GAUSSIAN JACOBSTHAL AND GAUSSIAN JACOBSTHAL LUCAS NUMBERS

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ABSTRACT. In this study we define and study the Gaussian Jacobsthal and Gaussian Jacobsthal Lucas numbers. We give generating functions, Binet formulas, explicit formulas and Q matrix of these numbers. We also present explicit combinatorial and determinantal expressions, study negatively subscripted numbers and give various identities. Similar to the Jacobsthal and Jacobsthal Lucas numbers we give some interesting results for the Gaussian Jacobsthal and Gaussian Jacobsthal Lucas numbers.

1. INTRODUCTION

Horadam [1] defined the Jacobsthal and the Jacobsthal Lucas sequences J_n and j_n by the following recurrence relations

$$J_n = J_{n-1} + 2J_{n-2} \text{ for } n \geq 2$$

where $J_0 = 0$ and $J_1 = 1$, and

$$j_n = j_{n-1} + 2j_{n-2} \text{ for } n \geq 2$$

where $J_0 = 2$ and $J_1 = 1$ respectively.

The Gaussian Fibonacci sequence in [11] is $GF_0 = i$, $GF_1 = 1$ and $GF_n = GF_{n-1} + GF_{n-2}$ for $n > 1$. One can see that

$$GF_n = F_n + iF_{n-1}$$

where F_n is the usual nth Fibonacci number.

The Gaussian Lucas sequence in [11] is defined similar to Gaussian Fibonacci sequence as $GL_0 = 2-i$, $GL_1 = 1+2i$, and $GL_n = GL_{n-1}+GL_{n-2}$ for $n > 1$. Also it can be seen that

$$GL_n = L_n + iL_{n-1}$$

where L_n is the usual nth Lucas number.

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The complex Fibonacci numbers and Gaussian Fibonacci numbers are studied by some other authors [5, 6, 7, 8, 13]. The complex Fibonacci polynomials were defined and studied in [10] by Horadam. Harman [6] give a new approach toward the extension of Fibonacci numbers into the complex plane. Before this study there were two different methods for defining such numbers studied by Horadam [9] and Berzsenyi [3]. Harman [6] generalized both of the methods. Good [4] points out that the square root of the Golden Ratio is the real part of a simple periodic continued fraction but using (complex) Gaussian integers $a + ib$ instead of the natural integers. The authors in [2] defined the Bivariate Gaussian Fibonacci and Bivariate Gaussian Lucas Polynomials $GF_n(x, y)$ and $GL_n(x, y)$. They give generating function, Binet formula, explicit formula and partial derivation of these polynomials. Special cases of these bivariate polynomials are Gaussian Fibonacci polynomials $F_n(x, 1)$, Gaussian Lucas polynomials $L_n(x, 1)$, Gaussian Fibonacci numbers $F_n(1, 1)$ and Gaussian Lucas numbers $L_n(1, 1)$ defined in [11].

In this study we define and study the Gaussian Jacobsthal and Gaussian Jacobsthal Lucas numbers. We give generating functions, Binet formulas, explicit formulas and Q matrix of these numbers. We also present explicit combinatorial and determinantal expressions, study negatively subscripted numbers and give various identities. Similar to the Jacobsthal and Jacobsthal Lucas numbers we give some interesting results for the Gaussian Jacobsthal and Gaussian Jacobsthal Lucas numbers.

2. The Gaussian Jacobsthal and Gaussian Jacobsthal Lucas Numbers

Definition 1. *The Gaussian Jacobsthal numbers $\{GJ_n\}_{n=0}^{\infty}$ are defined by the following recurrence relation*

$$GJ_{n+1} = GJ_n + 2GJ_{n-1}, \quad n \geq 1 \tag{2.1}$$

with initial conditions $GJ_0 = \frac{i}{2}$ and $GJ_1 = 1$.

It can be easily seen that

$$GJ_n = J_n + iJ_{n-1}$$

where J_n is the n'th Jacobsthal number.

Definition 2. *The Gaussian Jacobsthal Lucas numbers $\{Gj_n\}_{n=0}^{\infty}$ are defined by the following recurrence relation*

$$Gj_{n+1} = Gj_n + 2Gj_{n-1} \quad n \geq 1 \tag{2.2}$$

with initial conditions $Gj_0 = 2 - \frac{i}{2}$ and $Gj_1 = 1 + 2i$.

Also

$$Gj_n = j_n + ij_{n-1}$$

where j_n is the nth Jacobsthal Lucas number.

For later use the first few terms of the sequences are as shown in the following tables

n	GJ_n
0	$\frac{i}{2}$
1	1
2	$1+i$
3	$3+i$
4	$5+3i$
5	$11+5i$
$\vdots$	$\vdots$

and

n	Gj_n
0	$2-\frac{i}{2}$
1	$1+2i$
2	$5+i$
3	$7+5i$
4	$17+7i$
5	$31+17i$
$\vdots$	$\vdots$

3. Some Properties of The Gaussian Jacobsthal and Gaussian Jacobsthal Lucas Numbers

Theorem 1. *The generating function for Gaussian Jacobsthal numbers is*

$$g(t) = \sum_{n=0}^{\infty} GJ_n t^n = \frac{2t + i(1-t)}{2 - 2t - 4t^2}$$

and for Gaussian Jacobsthal Lucas numbers is

$$h(t) = \sum_{n=0}^{\infty} Gj_n t^n = \frac{4 - 2t + i(-1+5t)}{2 - 2t - 4t^2}.$$

Proof. Let $g(t)$ be the generating function of the Gaussian Jacobsthal sequence GJ_n then

$$\begin{aligned} g(t) - tg(t) - 2t^2 g(t) &= GJ_0 + t\,(GJ_1 - GJ_0) \\ &\quad + \sum_{n=2}^{\infty} t^n \,(GJ_n - GJ_{n-1} - 2GJ_{n-2}) \\ &= t + i\left(\frac{1}{2} - \frac{t}{2}\right). \end{aligned}$$

By taking $g(t)$ parenthesis we get

$$g(t) = \frac{2t + i\,(1-t)}{2 - 2t - 4t^2}.$$

The proof is completed. □

Now we can get the Binet formula of the Gaussian Jacobsthal numbers and the Gaussian Jacobsthal Lucas numbers.

Let α and β be the roots of the characteristic equation

$$t^2 - t - 2 = 0$$

of the recurrence relation (2.1). Then

$$\alpha = 2,\ \beta = -1.$$

Note that $\alpha + \beta = 1$ and $\alpha\beta = -2$. Now we can give the Binet formula for the Gaussian Jacobsthal numbers and the Gaussian Jacobsthal Lucas numbers.

Theorem 2.

$$GJ_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} + i\frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta}$$

and

$$Gj_n = \alpha^n + \beta^n + i\left(\alpha^{n-1} + \beta^{n-1}\right).$$

Proof. Theorem can be proved by mathematical induction on n. □

Theorem 3. *The explicit formula of Gaussian Jacobsthal numbers is*

$$\begin{aligned} GJ_n &= \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-k-1}{k} 2^k \\ &\quad + i \sum_{k=0}^{\lfloor \frac{n-2}{2} \rfloor} \binom{n-k-2}{k} 2^k. \end{aligned}$$

Proof. By mathematical induction on n. □

Theorem 4. *The explicit formula of Gaussian Jacobsthal Lucas numbers is*

$$\begin{aligned} Gj_n &= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{n-k}\binom{n-k}{k}2^k \\ &\quad +i\sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{n-1}{n-k-1}\binom{n-k-1}{k}2^k. \end{aligned}$$

Proof. By mathematical induction on n. □

Theorem 5. *Let D_n denote the $n \times n$ tridiagonal matrix as*

$$D_n = \begin{bmatrix} 1 & i & 0 & \cdots & 0 \\ -1 & 1 & 2 & \ddots & \vdots \\ 0 & -1 & 1 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 2 \\ 0 & \cdots & 0 & -1 & 1 \end{bmatrix}, n \geq 1$$

and let $D_0 = \frac{i}{2}$. Then

$$\det D_n = GJ_n, n \geq 0.$$

Proof. By induction on n we can prove the theorem. For $n = 1$ and $n = 2$,

$$\begin{aligned} \det D_1 &= 1 = GJ_1 \\ \det D_2 &= 1 + i = GJ_2. \end{aligned}$$

Assume that

$$\det D_{n-1} = GJ_{n-1}$$

and

$$\det D_{n-2} = GJ_{n-2}.$$

Then

$$\begin{aligned} \det D_n &= \det D_{n-1} + 2 \det D_{n-2} \\ &= GJ_{n-1} + 2GJ_{n-2} \\ &= GJ_n. \end{aligned}$$

□

Theorem 6. *Let H_n denote the $n \times n$ tridiagonal matrix defined as*

$$H_n = \begin{bmatrix} 2-\frac{i}{2} & -\frac{1}{2} & 0 & \cdots & 0 \\ 1 & i & 2 & \ddots & \vdots \\ 0 & -1 & 1 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 2 \\ 0 & \cdots & 0 & -1 & 1 \end{bmatrix}, \ n \geq 1.$$

Then

$$\det H_n = Gj_{n-1}, n \geq 0.$$

Now we introduce the matrix Q and P that plays the role of the Q-matrix in Fibonacci numbers theory. Let Q and P denote the 2×2 matrices defined as

$$Q = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} \text{ and } P = \begin{bmatrix} 1+i & 1 \\ 1 & \frac{i}{2} \end{bmatrix}.$$

Then we can give the following theorem:

Theorem 7. *Let $n \geq 1$. Then*

$$Q^n P = \begin{bmatrix} GJ_{n+2} & GJ_{n+1} \\ GJ_{n+1} & GJ_n \end{bmatrix}$$

where GJ_n is the nth Gaussian Jacobsthal number.

Proof. We can prove the theorem by induction on n. For $n = 1$

$$\begin{aligned} \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1+i & 1 \\ 1 & \frac{i}{2} \end{bmatrix} &= \begin{bmatrix} 3+i & 1+i \\ 1+i & 1 \end{bmatrix} \\ &= \begin{bmatrix} GJ_3 & GJ_2 \\ GJ_2 & GJ_1 \end{bmatrix}. \end{aligned}$$

Assume that the theorem holds for $n = k$, that is

$$\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}^k \begin{bmatrix} 1+i & 1 \\ 1 & \frac{i}{2} \end{bmatrix} = \begin{bmatrix} GJ_{k+2} & GJ_{k+1} \\ GJ_{k+1} & GJ_k \end{bmatrix}.$$

Then for $n = k+1$ we have

$$\begin{aligned} \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}^{k+1} \begin{bmatrix} 1+i & 1 \\ 1 & \frac{i}{2} \end{bmatrix} &= \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}^k \begin{bmatrix} 1+i & 1 \\ 1 & \frac{i}{2} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} GJ_{k+2} & GJ_{k+1} \\ GJ_{k+1} & GJ_k \end{bmatrix} \\ &= \begin{bmatrix} GJ_{k+3} & GJ_{k+2} \\ GJ_{k+2} & GJ_{k+1} \end{bmatrix}. \end{aligned}$$

□

We can extend the definition of Gaussian Jacobsthal numbers and Gaussian Jacobsthal Lucas numbers to negative subscripts. We can prove the following theorem by induction on n.

Theorem 8. *For $n \geq 1$*

$$\begin{aligned} GJ_{-n} &= J_{-n} + iJ_{-n-1} \\ &= (-1)^{n-1}(J_n - J_{n+1}) \end{aligned}$$

and

$$\begin{aligned} Gj_{-n} &= j_{-n} + ij_{-n-1} \\ &= (-1)^{n-1}(j_n - ij_{n+1}) \end{aligned}$$

where J_n and j_n are the Jacobsthal and Jacobsthal Lucas numbers defined above.

Theorem 9. *(Cassini Identity) For $n \geq 1$*

$$GJ_{n-1}GJ_{n+1} - GJ_n^2 = (-1)^n (3-i)\, 2^{n-2}.$$

Proof. We can prove the theorem by induction on n.

For $n = 1$

$$\begin{aligned} GJ_0GJ_2 - GJ_1^2 &= \frac{i}{2}(1+i) - (1)^2 \\ &= -\frac{3}{2} + \frac{1}{2}i \\ &= -\frac{1}{2}(3-i) \\ &= (-1)^1 2^{-1}(3-i) \end{aligned}$$

and thus the theorem holds. Suppose that the theorem is true for an arbitrary positive integer k, that is

$$GJ_{k-1}GJ_{k+1} - GJ_k^2 = (-1)^k (3-i)\, 2^{k-2}.$$

Then for $k+1$

$$\begin{aligned} & GJ_kGJ_{k+2} - GJ_{k+1}^2 \\ =\ & (GJ_{k+1} - 2GJ_{k-1})(GJ_{k+1} + 2GJ_k) - GJ_{k+1}^2 \\ =\ & GJ_{k+1}^2 + 2GJ_{k+1}GJ_k - 2GJ_{k-1}GJ_{k+1} - 4GJ_{k-1}GJ_k - GJ_{k+1}^2 \\ =\ & 2GJ_{k+1}GJ_k - 2GJ_{k-1}GJ_{k+1} - 4GJ_{k-1}GJ_k \\ =\ & 2GJ_{k+1}GJ_k - 4GJ_{k-1}GJ_k + (-1)^{k+1}2(3-i)\,2^{k-2} - 2GJ_k^2 \\ =\ & 2GJ_{k+1}GJ_k - 4GJ_{k-1}GJ_k + (-1)^{k+1}(3-i)\,2^{k-1} - 2GJ_k^2 \\ =\ & 2GJ_k(GJ_{k+1} - GJ_k - 2GJ_{k-1}) + (-1)^{k+1}(3-i)\,2^{k-1} \\ =\ & (-1)^{k+1}(3-i)\,2^{k-1} \end{aligned}$$

This completes the proof. □

Theorem 10. *For* $n \geq 1$

$$Gj_{n-2}Gj_{n+2} - Gj_n^2 = (-1)^n 9\,(3-i)\,.$$

Proof. Theorem can be carried out by induction on n. □

Theorem 11. *For* $n \geq 1$

$$Gj_n^2 - 9GJ_n^2 = (3-i)\,(-1)^n\,2^{n+1}$$

Proof. Theorem can be proved by mathematical induction on n. □

Theorem 12. *For* $n \geq 1$

$$Gj_n = GJ_{n+1} + 2GJ_{n-1}.$$

Proof.

$$\begin{aligned} Gj_n &= Gj_{n-1} + 2Gj_{n-2} \\ &= GJ_n + 2GJ_{n-2} \\ &\quad +2(GJ_{n-1} + 2GJ_{n-3}) \\ &= GJ_n + 2GJ_{n-2} \\ &\quad 2GJ_{n-1} + 4GJ_{n-3} \\ &= GJ_n + 2GJ_{n-1} \\ &\quad +2(GJ_{n-2} + 2GJ_{n-1}) \\ &= GJ_{n+1} + 2GJ_{n-1} \end{aligned}$$

□

Theorem 13. *The sum of the Gaussian Jacobsthal and the Gaussian Jacobsthal Lucas numbers are given as:*

(i) $\sum_{k=0}^{n} GJ_k = \frac{1}{2}\left[GJ_{n+2} - 1)\right]$

(ii) $\sum_{k=0}^{n} Gj_k = \frac{1}{2}\left[Gj_{n+2} - (1+2i)\right]$

Proof. (i) For $n \geq 2$ we have

$$GJ_{n-1} = \frac{1}{2}GJ_{n+1} - \frac{1}{2}GJ_n.$$

Then from this equation

$$\begin{aligned}
GJ_0 &= \frac{1}{2}GJ_2 - \frac{1}{2}GJ_1 \\
GJ_1 &= \frac{1}{2}GJ_3 - \frac{1}{2}GJ_2 \\
GJ_2 &= \frac{1}{2}GJ_4 - \frac{1}{2}GJ_3 \\
&\vdots \\
GJ_{n-1} &= \frac{1}{2}GJ_{n+1} - \frac{1}{2}GJ_n \\
GJ_n &= \frac{1}{2}GJ_{n+2} - \frac{1}{2}GJ_{n+1}.
\end{aligned}$$

By adding both sides of the equations we get

$$\sum_{k=0}^{n} GJ_k = \frac{1}{2}\left[GJ_{n+2} - 1\right].$$

This completes the proof. □

Theorem 14. *For $m \geq 0$ and $n \geq 0$*

$$GJ_m Gj_n + GJ_n Gj_m = 2\left(J_{n+m-2} + GJ_{n+m} + iJ_{n+m-1}\right)$$

Proof. By the Binet formulas of the numbers we have

$$\begin{aligned}
&\left(\frac{\alpha^m - \beta^m}{\alpha - \beta} + i\frac{\alpha^{m-1} - \beta^{m-1}}{\alpha - \beta}\right)\left(\alpha^n + \beta^n + i\left(\alpha^{n-1} + \beta^{n-1}\right)\right) \\
&+ \left(\frac{\alpha^n - \beta^n}{\alpha - \beta} + i\frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta}\right)\left(\alpha^m + \beta^m + i\left(\alpha^{m-1} + \beta^{m-1}\right)\right) \\
=\ & \frac{2}{\alpha^2\beta^2(\alpha - \beta)}\left(\alpha^2\beta^{m+n} - \alpha^{m+n}\beta^2 - 2i\alpha^2\beta^{m+n+1}\right. \\
&\left. + 2i\alpha^{m+n+1}\beta^2 - \alpha^2\beta^{m+n+2} + \alpha^{m+n+2}\beta^2\right) \\
=\ & \frac{2}{(\alpha - \beta)}\left(\beta^{m+n-2} - \alpha^{m+n-2} - 2i\beta^{m+n-1}\right. \\
&\left. + 2i\alpha^{m+n-1} - \beta^{m+n} + \alpha^{m+n}\right) \\
=\ & -2\left(-J_{n+m-2} - J_{n+m} - iJ_{n+m-1} - iJ_{n+m-1}\right) \\
=\ & 2\left(J_{n+m-2} + GJ_{n+m} + iJ_{n+m-1}\right)
\end{aligned}$$

□

Similar to the Jacobsthal and Jacobsthal Lucas numbers we can give the following interesting results for the Gaussian Jacobsthal and Gaussian Jacobsthal Lucas numbers. These theorems can be carried out by induction on n.

Theorem 15. *For* $n \geq 0$

$$GJ_nGj_n = GJ_{2n} + iJ_{2n-1} - J_{2n-2}$$

where J_n *is the nth Jacobsthal number.*

Theorem 16. *For* $n \geq 0$

$$Gj_{n+1} + 2Gj_{n-1} = 9GJ_n$$

Theorem 17. *For* $n \geq 1$

$$\begin{aligned} Gj_{n+1} + Gj_n &= 3\,(GJ_{n+1} + GJ_n) \\ &= 3\,(2+i)\,2^{n-1} \end{aligned}$$

Theorem 18. *For* $n \geq 0$

$$3GJ_n + Gj_n = (2+i)\,2^n$$

Theorem 19. *For* $n \geq 0$

$$GJ_n + Gj_n = 2GJ_{n+1}$$

Theorem 20. *For* $m \geq 0$ *and* $n \geq 0$

$$Gj_mGj_n + 9GJ_nGJ_m = 4\left(3.2^{m-1} + ij_{n+m-1}\right)$$

where j_n *is the nth Jacobsthal Lucas number.*

Theorem 21. *For* $n \geq 1$

$$Gj_n^2 + 9GJ_n^2 = 4\left(3.2^{n-1} + ij_{2n-1}\right)$$

where j_n *is the nth Jacobsthal Lucas number.*

Theorem 22. *For* $m \geq 0$ *and* $n \geq 0$

$$GJ_mGj_n - GJ_nGj_m = (-1)^{m-1}\,(3-i)\,2^nJ_{m-n}$$

where J_n *is the nth Jacobsthal number.*

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