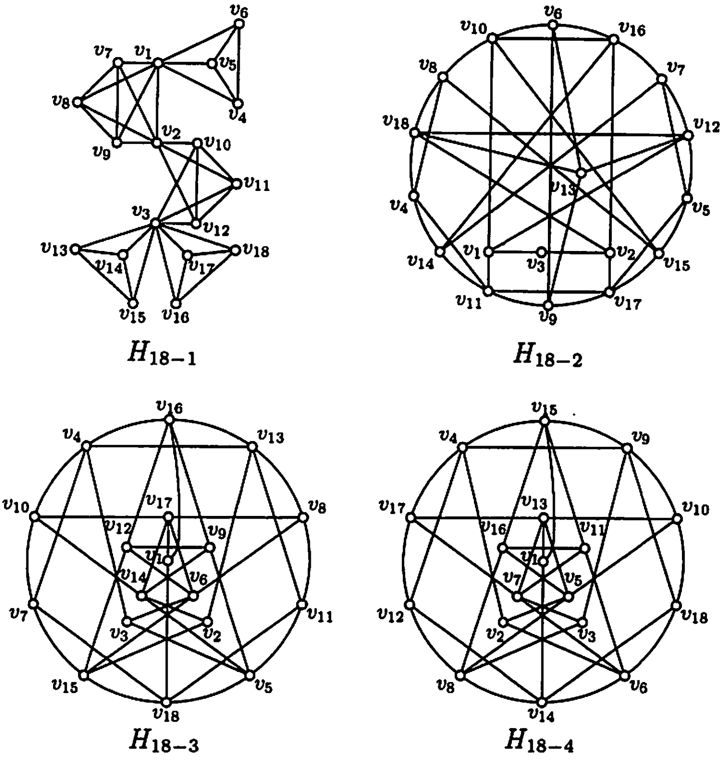

Fig. 2. The graphs H_{18-i} for $1 \leq i \leq 4$.

a graph without C_4, both H_{18-3} and H_{18-4} are the unique extremal graph without C_4. Hence, we have K_{18} is 4-colorable to (C_6, C_4, C_4, C_4), that is, $R(C_6, C_4, C_4, C_4) \geq 19$. □

Lemma 3.2. $R(C_6, C_6, C_4, C_4) \geq 18$.
Proof. We show a 4-coloring of the edges of K_{17} where $G_i \cong H_{17-i}$ for $1 \leq i \leq 4$ as shown in Fig. 3. We can easily find that $C_6 \not\subseteq H_{17-1}$ and $C_6 \not\subseteq H_{17-2}$. H_{17-3} is a graph without containing C_4, and H_{17-4} is isomorphic to H_{17-3}. Hence, we have K_{17} is 4-colorable to (C_6, C_6, C_4, C_4), that is, $R(C_6, C_6, C_4, C_4) \geq 18$. □

Lemma 3.3. $R(C_6, C_6, C_6, C_4) \geq 18$.
Proof. We show a 4-coloring of the edges of K_{17} where $G_i \cong H_{17-i}$ for $1 \leq i \leq 2$ as shown in Fig. 3, and $G_i \cong H_{17-(i+2)}$ for $3 \leq i \leq 4$ as shown

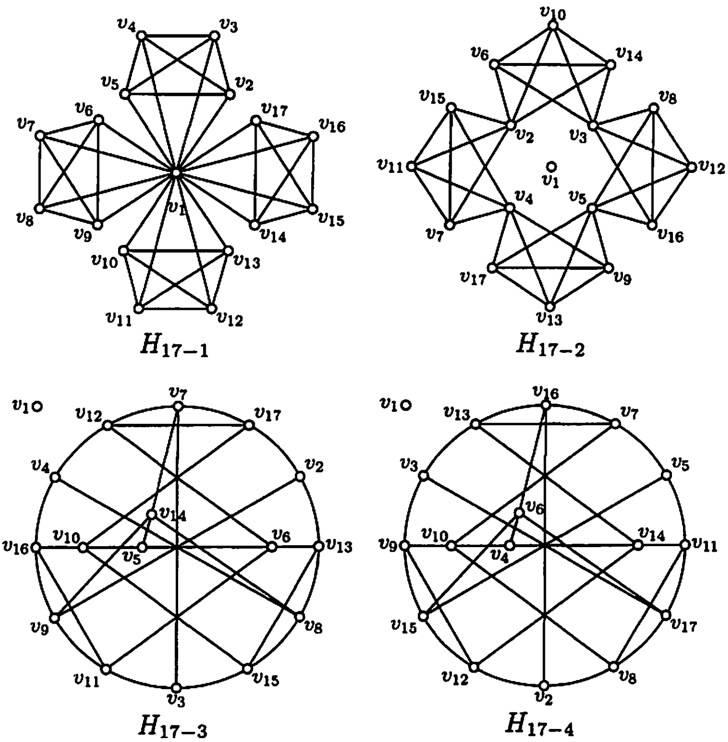

Fig. 3. The graphs H_{17-i} for $1 \leq i \leq 4$.

in Fig. 4. Similarly, we have $C_6 \not\subseteq H_{17-1}$ and $C_6 \not\subseteq H_{17-2}$. In addition, $C_6 \not\subseteq H_{17-5}$ and H_{17-6} is a graph without containing C_4. Hence, we have K_{17} is 4-colorable to (C_6, C_6, C_6, C_4), that is, $R(C_6, C_6, C_6, C_4) \geq 18$. □

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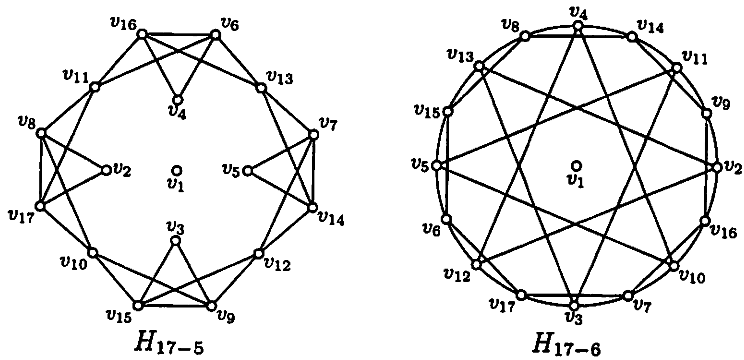

Fig. 4. The graphs H_{17-5} and H_{17-6}.

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