The Diameters of Almost All Bi-Cayley Graphs *

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Abstract

Let G be a finite group of order n and S (possibly, contains the identity element) be a subset of G. The Bi-Cayley graph $BC(G,S)$ of G is a bipartite graph with vertex set $G \times \{0,1\}$ and edge set $\{\{(g,0),(gs,1)\}|g \in G, s \in S\}$. Let p $(0 < p < 1)$ be a fixed number. We define $B = \{X = BC(G,S), S \subseteq G\}$ as a sample space and assign a probability measure by requiring $P_r(X) = p^k q^{n-k}$, for $X = BC(G,S)$ with $|S| = k$. Here it is shown that the probability of the set of Bi-Cayley graph of G with diameter 3 approaches 1 as the order n of G approaches infinity.

Keywords: Bi-Cayley graph; Random; Diameter.

1 Introduction

Let G be a finite group and S be a subset of $G \setminus \{1\}$, then we can define a directed graph $D(G,S)$ with vertex set G and arc set $\{(g,h) : g^{-1}h \in S\}$. If $S = S^{-1}$, then $D(G,S)$ corresponds to an undirected graph which we call a Cayley graph and is denoted by $C(G,S)$. To study semi-symmetric graphs, Xu defined the Bi-Cayley graph in [6]. For a finite group G and a subset S (possibly, contains the identity element) of G, the Bi-Cayley graph $X = BC(G,S)$ of G with respect to S is defined as the bipartite graph with vertex set $G \times \{0,1\}$ and edge set $\{\{(g,0),(gs,1)\}|g \in G, s \in S\}$.

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It is known that almost all graphs have diameter 2 [1] and almost all bipartite graphs have diameter 3 [7]. For special class of graphs, it was proved that almost all Cayley digraphs have diameter 2 in [3], and this was extended to Cayley graphs in [2]. Here, we will establish a model of random Bi-Cayley graphs and prove that almost all Bi-Cayley graphs have diameter 3.

Throughout this paper we assume that G is a finite group with order n, p $(0 < p < 1)$ is a fixed number and $q = 1 - p$. We consider labelled graphs.

To define the random Bi-Cayley graphs of G, we illustrate the so-called random subset of a set T. When we say that S is a random subset of T, we mean that $P_r(a \in S) = p$ and the events $a_i \in S (i = 1, 2, \cdots, |T|)$ are mutually independent. In particular, a random Bi-Cayley graph of a group G is a Bi-Cayley graph associated with a random subset S (possibly, contains the identity element) of G. Now we can introduce our model. Let $\mathcal{P}(G, p)$ be the probability space $(B, 2^B, P)$ where $B = \{X = BC(G, S) | S \subseteq G\}, 2^B$ is the power set of B, and P is probability measure (with respect to p) on B, $P_r(X) = p^k q^{n-k}$, for $X = BC(G, S)$ with $|S| = k$.

Remark 1.1. *If $S = G$, then $X = BC(G, S)$ is a complete Bi-Cayley graph, and it is the only Bi-Cayley graph with diameter 2 in all Bi-Cayley graphs of G.*

Remark 1.2. *If $p = q = \frac{1}{2}$, then each Bi-Cayley graph of G is assigned the same probability $\frac{1}{2^n}$, and so the probability of an event in this case is the ratio of the number of Bi-Cayley graphs contained in this event to the number $|B|$.*

Let Q be a graph property. We say that almost all Bi-Cayley graphs have property Q if for any group of order n, the probability of the event that a bi-Cayley graph of G has the property Q approaches 1 as n approaches infinity.

2 Main Results

Let $X = BC(G, S)$. We call S the symbol set of $BC(G, S)$. When we talk about events and probability, we mean events and probability in the space $\mathcal{P}(G, p)$ defined in the previous section.

The following theorem is one of our main results.

Theorem 2.1. *Almost all Bi-Cayley graphs have diameter 3.*

To prove the above theorem, we establish a sequence of Lemmas.

Lemma 2.1. *Let $X = BC(G, S)$. Then $Aut[X]$ acts transitively on the sets of its bipartition of X, thus $Aut[X]$ has at most two orbits on the vertex set of X.*

Proof. See [6] for details. □

Recall that the diameter of a graph X, denoted by $diam(X)$, is the maximum distance $d(x, y)$ between any two vertices of X. That is

$$diam(X) = max\{d(x, y)|x, y \in V(X)\}.$$

Clearly, a Bi-Cayley graph has diameter k if every pair of its vertices are connected by a path of length not greater than k and at least one pair is not connected by a shorter path. The necessary and sufficient condition for any Bi-Cayley graph to have diameter 3 is that every pair of vertices in the same part of its bipartition should be joined by a 2-path. Because, if this is not true for some pair of vertices in the same part of its bipartition, the diameter is at least 4. Next, suppose that every pair of vertices in the same part of its bipartition is joined by a 2-path and consider the pair of vertices $(x, 0), (y, 1)$. If the edge $\{(x, 0), (y, 1)\}$ is present, a 1-path suffices, (only in the complete graph is this true for every pair $(x, 0), (y, 1)$). If the edge $\{(x, 0), (y, 1)\}$ is absent, some edge $\{(x, 0), (z, 1)\}$ must be present, since $(x, 0)$ cannot be isolated. But $(z, 1)$ and $(y, 1)$ are joined by a 2-path and hence $(x, 0)$ and $(y, 1)$ are joined by a 3-path. For Bi-Cayley graph $X = BC(G, S)$, it is to see that the left multiplication $L_a : (g, i) \to (ag, i), g \in G, (i = 0, 1)$ for any element $a \in G$, is clearly an automorphism of $BC(G, S)$. All these left multiplications constitute a group L_G which acts transitively on $G \times \{0\}$ and $G \times \{1\}$, respectively.

We then have the following:

Lemma 2.2. *Let $X = BC(G, S)$. Then $diam(X) = 3$ if and only if for every $y \in G$, $(1, i)$ and (y, i) $(i = 0, 1)$ are joined by a 2-path.*

Now we consider a set of events. For $g \in G$, we use $E(g)$ to denote the event that g is contained in the symbol set S of a Bi-Cayley graph $BC(G, S)$ of G. Then by our assumption, $P_r(E(g)) = p$ for any $g \in G$. For distinct elements $g_i (0 \leq i \leq n)$, the events $E(g_1), E(g_2), \cdots, E(g_n)$ are mutually independent.

The following events will play important roles in discussions below.

E_n: *diam*(X) is at most 3.

$E_n(y)$: $(1,0)$ and $(y,0)$ are joined by a 2-path and $(1,1)$ and $(y,1)$ are also joined by a 2-path in a Bi-Cayley graph of G.

$E(s_1, s_2, y)$: $s_1, s_2, y^{-1}s_1$, and $s_2 y$ are contained in the symbol set of a Bi-Cayley graph of G.

$E(s_2, s_1, y)$: $s_1, s_2, y^{-1}s_2$, and $s_1 y$ are contained in the symbol set of a Bi-Cayley graph of G.

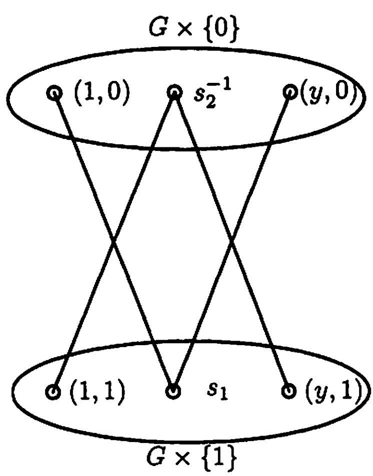

Fig.1

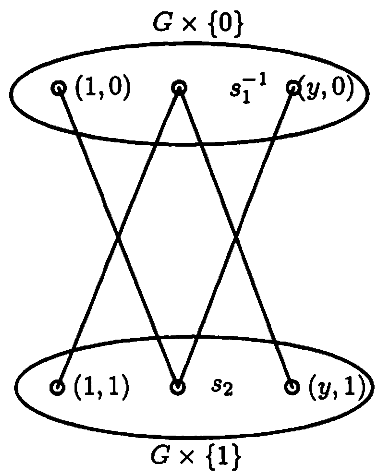

Fig.2

Lemma 2.3. *The following statements hold:*

(a) $E_n = \cap_{y \in G} E_n(y)$ *or equivalently,* $\overline{E_n} = \bigcup_{y \in G} \overline{E_n(y)}$.

(b) $\overline{E_n(y)} \subseteq \bigcap_{\{s_1, s_2\} \subseteq G} \{\overline{E(s_1, s_2, y)} \bigcap \overline{E(s_2, s_1, y)}\}$.

Proof. (a) follows from Lemma 2.1. To prove (b), it suffices to note that if the event $E_n(y)$ occurs, then some s_1, s_2, and $y^{-1}s_1$, $s_2 y$, or s_1, s_2, and $y^{-1}s_2$, $s_1 y$ are contained in the symbol set(as *Fig*.1 and *Fig*.2 show), and so there is a 2-path between the pair of vertices in the same part of its bipartition of $BC(G,S)$, clearly, $E_n(y) \supseteq \bigcup_{\{s_1,s_2\}\subseteq G}\{E(s_1,s_2,y)\bigcup E(s_2,s_1,y)\}$. This completes the proof. □

Now we estimate the probability of the event $\overline{E(s_1,s_2,y)}\bigcap\overline{E(s_2,s_1,y)}$.

Lemma 2.4. *The probability of the event $\overline{E(s_1,s_2,y)}\bigcap\overline{E(s_2,s_1,y)}$ has three cases:*
(a) If $E(s_1,s_2,y)$ and $E(s_2,s_1,y)$ occur:

$$P_r(\overline{E(s_1,s_2,y)}\bigcap\overline{E(s_2,s_1,y)})$$
$$=\begin{cases} 1-(2p^4-p^6) & \text{if } s_1,s_2,y^{-1}s_1,s_2y,y^{-1}s_2 \text{ and } s_1y \text{ are} \\ & \text{mutually different.} \\ 1-(p^3+p^4-p^5) & \text{if } s_1=s_2y \text{ and } s_2\neq y^{-1}s_1 \text{ or } s_2=y^{-1}s_1 \\ & \text{and } s_1\neq s_2y \text{ and } s_1,s_2,y^{-1}s_2,s_1y \text{ are} \\ & \text{mutually different.} \\ 1-(p^3+p^4-p^5) & \text{if } s_1=y^{-1}s_2 \text{ and } s_2\neq s_1y \text{ or } s_2=s_1y \\ & \text{and } s_1\neq y^{-1}s_2 \text{ and } s_1,s_2,y^{-1}s_1,s_2y \text{ are} \\ & \text{mutually different.} \\ 1-(2p^3-p^4) & \text{if } s_1\neq s_2;\ s_1=s_2y, s_2\neq y^{-1}s_1 \text{ or} \\ & s_2=y^{-1}s_1, s_1\neq s_2y \text{ and } s_1=y^{-1}s_2, \\ & s_2\neq s_1y \text{ or } s_2=s_1y,\ s_1\neq y^{-1}s_2. \\ 1-p^2 & \text{if } s_1\neq s_2; s_1=s_2y \text{ and } s_2=y^{-1}s_1 \\ & \text{and } s_1=y^{-1}s_2 \text{ and } s_2=s_1y. \\ 1-(p^3+p^2-p^4) & \text{if } s_1=s_2,\ y^{-1}s_1=s_2y \text{ and } y^{-1}s_2\neq s_1y \\ & \text{or } s_1=s_2,\ y^{-1}s_2=s_1y \text{ and } y^{-1}s_1\neq s_2y. \\ 1-(2p^2-p^3) & \text{if } s_1=s_2,\ y^{-1}s_1=s_2y \text{ and } y^{-1}s_2=s_1y. \end{cases}$$

(b)If $E(s_1,s_2,y)$ occurs and $E(s_2,s_1,y)$ does not occur:

$$P_r(\overline{E(s_1,s_2,y)}\bigcap\overline{E(s_2,s_1,y)})$$
$$=\begin{cases} 1-p^4 & \text{if } s_1,s_2,y^{-1}s_1 \text{ and } s_2y \text{ are mutually different.} \\ 1-p^3 & \text{if } s_1\neq s_2 \text{ and } s_1=s_2y \text{ and } s_2\neq y^{-1}s_1 \\ & \text{or } s_2=y^{-1}s_1 \text{ and } s_1\neq s_2y \text{ or } s_2y=y^{-1}s_1. \\ 1-p^2 & \text{if } s_1\neq s_2 \text{ and } s_1=s_2y \text{ and } s_2=y^{-1}s_1. \\ 1-p^3 & \text{if } s_1=s_2,y^{-1}s_1\neq s_2y. \\ 1-p^2 & \text{if } s_1=s_2,y^{-1}s_1=s_2y. \end{cases}$$

(c)If $E(s_2,s_1,y)$ occurs and $E(s_1,s_2,y)$ does not occur:

$$P_r(\overline{E(s_2,s_1,y)}\bigcap\overline{E(s_1,s_2,y)})$$

$$=\begin{cases} 1-p^4 & \textit{if } s_1,s_2,y^{-1}s_2 \textit{ and } s_1y \textit{ are mutually different.}\\ 1-p^3 & \textit{if } s_1\neq s_2 \textit{ or } s_2=s_1y \textit{ and } s_1\neq y^{-1}s_2\\ & \textit{or } s_1=y^{-1}s_2 \textit{ and } s_2\neq s_1y \textit{ or } s_1y=y^{-1}s_2.\\ 1-p^2 & \textit{if } s_1\neq s_2 \textit{ and } s_2=s_1y \textit{ and } s_1=y^{-1}s_2.\\ 1-p^3 & \textit{if } s_1=s_2, y^{-1}s_2\neq s_1y.\\ 1-p^2 & \textit{if } s_1=s_2, y^{-1}s_1=s_2y.\end{cases}$$

Proof. We have the following cases to prove it.

Case1: Assume $E(s_1,s_2,y)$ and $E(s_2,s_1,y)$ occur:

Case1.1. If $s_1,s_2,y^{-1}s_1,s_2y,y^{-1}s_2$, and s_1y are mutually different, then $P_r(\{E(s_1)\cap E(s_2)\cap E(y^{-1}s_1)\cap E(s_2y)\}\bigcup\{E(s_1)\cap E(s_2)\cap E(y^{-1}s_2)\cap E(s_1y)\}) = P_r(\{E(s_1)\cap E(s_2)\cap E(y^{-1}s_1)\cap E(s_2y)\})+P_r(\{E(s_1)\cap E(s_2)\cap E(y^{-1}s_2)\cap E(s_1y)\})-P_r(\{E(s_1)\cap E(s_2)\cap E(y^{-1}s_1)\cap E(s_2y)\cap E(y^{-1}s_2)\cap E(s_1y)\}=2p^4-p^6$.

Case1.2. If $s_1=s_2y$ and $s_2\neq y^{-1}s_1$ or $s_2=y^{-1}s_1$ and $s_1\neq s_2y$ and $s_1,s_2,y^{-1}s_2,s_1y$ are mutually different, then $P_r(\{E(s_1)\cap E(s_2)\cap E(y^{-1}s_1)\cap E(s_2y)\}\bigcup\{E(s_1)\cap E(s_2)\cap E(y^{-1}s_2)\cap E(s_1y)\}) = P_r(\{E(s_1)\cap E(s_2)\cap E(y^{-1}s_1)\cap E(s_2y)\})+P_r(\{E(s_1)\cap E(s_2)\cap E(y^{-1}s_2)\cap E(s_1y)\})-P_r(\{E(s_1)\cap E(s_2)\cap E(y^{-1}s_1)\cap E(s_2y)\cap E(y^{-1}s_2)\cap E(s_1y)\}=p^3+p^4-p^5$.

Case1.3. If $s_1=y^{-1}s_2$ and $s_2\neq s_1y$ or $s_2=s_1y$ and $s_1\neq y^{-1}s_2$and $s_1,s_2,y^{-1}s_1,s_2y$ are mutually different, then $P_r(\{E(s_1)\cap E(s_2)\cap E(y^{-1}s_1)\cap E(s_2y)\}\bigcup\{E(s_1)\cap E(s_2)\cap E(y^{-1}s_2)\cap E(s_1y)\}) = P_r(\{E(s_1)\cap E(s_2)\cap E(y^{-1}s_1)\cap E(s_2y)\})+P_r(\{E(s_1)\cap E(s_2)\cap E(y^{-1}s_2)\cap E(s_1y)\})-P_r(\{E(s_1)\cap E(s_2)\cap E(y^{-1}s_1)\cap E(s_2y)\cap E(y^{-1}s_2)\cap E(s_1y)\}=p^3+p^4-p^5$.

Case1.4. If $s_1\neq s_2$;$s_1=s_2y$, $s_2\neq y^{-1}s_1$ or $s_2=y^{-1}s_1$,$s_1\neq s_2y$ and $s_1=y^{-1}s_2$,$s_2\neq s_1y$ or $s_2=s_1y$,$s_1\neq y^{-1}s_2$, then $P_r(\{E(s_1)\cap E(s_2)\cap E(y^{-1}s_1)\cap E(s_2y)\}\bigcup\{E(s_1)\cap E(s_2)\cap E(y^{-1}s_2)\cap E(s_1y)\}) = P_r(\{E(s_1)\cap E(s_2)\cap E(y^{-1}s_1)\cap E(s_2y)\})+P_r(\{E(s_1)\cap E(s_2)\cap E(y^{-1}s_2)\cap E(s_1y)\})-P_r(\{E(s_1)\cap E(s_2)\cap E(y^{-1}s_1)\cap E(s_2y)\cap E(y^{-1}s_2)\cap E(s_1y)\}=2p^3-p^4$.

Case1.5. If $s_1\neq s_2$; $s_1=s_2y$ and $s_2=y^{-1}s_1$ and $s_1=y^{-1}s_2$ and $s_2=s_1y$ or $s_2=s_1y$, then $P_r(\{E(s_1)\cap E(s_2)\cap E(y^{-1}s_1)\cap E(s_2y)\}\bigcup\{E(s_1)\cap E(s_2)\cap E(y^{-1}s_2)\cap E(s_1y)\}) = P_r(\{E(s_1)\cap E(s_2)\cap E(y^{-1}s_1)\cap E(s_2y)\})+P_r(\{E(s_1)\cap E(s_2)\cap E(y^{-1}s_2)\cap E(s_1y)\})-P_r(\{E(s_1)\cap E(s_2)\cap E(y^{-1}s_1)\cap E(s_2y)\cap E(y^{-1}s_2)\cap E(s_1y)\}=p^2$.

Case1.6. If $s_1 = s_2$, $y^{-1}s_1 = s_2y$ and $y^{-1}s_2 \neq s_1y$ or $s_1 = s_2$, $y^{-1}s_2 = s_1y$ and $y^{-1}s_1 \neq s_2y$, then $P_r(\{E(s_1) \cap E(s_2) \cap E(y^{-1}s_1) \cap E(s_2y)\} \bigcup \{E(s_1) \cap E(s_2) \cap E(y^{-1}s_2) \cap E(s_1y)\}) = P_r(\{E(s_1) \cap E(s_2) \cap E(y^{-1}s_1) \cap E(s_2y)\}) + P_r(\{E(s_1) \cap E(s_2) \cap E(y^{-1}s_2) \cap E(s_1y)\}) - P_r(\{E(s_1) \cap E(s_2) \cap E(y^{-1}s_1) \cap E(s_2y)\} \cap E(y^{-1}s_2) \cap E(s_1y)\} = p^3 + p^2 - p^4$.

Case1.7. If $s_1 = s_2$, $y^{-1}s_1 = s_2y$ and $y^{-1}s_2 = s_1y$, then $P_r(\{E(s_1) \cap E(s_2) \cap E(y^{-1}s_1) \cap E(s_2y)\} \bigcup \{E(s_1) \cap E(s_2) \cap E(y^{-1}s_2) \cap E(s_1y)\}) = P_r(\{E(s_1) \cap E(s_2) \cap E(y^{-1}s_1) \cap E(s_2y)\}) + P_r(\{E(s_1) \cap E(s_2) \cap E(y^{-1}s_2) \cap E(s_1y)\}) - P_r(\{E(s_1) \cap E(s_2) \cap E(y^{-1}s_1) \cap E(s_2y) \cap E(y^{-1}s_2) \cap E(s_1y)\} = 2p^2 - p^3$.

$2p^4 - p^6, p^3 + p^4 - p^5, 2p^3 - p^4, p^3 + p^2 - p^4$, and $2p^2 - p^3$ are strictly monotone increasing function of p on (0,1), thus they not more than 1 and not less than 0. Thus (a) is proved.

Case2: Assume $E(s_1, s_2, y)$ occurs and $E(s_2, s_1, y)$ does not occur:

Case2.1. If $s_1, s_2, y^{-1}s_1$ and s_2y are mutually different, then $P_r(\{E(s_1) \cap E(s_2) \cap E(y^{-1}s_1) \cap E(s_2y)\} \bigcup \{E(s_1) \cap E(s_2) \cap E(y^{-1}s_2) \cap E(s_1y)\}) = P_r(\{E(s_1) \cap E(s_2) \cap E(y^{-1}s_1) \cap E(s_2y)\}) = p^4$.

Case2.2. If $s_1 \neq s_2$ and $s_1 = s_2y$ and $s_2y \neq y^{-1}s_1$ or $s_2 = y^{-1}s_1$ and $s_1 \neq s_2y$ or $s_2y = y^{-1}s_1$, then $P_r(\{E(s_1) \cap E(s_2) \cap E(y^{-1}s_1) \cap E(s_2y)\} \bigcup \{E(s_1) \cap E(s_2) \cap E(y^{-1}s_2) \cap E(s_1y)\}) = P_r(\{E(s_1) \cap E(s_2) \cap E(y^{-1}s_1) \cap E(s_2y)\}) = p^3$.

Case2.3. If $s_1 \neq s_2$ and $s_1 = s_2y$ and $s_2 = y^{-1}s_1$, then $P_r(\{E(s_1) \cap E(s_2) \cap E(y^{-1}s_1) \cap E(s_2y)\} \bigcup \{E(s_1) \cap E(s_2) \cap E(y^{-1}s_2) \cap E(s_1y)\}) = P_r(\{E(s_1) \cap E(s_2) \cap E(y^{-1}s_1) \cap E(s_2y)\}) = p^2$.

Case2.4. If $s_1 = s_2, y^{-1}s_1 \neq s_2y$, then $P_r(\{E(s_1) \cap E(s_2) \cap E(y^{-1}s_1) \cap E(s_2y)\} \bigcup \{E(s_1) \cap E(s_2) \cap E(y^{-1}s_2) \cap E(s_1y)\}) = P_r(\{E(s_1) \cap E(s_2) \cap E(y^{-1}s_1) \cap E(s_2y)\}) = p^3$.

Case2.5. If $s_1 = s_2, y^{-1}s_1 = s_2y$, then $P_r(\{E(s_1) \cap E(s_2) \cap E(y^{-1}s_1) \cap E(s_2y)\} \bigcup \{E(s_1) \cap E(s_2) \cap E(y^{-1}s_2) \cap E(s_1y)\}) = P_r(\{E(s_1) \cap E(s_2) \cap E(y^{-1}s_1) \cap E(s_2y)\}) = p^2$. Thus (b) is proved.

Case3: Assume $E(s_2, s_1, y)$ occurs and $E(s_1, s_2, y)$ does not occur :

Case3.1. If $s_1, s_2, y^{-1}s_2$ and s_1y are mutually different, then $P_r(\{E(s_1) \cap E(s_2) \cap E(y^{-1}s_1) \cap E(s_2y)\} \bigcup \{E(s_1) \cap E(s_2) \cap E(y^{-1}s_2) \cap E(s_1y)\}) = P_r(\{E(s_1) \cap E(s_2) \cap E(y^{-1}s_2) \cap E(s_1y)\}) = p^4$.

Case3.2. If $s_1 \neq s_2$ and $s_2 = s_1y$ and $s_1 \neq y^{-1}s_2$ or $s_1 = y^{-1}s_2$ and $s_2 \neq s_1y$ or $s_1y = y^{-1}s_2$, then $P_r(\{E(s_1) \cap E(s_2) \cap E(y^{-1}s_1) \cap$

$E(s_2y)\}\bigcup\{E(s_1)\cap E(s_2)\cap E(y^{-1}s_2)\cap E(s_1y)\}) = P_r(\{E(s_1)\cap E(s_2)\cap E(y^{-1}s_2)\cap E(s_1y)\}) = p^3$.

Case3.3. If $s_1\neq s_2$ and $s_2=s_1y$ and $s_1=y^{-1}s_2$ and $s_1y=y^{-1}s_2$, then $P_r(\{E(s_1)\cap E(s_2)\cap E(y^{-1}s_1)\cap E(s_2y)\}\bigcup\{E(s_1)\cap E(s_2)\cap E(y^{-1}s_2)\cap E(s_1y)\}) = P_r(\{E(s_1)\cap E(s_2)\cap E(y^{-1}s_2)\cap E(s_1y)\}) = p^2$.

Case3.4. If $s_1=s_2, y^{-1}s_2\neq s_1y$, then $P_r(\{E(s_1)\cap E(s_2)\cap E(y^{-1}s_1)\cap E(s_2y)\}\bigcup\{E(s_1)\cap E(s_2)\cap E(y^{-1}s_2)\cap E(s_1y)\}) = P_r(\{E(s_1)\cap E(s_2)\cap E(y^{-1}s_2)\cap E(s_1y)\}) = p^3$.

Case3.5. If $s_1=s_2, y^{-1}s_2=s_1y$, then $P_r(\{E(s_1)\cap E(s_2)\cap E(y^{-1}s_1)\cap E(s_2y)\}\bigcup\{E(s_1)\cap E(s_2)\cap E(y^{-1}s_2)\cap E(s_1y)\}) = P_r(\{E(s_1)\cap E(s_2)\cap E(y^{-1}s_2)\cap E(s_1y)\}) = p^2$.

Thus (c) is proved. Since $y\neq 1$, above is sufficient. This completes the proof.

□

For $s_1, s_2\in G$, set $T(s_1,s_2)=\{s_1, y^{-1}s_1, s_2, s_2y, ys_1, ys_2, ys_2y, y^{-1}s_1y^{-1}, s_1y^{-1}, s_2y^{-1}, y^{-1}s_2, s_1y, ys_1y, y^{-1}s_2y^{-1}\}$. By a trivial check, we can deduce the following:

Lemma 2.5. *Let $s_1,s_2,s_3,s_4\in G$. If $s_3,s_4\notin T(s_1,s_2)$, then $\{s_1,s_2,y^{-1}s_1,s_2y, y^{-1}s_2,s_1y\}\cap\{s_3,s_4,y^{-1}s_3,s_4y,y^{-1}s_4,s_3y\}=\varnothing$.*

Lemma 2.6. *[4] If $\{E_{i1},E_{i2},\cdots,E_{ik},E_{j1},E_{j2},\cdots,E_{js},\cdots,E_{l1},E_{l2},\cdots,E_{lt}\}$ is an independent collection of events, then $\{\cap_{m=1}^{k}E_{im},\cap_{m=1}^{s}E_{jm},\cdots,\cap_{m=1}^{t}E_{lm}\}$ is also an independent collection of events.*

Now we estimate the probability of the event $\overline{E_n(y)}$.

Lemma 2.7. $P_r(\overline{E_n(y)})\leq(1-p^4)^{\lfloor\frac{(n+11)}{14}\rfloor}$.

Proof. Let s_1,s_2 be any elements in $C_0=G\setminus\{y\}$. Choose any elements s_3,s_4 in $C_1=C_0\setminus T(s_1,s_2)$. Generally, if $\{s_1,s_2\},\{s_3,s_4\},\cdots,\{s_i,s_{i+1}\}$ have been chosen, $C_i=C_{i-1}\setminus T(s_{2i-1},s_{2i})$, and $|C_i|\geq 2$, choose two distinct elements in C_i as s_{2i+1} and s_{2i+2}. According to the above rule, we can clearly choose at least $\lfloor\frac{(n-3)}{14}\rfloor+1$ such $\{s_i,s_{i+1}\}$ tuples. Let $\{s_1,s_2\},\{s_3,s_4\},\cdots,\{s_{2k+1},s_{2k+2}\}$ be chosen in the above way and $k\geq\lfloor\frac{(n+11)}{14}\rfloor$, and denote $I=\{i|1\leq i\leq 2k+1 \text{ and } i \text{ is an odd number}\}$.

By Lemma 2.5, we know that $\{E(s_i, s_{i+1}, y) \bigcup E(s_{i+1}, s_i, y); i \in I\}$ is a set of independent events. By Lemma 2.3,

$$\overline{E_n(y)} \subseteq \bigcap_{i \in I} \{\overline{E(s_i, s_{i+1}, y)} \bigcap \overline{E(s_{i+1}, s_i, y)}\}$$

. Combining Lemma 2.3, Lemma 2.4, and Lemma 2.6, we have

$$\begin{aligned}
P_r(\overline{E_n(y)}) &\leq P_r(\bigcap_{i \in I} \{\overline{E(s_i, s_{i+1}, y)}) \bigcap \overline{E(s_{i+1}, s_i, y)}\}) \\
&= \amalg_{i \in I} P_r(\{\overline{E(s_i, s_{i+1}, y)}) \bigcap \overline{E(s_{i+1}, s_i, y)}\}) \\
&\leq (\mu)^{\lfloor \frac{(n+11)}{14} \rfloor}, \\
&\mathit{where}\ \mu = \max\{1 - (2p^4 - p^6), 1 - (2p^2 - p^3), \\
&1 - (p^2 + p^3 - p^4), 1 - (p^3 + p^4 - p^5), 1 - p^4, \\
&1 - (2p^3 - p^6)\} = 1 - p^4.
\end{aligned}$$

This completes the proof.

□

Now we are in the position to prove Theorem 2.1.

Proof of theorem 2.1. By Lemma 2.3, Lemma 2.4, and Lemma 2.7, we have

$$P_r(\overline{E_n}) = P_r(\bigcup_{y \in G} \overline{E_n(y)}) \leq \sum_{y \in G} P_r(\overline{E_n(y)}) \leq n(1 - p^4)^{\lfloor \frac{(n+11)}{14} \rfloor}.$$

Thus $\lim_{n \to \infty} P_r(\overline{E_n}) = 0$ and hence $\lim_{n \to \infty} P_r(E_n) = 1$. The theorem follows by noting that there is only one Bi-Cayley graph $BC(G, S)$ of G with diameter 2.

Remark 2.1. *Theorem 2.1 holds not only for fixed p but also for those p's having relation with the order of G under the condition that $n(1 - p^4)^{\lfloor \frac{(n+11)}{14} \rfloor} \to 0$ as $n \to \infty$.*

Corollary 2.1. *Almost all Bi-Cayley graphs are connected.*

References

[1] B.Bollabas. Random Graphs (*Second Edition*), Cambridge Press, England, 2001.

[2] J.Meng and Q.Huang. Almost all Cayley graphs have diameter 2, Discrete Math., 178: 267-269, 1998.

[3] J.Meng and X.liu. The diameters of almost all Cayley digraphs, Acta Math. Appl. Sinica(English Ser.), 13: 400-413, 1997.

[4] P. Billingsley. Probability and Measure, John Wiley &Sons, New York, 1979.

[5] F.X.Liu and J.Meng. Super-edge-connected and optimally Super-edge-connected Bi-Cayley graphs. To appear.

[6] M.Y.Xu, Introduction of finite groups II, Science press, Beijing, 1999.

[7] V.L. kiee, D.G. larman and E.M. wright, The diameter of almost all bipartite graphs, Studia.Sci. Math., Hungar, 15: 39-43, 1980.