Edge-colorings of complete bipartite graphs without large rainbow trees*

Zemin Jin, Lifen Li
Department of Mathematics, Zhejiang Normal University
Jinhua 321004, P.R. China

Abstract

Let $\mathcal{G}$ be a family of graphs. The anti-Ramsey number $AR(n, \mathcal{G})$ for $\mathcal{G}$ is the maximum number of colors in an edge coloring of K_n that has no rainbow copy of any graph in $\mathcal{G}$. In this paper, we determine the bipartite anti-Ramsey number for the family of trees with k edges.

Key Words: anti-Ramsey number, tree, rainbow.

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1 Introduction

An edge-colored graph is called *rainbow* if any of its two edges have distinct colors. Let $\mathcal{G}$ be a family of graphs. The *anti-Ramsey number* $AR(n, \mathcal{G})$ for $\mathcal{G}$ is the maximum number of colors in an edge coloring of K_n that has no rainbow copy of any graph in $\mathcal{G}$. The *Turán number* $ex(n, \mathcal{G})$ is the maximum number of edges of a simple graph without a copy of any graph in $\mathcal{G}$. Clearly, by taking one edge of each color in an edge coloring of K_n, one can show that $AR(n, \mathcal{G}) \leq ex(n, \mathcal{G})$. When $\mathcal{G}$ consists of a single graph H, we write $AR(m, H)$ and $ex(n, H)$ for $AR(m, \{H\})$ and $ex(n, \{H\})$, respectively.

Anti-Ramsey number was introduced by Erdős et al. in [5], which is showed to be connected not so much to Ramsey theory than to Turán numbers. The anti-Ramsey numbers for some special graph classes have been determined. As conjectured by Erdős et al. [5], the anti-Ramsey

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Email: zeminjin@gmail.com (Jin), lilifen_0408@126.com (Li).

number $AR(n, C_k)$ for cycles was determined for $k \leq 6$ in [1, 5, 8], and later completely solved in [13]. The anti-Ramsey number $AR(n, P_{k+1})$ for paths was determined in [15]. Independently, the authors of [12] and [14] considered the anti-Ramsey number for complete graphs. The anti-Ramsey numbers for other graph classes have been studied, including small bipartite graphs [2], stars [6], subdivided graphs [7], trees with order k [9], graphs with independent cycles [10] and matchings [4, 14]. The bipartite analogue of the anti-Ramsey number was studied for even cycles [3], stars [6] and matchings [11].

Naturally, the host graph K_n in the anti-Ramsey number can be generalized to any graph G. The anti-Ramsey number $AR(G, \mathcal{G})$ for the family $\mathcal{G}$ in G is the maximum number of colors in an edge coloring of G that has no rainbow copy of any graph in $\mathcal{G}$.

In this paper, we consider the bipartite anti-Ramsey number $AR(K_{a,b}, \mathcal{T}_k)$, where $\mathcal{T}_k$ denote the family of trees with k edges. Let $\mathcal{L}(K_{a,b}, k)$ denote the family of subgraphs of $K_{a,b}$ every two components of which together have at most k vertices. Let $l(K_{a,b}, k)$ be the maximum size of a graph in $\mathcal{L}(K_{a,b}, k)$.

Let G be a graph and c be a coloring of $E(G)$. A representing subgraph of c is spanning subgraph L of G which has exactly one edge of each color of c. For an edge $e \in E(G)$, denote by $c(e)$ the color assigned to the edge e.

Let $H \subseteq G$. A coloring c of G is induced by H if c assigns distinct colors to each edge of H and assigns one additional color to all of $E(G) \backslash E(H)$.

2 Main Theorem

Obviously, $AR(K_{a,b}, \mathcal{T}_k) = ab$ if $k \geq a + b$. The following lemma is obvious.

Lemma 2.1 [9] *Let $G \subseteq K_{a,b}$ where every two components together have at most k vertices, then a coloring of $E(K_{a,b})$ induced by G has no any rainbow trees with k edges.*

Lemma 2.2 [9] *Let G be a connected graph. Then G contains a vertex w such that for all $e \subseteq E(G)$, the component of $G - e$ containing w has at least $\frac{|V(G)|}{2}$ vertices.*

As in [9] for the complete graphs, we have the following analogue result. In fact, the following result holds also for any host graphs with a component of order at least $k + 1$.

Theorem 2.3 *For $a + b > k$, $AR(K_{a,b}, \mathcal{T}_k) = l(K_{a,b}, k) + 1$.*

Proof. By Lemma 2.1, we have the lower bound. So here we only need to show that $AR(K_{a,b}, \mathscr{T}_k) \le l(K_{a,b}, k) + 1$.

Let c be a coloring of $E(K_{a,b})$ which avoids rainbow trees with k edges. Let H be a representing subgraph of c that has a largest possible component, denoted by F_1. It is obviously that every component of H has at most k vertices. It suffices to show that $|E(H)| \le l(K_{a,b}, k) + 1$.

By Lemma 2.2, F_1 contain a vertex w such that for all $e \in E(F_1)$, the component containing w in $F_1 - e$ has at least $\left\lceil \frac{|E(F_1)|}{2} \right\rceil$ vertices. Let F_2 be a component in $H - F_1$ and let $v \in V(F_2)$. Since H is a representing subgraph of c and $wv \notin E(H)$, there is an edge $e' \in E(H)$ with color $c(wv)$, and $H' = H - e' + wv$ is also a representing subgraph of c. The edge e' must be a cut edge of F_1, since otherwise H' has a component with order larger than F_1.

Let F_3 and F_4 be the two components of $F_1 - e'$ where $w \in F_3$. So $|V(F_3)| \ge |V(F_4)|$. From the choice of H, we have that $|V(F_3)| \ge |V(F_4)| \ge |V(F_2)|$. This implies that F_3 and F_4 are the two largest components of $H - e$. From $|V(F_3)| + |V(F_4)| = |V(F_1)| \le k$, any two components of $H - e$ together has at most k vertices. Hence $H - e \in \mathscr{L}(K_{a,b}, k)$ and $|E(H - e)| \le l(K_{a,b}, k)$. Then $|E(H)| \le l(K_{a,b}, k) + 1$. ∎

3 Computing $l(K_{a,b}, k)$

Lemma 3.1 *Let $a, b, c, d \ge 0$ and $a + b = c + d$. If $|a - b| \le |c - d|$, then $|E(K_{a,b})| \ge |E(K_{c,d})|$.*

Let $|a-b| \le 1$ and $a+b > p$. Let $\mathcal{J}_{a,b,p} = rK_{\lceil \frac{p}{2} \rceil, \lfloor \frac{p}{2} \rfloor} + K_{\lceil \frac{s}{2} \rceil, \lfloor \frac{s}{2} \rfloor} \subseteq K_{a,b}$, where $r = \lfloor \frac{a+b}{p} \rfloor$ and $s = a + b - rp$.

Lemma 3.2 *Let $|a - b| \le 1, a + b > p$ and $G \subseteq K_{a,b}$. If each component of G has at most p vertices, then $|E(G)| \le |E(\mathcal{J}_{a,b,p})|$. Furthermore, the equality holds if and only if $G = \mathcal{J}_{a,b,p}$.*

Proof. Let $G \subseteq K_{a,b}$ be a graph with the largest number of edges where every component has at most p vertices. We choose G to have as many components of p vertices as possible. Then the followings must hold.

(1) Each non-trivial component of G is complete bipartite.

(2) Every two components of G together have at least p vertices.

The following claims are easy to verify.

Claim 1. For every two components of G, there is one component with p vertices.

Claim 2. There is a component G_0 with s (if $s > 0$) vertices in G, and each other component of G has p **Claim 3.** If $s > 0$, then $G_0 = K_{\lceil \frac{s}{2} \rceil, \lfloor \frac{s}{2} \rfloor}$.

Claim 4. Each component of $G-G_0$ is $K_{\lceil \frac{k}{2}\rceil,\lfloor \frac{k}{2}\rfloor}$.

The lemma follows clearly from the claims above. ∎

Let $H_m = K_{\lceil \frac{m}{2}\rceil,\lfloor \frac{m}{2}\rfloor} + \mathcal{J}_{a-\lceil \frac{m}{2}\rceil, b-\lfloor \frac{m}{2}\rfloor, k-m} \subseteq K_{a,b}$, where $\lceil \frac{k}{2}\rceil \le m \le k-1$, $b \le a \le b+1$ and $a+b > k$. Clearly, $H_m \in \mathcal{L}(K_{a,b},k)$ and H_m contains a component with m vertices.

Lemma 3.3 *Let* $\lceil \frac{k}{2}\rceil \le m \le k-1$, $b \le a \le b+1$, $a+b > k$ *and* $G \in \mathcal{L}(K_{a,b},k) \subseteq K_{a,b}$. *If there is a component with* m *vertices in* G, *then* $|E(G)| \le |E(H_m)|$.

Proof. We choose G with the largest number of edges. Note that each non-trivial component of G is complete bipartite. Denote by A and B the parts of $K_{a,b}$, where $|A| = a$ and $|B| = b$. Let G_0 be a component with m vertices in G and let $G_0 = K_{e_0,f_0}$ where $|V(G_0)\cap A| = e_0$ and $|V(G_0)\cap B| = f_0$. Then each component of $G-G_0$ contains at most $k-m$ vertices.

Now we prove that $|e_0 - f_0| \le 1$. Suppose that $e_0 \ge f_0 + 2$. From $b \le a \le b+1$, there exists a component $G_1 = K_{e_1,f_1}$ in $G-G_0$, where $|V(G_1)\cap A| = e_1$ and $|V(G_1)\cap B| = f_1$, such that $f_1 \ge e_1 + 1$.

Let $G_0' = K_{e_0-1,f_0+1}$ and $G_1' = K_{e_1+1,f_1-1}$, where $V(G_0 \cup G_1) = V(G_0' \cup G_1')$. It is obvious that $G' = G - G_0 - G_1 + G_0' + G_1'$ contradicts the choice of G. So $e_0 - f_0 \le 1$. Also, from $b \le a$, we can show that $0 \le e_0 - f_0 \le 1$. Clearly, $G_0 = K_{\lceil \frac{m}{2}\rceil,\lfloor \frac{m}{2}\rfloor}$. From Lemma 3.2, we can show that $G - G_0 = \mathcal{J}_{a-\lceil \frac{m}{2}\rceil, b-\lfloor \frac{m}{2}\rfloor, k-m}$, i.e., $G = H_m$. ∎

Lemma 3.4 *Let* $\lceil \frac{k}{2}\rceil \le m \le k-1$, $b \le a \le b+1$ *and* $a+b > k \ge 4$. *If* $k+1 \le a+b \le 2k-4$, *then* $|E(H_m)| \le |E(H_{k-1})|$.

Proof. Let $f(m) = -\frac{m^2}{2} + \frac{3k-4}{4}m - \frac{k^2}{4} - \frac{k}{2}$. From the definition of H_m, we have $|E((H_{k-1}))| - |E((H_m))| \ge \lceil \frac{k-1}{2}\rceil \lfloor \frac{k-1}{2}\rfloor - \lceil \frac{m}{2}\rceil \lfloor \frac{m}{2}\rfloor - \frac{a+b-m}{k-m}\lceil \frac{k-m}{2}\rceil \lfloor \frac{k-m}{2}\rfloor \ge f(m) \ge 0$ for $\frac{k}{2} \le m \le k-2$. ∎

Lemma 3.5 *Let* $\lceil \frac{k}{2}\rceil \le m \le k-1$, $b \le a \le b+1$ *and* $a+b > k \ge 4$. *If* $a+b \ge \frac{k^2}{2} + 2k + 2$, *then* $|E(H_m)| \le |E(H_{\lceil \frac{k}{2}\rceil})|$.

Proof. We only need to show that $|E(H_{m+1})| - |E(H_m)| \le 0$ for $\lceil \frac{k}{2}\rceil \le m \le k-2$. By the definition of H_m, we have

$$\begin{aligned} & |E(H_{m+1})| - |E(H_m)| \\ = & \left\lceil \frac{m+1}{2}\right\rceil \left\lfloor \frac{m+1}{2}\right\rfloor + \left\lceil \frac{k-m-1}{2}\right\rceil \left\lfloor \frac{k-m-1}{2}\right\rfloor \\ & + \left\lfloor \frac{a+b-k}{k-m-1}\right\rfloor \left\lceil \frac{k-m-1}{2}\right\rceil \left\lfloor \frac{k-m-1}{2}\right\rfloor \end{aligned}$$

$$+\left\lceil\frac{S_{m+1}}{2}\right\rceil\left\lfloor\frac{S_{m+1}}{2}\right\rfloor-\left(\left\lceil\frac{m}{2}\right\rceil\left\lfloor\frac{m}{2}\right\rfloor+\left\lceil\frac{k-m}{2}\right\rceil\left\lfloor\frac{k-m}{2}\right\rfloor\right)$$
$$-\left\lfloor\frac{a+b-k}{k-m}\right\rfloor\left\lceil\frac{k-m}{2}\right\rceil\left\lfloor\frac{k-m}{2}\right\rfloor-\left\lceil\frac{S_m}{2}\right\rceil\left\lfloor\frac{S_m}{2}\right\rfloor,$$

where $S_m = a+b-k-\left\lfloor\frac{a+b-k}{k-m}\right\rfloor(k-m)$. Clearly, $\left\lceil\frac{S_{m+1}}{2}\right\rceil\left\lfloor\frac{S_{m+1}}{2}\right\rfloor \le \left\lceil\frac{k-m-1}{2}\right\rceil\left\lfloor\frac{k-m-1}{2}\right\rfloor$.

We distinguish the following cases.

Case 2.1. $k,m \equiv 0 \pmod 2$.

Then we have

$$\begin{aligned}
&|E(H_{m+1})|-|E(H_m)|\\
\le\ & \frac{m-k}{2}\left(\left\lfloor\frac{a+b-k}{k-m}\right\rfloor-\frac{2m-k}{k-m}\right)\\
&+\frac{m-k}{2}\left(\left\lfloor\frac{a+b-k}{k-m-1}\right\rfloor-\left\lfloor\frac{a+b-k}{k-m}\right\rfloor+1\right)\frac{m-k+2}{2}.
\end{aligned}$$

Since $m \le k-2$, we only need to show that $g(m)=\left\lfloor\frac{a+b-k}{k-m}\right\rfloor-\frac{2m-k}{k-m}+\left(\left\lfloor\frac{a+b-k}{k-m-1}\right\rfloor-\left\lfloor\frac{a+b-k}{k-m}\right\rfloor+1\right)\frac{m-k+2}{2}\ge 0$.

Clearly, $\left\lfloor\frac{a+b-k}{k-m-1}\right\rfloor \ge \frac{a+b-2k+m+2}{k-m-1}$, $\left\lfloor\frac{a+b-k}{k-m-1}\right\rfloor \le \frac{a+b-k}{k-m-1}$ and $\left\lfloor\frac{a+b-k}{k-m}\right\rfloor \ge \frac{a+b-2k+m+1}{k-m}$. Then

$$\begin{aligned}
g(m)=\ & \left\lfloor\frac{a+b-k}{k-m-1}\right\rfloor+1+\left(\left\lfloor\frac{a+b-k}{k-m-1}\right\rfloor-\left\lfloor\frac{a+b-k}{k-m}\right\rfloor\right)\frac{m-k}{2}\\
&-\frac{2m-k}{k-m}+\frac{m-k}{2}\\
\ge\ & \frac{a+b-k+1}{k-m-1}-\frac{a+b-k(k-m)}{2(k-m-1)}-\frac{3k-2m-1}{2}-\frac{2m-k}{k-m}\\
\ge\ & \frac{a+b+k(k-m)-4m+2}{2(k-m-1)}-\frac{3k-2m-1}{2}\\
\ge\ & \frac{a+b-\frac{k^2}{2}+2}{k-2}>0.
\end{aligned}$$

Thus $|E(H_{m+1})|-|E(H_m)|\le 0$.

Case 2.2. $m\equiv 0 \pmod 2$ and $k \equiv 1 \pmod 2$.

Then we have

$$\begin{aligned}
&|E(H_{m+1})|-|E(H_m)|\\
\le\ & \left(\left\lfloor\frac{a+b-k}{k-m}\right\rfloor-\frac{2m-k+1}{k-m-1}\right)\frac{m-k+1}{2}
\end{aligned}$$

$$+\left(\left\lfloor\frac{a+b-k}{k-m-1}\right\rfloor-\left\lfloor\frac{a+b-k}{k-m}\right\rfloor+1\right)\frac{(k-m-1)^2}{4}$$
$$=\frac{m-k+1}{2}\left(\left\lfloor\frac{a+b-k}{k-m}\right\rfloor-\frac{2m-k+1}{k-m-1}\right)$$
$$+\frac{m-k+1}{2}\left(\left(\left\lfloor\frac{a+b-k}{k-m-1}\right\rfloor-\left\lfloor\frac{a+b-k}{k-m}\right\rfloor+1\right)\frac{m-k+1}{2}\right).$$

Since $m-k+1<0$, we only need to show that $\left\lfloor\frac{a+b-k}{k-m}\right\rfloor-\frac{2m-k+1}{k-m-1}$ $+\left(\left\lfloor\frac{a+b-k}{k-m-1}\right\rfloor-\left\lfloor\frac{a+b-k}{k-m}\right\rfloor+1\right)\frac{m-k+1}{2}\geq 0$.

Clearly, $\left\lfloor\frac{a+b-k}{k-m-1}\right\rfloor\geq\frac{a+b-2k+m+2}{k-m-1}$, $\left\lfloor\frac{a+b-k}{k-m}\right\rfloor\geq\frac{a+b-2k+m+1}{k-m}$ and $\left\lfloor\frac{a+b-k}{k-m-1}\right\rfloor\leq\frac{a+b-k}{k-m-1}$. Since $m\equiv 0\,(\mathrm{mod}\ 2)$ and $k\equiv 1\,(\mathrm{mod}\ 2)$, we have $k\geq 5$ and $\frac{k+1}{2}\leq m\leq k-3$. Hence $k^2-km+k-4m-1>0$. Then

$$\left\lfloor\frac{a+b-k}{k-m}\right\rfloor-\frac{2m-k+1}{k-m-1}+\left(\left\lfloor\frac{a+b-k}{k-m-1}\right\rfloor-\left\lfloor\frac{a+b-k}{k-m}\right\rfloor+1\right)\frac{m-k+1}{2}$$
$$\geq\frac{a+b-2k+m+1}{2(k-m)}+\frac{k^2-km+k-4m-1}{2(k-m-1)}-\frac{3k-2m-1}{2}$$
$$\geq\frac{a+b-2k+m+1}{2(k-m)}+\frac{k^2-km+k-4m-1}{2(k-m-1)}-k+1$$
$$\geq\frac{a+b+k^2-km-k-3m-2k^2+2km+2k-2m}{2(k-m)}$$
$$\geq\frac{a+b-\frac{k^2}{2}-k-\frac{5}{2}}{2(k-m)}\geq 0.$$

Thus $|E(H_{m+1})|-|E(H_m)|\leq 0$.

Case 2.3. $m\equiv 1\,(\mathrm{mod}\ 2)$ and $k\equiv 0\,(\mathrm{mod}\ 2)$.

Then we have

$$|E(H_{m+1})|-|E(H_m)|$$
$$\leq\left(\left\lfloor\frac{a+b-k}{k-m}\right\rfloor-\frac{2m-k+2}{k-m-1}\right)\frac{m-k+1}{2}$$
$$+\left(\left\lfloor\frac{a+b-k}{k-m-1}\right\rfloor-\left\lfloor\frac{a+b-k}{k-m}\right\rfloor+1\right)\frac{(k-m-1)^2}{4}$$
$$=\frac{m-k+1}{2}\left(\left\lfloor\frac{a+b-k}{k-m}\right\rfloor-\frac{2m-k+2}{k-m-1}\right)$$
$$+\frac{m-k+1}{2}\left(\left(\left\lfloor\frac{a+b-k}{k-m-1}\right\rfloor-\left\lfloor\frac{a+b-k}{k-m}\right\rfloor+1\right)\frac{m-k+1}{2}\right).$$

Since $m-k+1<0$, we only need to show that $\left\lfloor \frac{a+b-k}{k-m} \right\rfloor - \frac{2m-k+2}{k-m-1}$ $+\left(\left\lfloor \frac{a+b-k}{k-m-1} \right\rfloor - \left\lfloor \frac{a+b-k}{k-m} \right\rfloor + 1\right) \frac{m-k+1}{2} \geq 0$.

Clearly, $\left\lfloor \frac{a+b-k}{k-m-1} \right\rfloor \geq \frac{a+b-2k+m+2}{k-m-1}$, $\left\lfloor \frac{a+b-k}{k-m} \right\rfloor \geq \frac{a+b-2k+m+1}{k-m}$ and $\left\lfloor \frac{a+b-k}{k-m-1} \right\rfloor \leq \frac{a+b-k}{k-m-1}$. Since $m \equiv 1 \pmod 2$ and $k \equiv 0 \pmod 2$, we have $k \geq 6$ and $\lceil \frac{k}{2} \rceil \leq m \leq k-3$. Hence $k^2-km+k-4m-3>0$. Thus

$$\left\lfloor \frac{a+b-k}{k-m} \right\rfloor - \frac{2m-k+2}{k-m-1} + \left(\left\lfloor \frac{a+b-k}{k-m-1} \right\rfloor - \left\lfloor \frac{a+b-k}{k-m} \right\rfloor + 1\right) \frac{m-k+1}{2}$$

$$\begin{aligned} &\geq \frac{a+b-2k+m+1}{2(k-m)} + \frac{k^2-km+k-4m-3}{2(k-m-1)} - \frac{3k-2m-1}{2} \\ &\geq \frac{a+b-2k+m+1+k^2-km+k-4m-3-(3k-2m-1)(k-m)}{2(k-m)} \\ &\geq \frac{a+b-\frac{k^2}{2}-2k-2}{2(k-m)} \geq 0. \end{aligned}$$

Thus $|E(H_{m+1})| - |E(H_m)| \leq 0$.

Case 2.4. $m, k \equiv 1 \pmod 2$.

Then we have

$$\begin{aligned} |E(H_{m+1})| - |E(H_m)| &\leq \left(\left\lfloor \frac{a+b-k}{k-m} \right\rfloor - \frac{2m-k+1}{k-m}\right) \frac{m-k}{2} \\ &+ \left(\left\lfloor \frac{a+b-k}{k-m-1} \right\rfloor - \left\lfloor \frac{a+b-k}{k-m} \right\rfloor + 1\right)\left(\frac{(k-m)^2}{4} - \frac{k-m}{2}\right) \\ &= \frac{m-k}{2}\left(\left\lfloor \frac{a+b-k}{k-m} \right\rfloor - \frac{2m-k+1}{k-m}\right) \\ &+ \frac{m-k}{2}\left(\left(\left\lfloor \frac{a+b-k}{k-m-1} \right\rfloor - \left\lfloor \frac{a+b-k}{k-m} \right\rfloor + 1\right) \frac{m-k+2}{2}\right). \end{aligned}$$

Since $m-k<0$, it suffices to show that $\left\lfloor \frac{a+b-k}{k-m} \right\rfloor - \frac{2m-k+1}{k-m}$ $+\left(\left\lfloor \frac{a+b-k}{k-m-1} \right\rfloor - \left\lfloor \frac{a+b-k}{k-m} \right\rfloor + 1\right)\left(\frac{m-k}{2}+1\right) \geq 0$.

Clearly, $\left\lfloor \frac{a+b-k}{k-m-1} \right\rfloor \geq \frac{a+b-2k+m+2}{k-m-1}$, $\left\lfloor \frac{a+b-k}{k-m-1} \right\rfloor \leq \frac{a+b-k}{k-m-1}$ and $\left\lfloor \frac{a+b-k}{k-m} \right\rfloor \geq \frac{a+b-2k+m+1}{k-m}$. Then

$$\left\lfloor \frac{a+b-k}{k-m} \right\rfloor - \frac{2m-k+1}{k-m} + \left(\left\lfloor \frac{a+b-k}{k-m-1} \right\rfloor - \left\lfloor \frac{a+b-k}{k-m} \right\rfloor + 1\right) \frac{m-k+2}{2}$$

$$
\begin{aligned}
&= \left\lfloor \frac{a+b-k}{k-m-1} \right\rfloor + \left(\left\lfloor \frac{a+b-k}{k-m-1} \right\rfloor - \left\lfloor \frac{a+b-k}{k-m} \right\rfloor \right) \frac{m-k}{2} \\
&\quad - \frac{2m-k+1}{k-m} + \frac{m-k}{2} + 1 \\
&\geq \frac{a+b+k(k-m)-2k+2}{2(k-m-1)} - \frac{2m-k+1}{k-m} - \frac{2k-m-1}{2} \\
&\quad + \frac{m-k}{2} \\
&\geq \frac{a+b+k(k-m)-2k+2}{2(k-m-1)} - \frac{2m-k+1}{k-m-1} - \frac{3k-2m-1}{2} \\
&\geq \frac{a+b+k^2-km-4m}{2(k-m-1)} - k + 1 \\
&= \frac{a+b-k^2+(k-6)m+4k-2}{2(k-m-1)} \\
&\geq \frac{a+b-\frac{k^2}{2}+2k-5}{k-2} \geq 0.
\end{aligned}
$$

Thus $|E(H_{m+1})| - |E(H_m)| \leq 0$.
This completes the proof. ∎

From Lemmas 3.3, 3.4 and 3.5, we have the following result.

Theorem 3.6 *Let $b \leq a \leq b+1$ and $a+b > k \geq 4$. Then*
(1) if $k+1 \leq a+b \leq 2k-4$, then $l(K_{a,b},k) = \left|E\left(K_{\lceil \frac{k-1}{2} \rceil, \lfloor \frac{k-1}{2} \rfloor}\right)\right| = \lceil \frac{k-1}{2} \rceil \lfloor \frac{k-1}{2} \rfloor$;
(2) if $a+b \geq \frac{k^2}{2}+2k+2$, then $l(K_{a,b},k) = \left|E\left(H_{\lceil \frac{k}{2} \rceil}\right)\right| = \left\lceil \frac{\lceil \frac{k}{2} \rceil}{2} \right\rceil \left\lfloor \frac{\lceil \frac{k}{2} \rceil}{2} \right\rfloor + \lceil \frac{s}{2} \rceil \lfloor \frac{s}{2} \rfloor + r \left\lfloor \frac{\lfloor \frac{k}{2} \rfloor}{2} \right\rfloor \left\lceil \frac{\lfloor \frac{k}{2} \rfloor}{2} \right\rceil$, where $r = \left\lfloor \frac{a+b-\lceil \frac{k}{2} \rceil}{\lfloor \frac{k}{2} \rfloor} \right\rfloor$ and $s = a+b-\lceil \frac{k}{2} \rceil - r \lfloor \frac{k}{2} \rfloor$.

4 Concluding remarks

From previous section we know that the function $|E(H_m)|$ is decreasing for $\lceil \frac{k}{2} \rceil \leq m \leq k-1$ when $2n \geq \frac{k^2}{2}+2k+2$. However, the property may not hold when $2(k-1) \leq 2n < \frac{k^2}{2}+2k+2$. Even the value curve of $|E(H_m)|$ is tortuous in some cases of n and k with $2(k-1) \leq 2n < \frac{k^2}{2}+2k+2$, see Figure 1 for some examples. It is possible that one can improve the bound $a+b \geq \frac{k^2}{2}+2k+2$ in Lemma 3.5 and Theorem 3.6 by carefully computing.

For $a+b = 2(k-1)$, we have the following partial result.

Lemma 4.1 *Let $\lceil \frac{k}{2} \rceil \leq m \leq k-1$, $a = b = k-1$ and $k \equiv 2 \,(mod\, 2)$. If $m \equiv 0 \,(mod\, 2)$, then $|E(H_m)| \leq |E(H_{k-2})|$.*

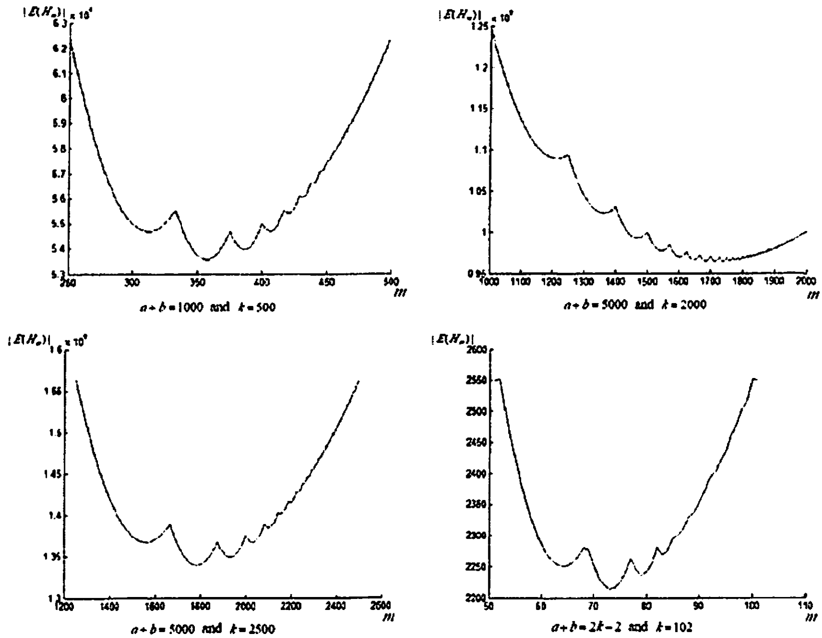

Figure 1: Examples for the value curve of $|E(H_m)|$

Proof. First we consider the case $k \equiv 2 \pmod 4$. It is easy to see that $|E(H_{k-1})| = |E(H_{\frac{k}{2}})| = \frac{k^2}{4} - \frac{k}{2}$ and $|E(H_{k-2})| = \frac{k^2}{4} - \frac{k}{2} + 1$. For $\frac{k}{2} + 1 \leq m \leq k-2$, it is to verfy that $|E(H_m)| \leq \frac{m^2}{4} + \left(k - 1 - \frac{m}{2}\right)\frac{k-m}{2} \leq \frac{m^2}{2} - \frac{3k-2}{4}m + \frac{k^2+k}{2} \leq \frac{k^2}{4} - \frac{k}{2} + 1 = |E(H_{k-2})|$. The case $k \equiv 0 \pmod 4$ can be proved in the similar way. ∎

However, we cannot put our hopes on $|E(H_{k-2})|$ for the case $a = b = k - 1$, since $|E(H_{\lceil \frac{k}{2} \rceil})| > |E(H_{k-2})|$ for $k \equiv 1 \pmod 2$. It is still very hard to solve the problem completely.

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