Roman domination subdivision number of a graph and its complement

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Abstract

A Roman dominating function of a graph G is a labeling $f : V(G) \longrightarrow \{0,1,2\}$ such that every vertex with label 0 has a neighbor with label 2. The Roman domination number $\gamma_R(G)$ of G is the minimum of $\sum_{v\in V(G)} f(v)$ over such functions. The *Roman domination subdivision number* $\mathrm{sd}_{\gamma_R}(G)$ is the minimum number of edges that must be subdivided (each edge in G can be subdivided at most once) in order to increase the Roman domination number. In this paper, we prove that if G is a graph of order $n \geq 4$ such that G and $\overline{G}$ have connected components of order at least 3, then $\mathrm{sd}_{\gamma_R}(G) + \mathrm{sd}_{\gamma_R}(\overline{G}) \leq \lfloor \frac{n}{2} \rfloor + 3$.

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1 Introduction

In this paper, G is a simple graph with vertex set $V(G)$ and edge set $E(G)$ (briefly V and E). For every vertex $v \in V$, the *open neighborhood* $N(v)$ is the set $\{u \in V(G) \mid uv \in E(G)\}$ and its *closed neighborhood* is $N[v] = N(v) \cup \{v\}$. Similarly, the *open neighborhood* of a set $S \subseteq V$ is the set $N(S) = \cup_{v \in S} N(v)$ and its *closed neighborhood* is $N[S] = N(S) \cup S$. The minimum and maximum vertex degrees in G are respectively denoted by $\delta(G)$ and $\Delta(G)$. Given graphs G and H, the *cartesian product* $G \Box H$ is the graph with vertex set $V(G) \times V(H)$ and edge set defined by making (u, v) and (u', v') adjacent if and only if either (1) $u = u'$ and $vv' \in E(H)$ or (2) $v = v'$ and $uu' \in E(G)$.

A subset S of vertices of G is a *dominating set* if $N[S] = V$. A *Roman dominating function* (RDF) on a graph $G = (V, E)$ is defined in [6, 7] as a function $f : V \longrightarrow \{0, 1, 2\}$ satisfying the condition that every vertex v for which $f(v) = 0$ is adjacent to at least one vertex u for which $f(u) = 2$. The *weight* of a RDF is the value $w(f) = \sum_{v \in V} f(v)$. The *Roman domination number* of a graph G, denoted by $\gamma_R(G)$, equals the minimum weight of a RDF on G. A $\gamma_R(G)$*-function* is a Roman dominating function of G with weight $\gamma_R(G)$. A Roman dominating function $f : V \longrightarrow \{0, 1, 2\}$ may be represented by the ordered partition (V_0^f, V_1^f, V_2^f) of V, where $V_i^f = \{v \in V \mid f(v) = i\}$. For a more thorough treatment of domination parameters and for terminology not presented here see [5, 8].

The *Roman domination subdivision number* of a graph G is the minimum number of edges that must be subdivided (where each edge in G can be subdivided at most once) in order to increase the Roman domination number of G. The Roman domination subdivision number was introduced by Atapour et al. in [1, 2] and denoted by $\mathrm{sd}_{\gamma_R}(G)$.

The complement $\overline{G}$ of a graph G has vertex set $V(G)$ and $xy \in E(\overline{G})$ if and only if $xy \notin E(G)$. For any graph parameter μ, bounds on $\mu(G) + \mu(\overline{G})$ and on $\mu(G)\mu(\overline{G})$ are called Nordhaus-Gaddum inequalities. Many Nordhaus-Gaddum bounds have been obtained on various domination parameters. For instance,

Theorem A. (Chambers et al. [3]) If G is an n-vertex graph, with $n \geq 3$, then

$$5 \leq \gamma_R(G) + \gamma_R(\overline{G}) \leq n + 3.$$

Furthermore, equality holds in the upper bound only when G or $\overline{G}$ is C_5 or $\frac{n}{2}K_2$.

In this paper, we prove that if G is a graph of order $n \geq 4$ such that G and $\overline{G}$ have connected components of order at least 3, then $\mathrm{sd}_{\gamma_R}(G) + \mathrm{sd}_{\gamma_R}(\overline{G}) \leq \lfloor \frac{n}{2} \rfloor + 3$.

We make use of the following results. Recall that a set $S \subseteq V$ is a 2-packing set of G if $N[u] \cap N[v] = \emptyset$ holds for any two distinct vertices $u, v \in S$.

Theorem B. (Cockayne et al. [4]). Let $f = (V_0^f, V_1^f, V_2^f)$ be a γ_R-function for a simple graph G, such that $|V_1^f|$ is minimum. Then V_1 is a 2-packing.

Theorem C. (Atapour et al. [1]) Let G be a simple connected graph of order $n \geq 3$. If $\gamma_R(G) = 2$ or 3, then $\mathrm{sd}_{\gamma_R}(G) = 1$.

Theorem D. (Atapour et al. [1]) If G contains a matching M such that $\lfloor \frac{\gamma_R(G)}{2} \rfloor + 1 \leq |M|$, then $\mathrm{sd}_{\gamma_R}(G) \leq \lfloor \frac{\gamma_R(G)}{2} \rfloor + 1$.

Theorem E. (Atapour et al. [1]) For every simple connected graph G of order $n \geq 3$, $\mathrm{sd}_{\gamma_R}(G) \leq \lceil \frac{n}{2} \rceil - 1$.

2 An upper bound for $\mathrm{sd}_{\gamma_R}(G) + \mathrm{sd}_{\gamma_R}(\overline{G})$

Theorem 1. Let G be a simple connected graph of order n. If $\gamma_R(G) = 4$, then $\mathrm{sd}_{\gamma_R}(G) \leq 2$. Furthermore, this bound is sharp.

Proof. Let f be a $\gamma_R(G)$-function such that V_1^f is minimum. Since $\gamma_R(G) = 4$, $n \geq 5$ which implies that $V_2^f \neq \emptyset$.

Case 1 $V_1^f = \{u, v\}$ and $V_2^f = \{w\}$.
By the choice of f, u and v are non-adjacent and have no common neighbors. Let $u_0 \in N(u) \cap V_0^f$ and $v_0 \in N(v) \cap V_0^f$ and let G' be obtained from G by subdividing the edges uu_0 and vv_0 with vertices u', v', respectively. Assume g is a $\gamma_R(G')$-function. We have the following subcases.

Subcase 1.1 $g(u') = 1$ (the case $g(v') = 1$ is similar).
Subcase 1.2 $g(u') = 2$ and $g(v') = 0$ (the case $g(v') = 2$ and $g(u') = 0$ is similar).
Subcase 1.3 $g(u') = g(v') = 2$.
Subcase 1.4 $g(u') = g(v') = 0$.
It is straightforward to see that in each case $\gamma_R(G') \geq 5$.

Case 2 $V_1^f = \emptyset$ and $V_2^f = \{x, y\}$.
We consider two subcases.

Subcase 2.1 $xy \notin E(G)$. Let $x_1 \in N(x) \setminus N(y)$ and $y_1 \in N(y) \setminus N(x)$. Note that since $\gamma_R(G) = 4$, the vertices x_1 and y_1 exist. Suppose that G' is obtained by subdividing the edges xx_1 and yy_1 with vertices x', y', respectively. Assume g is a $\gamma_R(G')$-function. A simple case checking similar to that given in Case 1 shows that $\gamma_R(G') \geq 5$.

Subcase 2.2 $xy \in E(G)$.
Then each of x and y have at least two private neighbors in V_0^f, otherwise $\gamma_R(G) \leq 3$, a contradiction. Suppose that x_1, x_2 are two private neighbors of x in V_0^f and that y_1, y_2 are two private neighbors of y in V_0^f. Consider two subcases.

Subcase 2.2.1 x and y have private neighbors x_1 and y_1 such that $x_1y_1 \notin E(G)$. Let G' be obtained from G by subdividing the edges xx_1 and yy_1 with vertices x' and y', respectively. A simple case checking similar to that given in Case 1 shows that $\gamma_R(G') \geq 5$.

Thus we may assume each private neighbor of x is adjacent to every private neighbor of y.

Subcase 2.2.2 x has two private neighbors x_1 and x_2 which are not adjacent.
Assume y has two private neighbors y_1 and y_2 which are not adjacent. Let G_1 be obtained from G by subdividing the edges xx_1 and yy_1. It is straightforward to see that $\gamma_R(G_1) \geq 5$. Therefore, we may assume that every pair of private neighbors of y are adjacent. Since $\gamma_R(G) = 4$, no vertex of V_0^f is adjacent to all vertices in V_0^f. Hence, no private neighbor of y is adjacent to all vertices in $N(x) \cap N(y)$. Let y_1 be a private neighbor of y and $z \in N(x) \cap N(y)$ such that $y_1z \notin E(G)$. Let G_2 be obtained from G by subdividing the edges xz and yy_1. It is easy to see that $\gamma_R(G_2) \geq 5$.

Therefore we may assume the subgraph induced by private neighbors of x and private neighbors of y is a complete graph. If $N(x) \cap N(y) \cap V_0^f = \emptyset$, then $G[V_0^f]$ is a complete graph which forces $\gamma_R(G) = 3$, a contradiction. Therefore we assume $z \in N(x) \cap N(y) \cap V_0^f \neq \emptyset$. Assume z is not adjacent to x_1, a private neighbor of x (the case z is not adjacent to a private neighbor of y is similar). Let G_3 be obtained from G by subdividing the edges yz and xx_1. Then $\gamma_R(G_3) \geq 5$. Finally, if every vertex in $N(x) \cap N(y) \cap V_0^f$ is adjacent to all private neighbors of x and y, then $G[V_0^f]$ is a complete graph and $\gamma_R(G) = 3$, a contradiction.

In order to prove that the bound is sharp, let G be the cartesian product $K_m \square P_2$, $m \geq 3$. Obviously, $\gamma_R(G) = 4$. It is easy to see that $\mathrm{sd}_{\gamma_R(G)} = 2$. This completes the proof. □

By Theorems C and 1 we have:

Corollary 2. Let G be a simple connected graph of order $n \geq 3$. If $\mathrm{sd}_{\gamma_R}(G) > 2$, then $\gamma_R(G) \geq 5$.

Theorem 3. If G and $\overline{G}$ are n-vertex graphs with $\gamma_R(G), \gamma_R(\overline{G}) \geq 5$, then G (respectively, $\overline{G}$) have a matching of size at least $\lfloor \frac{\gamma_R(G)}{2} \rfloor + 1$

(respectively, $\lfloor\frac{\gamma_R(\overline{G})}{2}\rfloor + 1$).

Proof. Since $\gamma_R(G), \gamma_R(\overline{G}) \geq 5$, G and $\overline{G}$ are connected. We consider two cases.

Case 1 For every $\gamma_R(G)$-function f for which $|V_1^f|$ is minimum, $V_1^f \neq \emptyset$. Obviously, $V_2^f \neq \emptyset$. Assume $V_1^f = \{v_1, \ldots, v_k\}$ and $V_2^f = \{u_1, \ldots, u_m\}$. By the choice of f and Theorem B, V_1^f is an independent set and $N(v_i) \cap N(v_j) \cap V_0^f = \emptyset$ for $i \neq j$. Let G' be obtained from G by removing $\deg(v_i) - 1$ edges at v_i for $1 \leq i \leq k$, all the edges at u_j which have one endpoint in $\cup_{i=1}^{j-1} N(u_i)$ for $j = 2, \ldots, m$ and the edges whose endpoints are both in V_0^f or both in V_2^f (see Figure 1). (Note that G' is not unique.) Let G_i be the connected component of G' containing u_i. It is straightforward to see that $\gamma_R(G) = \gamma_R(G') = \sum_{i=1}^m \gamma_R(G_i)$ and $\alpha'(G) \geq \alpha'(G') = \sum_{i=1}^m \alpha'(G_i)$, where α' denotes the matching number. Now we distinguish two subcases for each i.

Subcase 1.1 u_i has a private neighbor w_i in V_0^f, which is not adjacent to the vertices of $V_1^f \cap V(G_i)$. Let $M_i \in E(G_i)$ be the set consisting of $u_i w_i$ and all edges of G_i with one endpoint in $V_1^f \cap V(G_i)$. Obviously, M_i is a matching of G_i. Since $|V_2^f \cap V(G_i)| = 1$, we have $|M_i| = |V_1^f \cap V(G_i)| + 1$ and $\gamma_R(G_i) = 2 + |V_1^f \cap V(G_i)|$. Hence,

$$\alpha'(G_i) \geq |M_i| = \gamma_R(G_i) - 1 = \frac{\gamma_R(G_i)}{2} + \frac{|V_1^f \cap V(G_i)|}{2}.$$

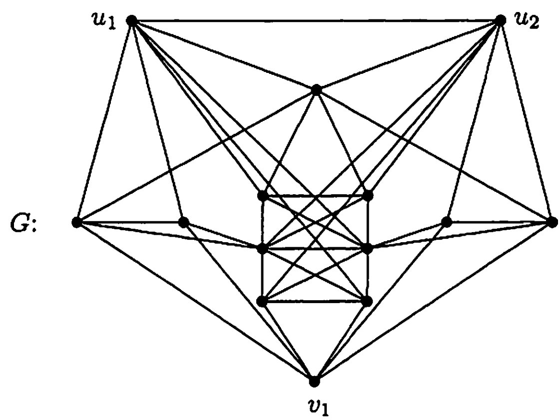
subcase 1.2 All private neighbors of u_i in V_0^f are adjacent to some vertices in $V_1^f \cap V(G_i)$. We claim that u_i has at least three private neighbors in V_0^f. First assume w is the only private neighbor of u_i. By assumptions, w has a neighbor w' in $V_1^f \cap V(G_i)$. Then the ordered partition $((V_0^f - \{w\}) \cup \{u_i, w'\}, V_1^f - \{w'\}, (V_2^f - \{u_i\}) \cup \{w\})$ defines a RDF of G of size less than $\gamma_R(G)$, a contradiction. Now assume w_1 and w_2 are the only private neighbors of u_i. By assumptions, w_1 (respectively, w_2) has a neighbor in $V_1^f \cap V(G_i)$, say w_1' (respectively, w_2'). Then the ordered partition $g = ((V_0^f - \{w_1, w_2\}) \cup \{u_i, w_1', w_2'\}, V_1^f - \{w_1', w_2'\}, (V_2^f - \{u_i\}) \cup \{w_1, w_2\})$ defines a $\gamma_R(G)$ with $|V_1^g| < |V_1^f|$, a contradiction. Therefore, u_i has at least three private neighbors in V_0^f. This forces that $\gamma_R(G_i) \geq 5$.

Let $M_i \in E(G_i)$ be the set consisting of all edges of G_i with one endpoint in $V_1^f \cap V(G_i)$. Since $|M_i| = |V(G_i) \cap V_1^f|$ and $\gamma_R(G_i) = 2 + |V(G_i) \cap V_1^f|$,

$$\alpha'(G_i) \geq |M_i| = \gamma_R(G_i) - 2 \geq \frac{\gamma_R(G_i)}{2} + \frac{1}{2}.$$

Now $M = \cup_{i=1}^{m} M_i$ is a matching of G and it is easy to see that

$$\alpha'(G) \geq |M| = \sum_{i=1}^{m} |M_i| \geq \lfloor \frac{\gamma_R(G)}{2} \rfloor + 1.$$



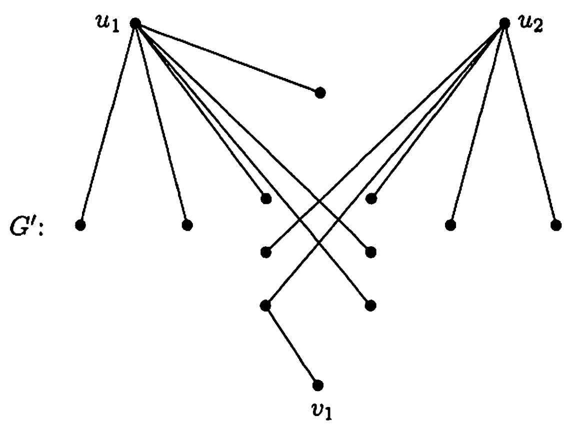


Figure 1: The graphs G and G' in Case 1: $V_1^f = \{v_1\}$ and $V_2^f = \{u_1, u_2\}$

Case 2 There exists a $\gamma_R(G)$-function f such that $V_1^f = \emptyset$. Then $\gamma_R(G) \geq 6$ and so $|V_2^f = \{u_1, \ldots, u_k\}| \geq 3$. Assume $v_i \in V_0^f$ is a private neighbor of u_i for each $1 \leq i \leq k$. First assume V_0^f is an independent set. Let $u, v \in V_2^f$ and let u_0 (respectively, v_0) be a private neighbor of u (respectively, v) in V_0^f. Obviously, $\{u_0, v_0\}$ is a dominating set for $\overline{G}$, which implies that $\gamma_R(\overline{G}) \leq 4$, a contradiction. Now assume V_0^f is not an independent set. Let $u_0, v_0 \in V_0^f$ and $u_0v_0 \in E(G)$. We consider two subcases.

subcase 2.1 u_0 and v_0 have distinct neighbors in V_2^f. Without loss of generality we assume $u_1u_0, u_2v_0 \in E(G)$. If $N(u_1) \cap V_0^f = \{u_0\}$ (the case $N(u_2) \cap V_0^f = \{v_0\}$ is similar), then u_0 is the private neighbor of u_1 in V_0^f and so the ordered partition $(V(G) - \{u_0, u_1\}, \emptyset, \{u_0, u_1\})$ defines a RDF of $\overline{G}$ of size 4, a contradiction. Let $\{u_0\} \subsetneqq N(u_1) \cap V_0^f$ and $\{v_0\} \subsetneqq N(u_2) \cap V_0^f$. If u_1, u_2 have distinct neighbors w_1, w_2 in $V_0^f - \{u_0, v_0\}$, respectively, then $M = \{u_iv_i \mid 3 \leq i \leq k\} \cup \{u_0v_0, u_1w_1, u_2w_2\}$ is a matching in G of size $\lfloor \frac{\gamma_R(G)}{2} \rfloor + 1$, as desired. If $u_1v_0, u_2u_0 \in E(G)$, then obviously u_1 and u_2 have distinct neighbors in $V_0^f - \{u_0, v_0\}$ and the result follows as before. Thus we may assume $u_1v_0 \notin E(G)$ or $u_2u_0 \notin E(G)$. Consider two subcases.

Subcase 2.2.1 $u_1v_0 \in E(G)$ and $u_2u_0 \notin E(G)$ (the case $u_1v_0 \notin E(G)$ and $u_2u_0 \in E(G)$ is similar). Then u_2 has a private neighbor w in V_0^f. If u_1 has a neighbor in $V_0^f - \{u_0, v_0\}$, then u_1 and u_2 have distinct neighbors in $V_0^f - \{u_0, v_0\}$ and the result follows as before. So let $N(u_1) \cap V_0^f = \{u_0, v_0\}$. If w is the only neighbor of u_2 in $V_0^f - \{u_0, v_0\}$, then the ordered partition $((V_0^f - \{v_0, w\}) \cup \{u_1, u_2\}, \{w\}, (V_2^f - \{u_1, u_2\}) \cup \{v_0\})$ defines a RDF of G of size less than $\gamma_R(G)$, a contradiction. Therefore we assume u_2 has at least two neighbors in $V_0^f - \{u_0, v_0\}$. If u_2 has two adjacent neighbors w_1, w_2 in $V_0^f - \{u_0, v_0\}$, then $M = \{u_iv_i \mid 3 \leq i \leq k\} \cup \{w_1w_2, u_1u_0, u_2v_0\}$ is a matching of G of size $\lfloor \frac{\gamma_R(G)}{2} \rfloor + 1$, as desired. Assume now $N(u_2) \cap (V_0^f - \{u_0, v_0\})$ is an independent set. If u_2 has a private neighbor w in $V_0^f - \{u_0, v_0\}$ such that $wv_0 \notin E(G)$, then $\{u_2, w\}$ is a dominating set for $\overline{G}$, hence $\gamma_R(\overline{G}) \leq 4$, a contradiction. Thus we may assume v_0 is adjacent to all private neighbors of u_2. Then the ordered partition $((V_0^f - \{v_0\}) \cup \{u_1, u_2\}, \emptyset, (V_2^f - \{u_1, u_2\}) \cup \{v_0\})$ defines a RDF of G of size less than $\gamma_R(G)$, a contradiction.

Subcase 2.2.2 $u_1v_0, u_2u_0 \notin E(G)$. Using an argument similar to that described in the first part of Case 2, we may assume $N(u_1) \cap (V_0^f - \{u_0, v_0\}) = N(u_2) \cap (V_0^f - \{u_0, v_0\}) = \{z\}$. If z has another neighbor in V_2^f, then the ordered partition $((V_0^f - \{u_0\}) \cup \{u_1\}, \{u_2\}, (V_2^f - \{u_1, u_2\}) \cup \{u_0\})$ defines a RDF of G of size less than $\gamma_R(G)$, a contradiction. Now assume $N(z) \cap V_2^f = \{u_1, u_2\}$. If $u_1u_2 \in E(G)$, then the ordered partition $((V_0^f - \{u_0\}) \cup \{u_1\}, \{u_0\}, V_2^f - \{u_1\})$ defines a RDF of G of size less than

$\gamma_R(G)$, a contradiction. Suppose that $u_1u_2 \notin E(G)$. It is easy to see that the assumption $zu_0 \in E(G)$ (respectively, $zv_0 \in E(G)$) leads to a contradiction. Thus we assume z is not adjacent to u_0 and v_0. Then obviously $\{u_1, z\}$ is a dominating set for $\overline{G}$, hence $\gamma_R(\overline{G}) \leq 4$, a contradiction.

subcase 2.2 u_0 and v_0 have precisely one common neighbor in V_2^f. Without loss of generality we may assume $u_1 = N(u_0) \cap N(v_0) \cap V_2^f$. If u_1 has a neighbor w in $V_0^f - \{u_0, v_0\}$, then $M = \{u_1w, u_0v_0\} \cup \{u_iv_i \mid 2 \leq i \leq k\}$ is a matching of G of size $\lfloor\frac{\gamma_R(G)}{2}\rfloor + 1$. Suppose that $N(u_1) \cap V_0^f = \{u_0, v_0\}$. If u_0 (respectively, v_0) has a neighbor w in $V_0^f - \{u_0, v_0\}$, then obviously u_0 and w (respectively, v_0 and w) have distinct neighbors in V_2^f and the result follows by Subcase 2.1. So we assume u_0 (respectively, v_0) does not have other neighbors in V_0^f. This implies that $\deg(u_0) = \deg(v_0) = 2$. Since $n \geq 5$, we have $k \geq 2$. Consider v_2, a private neighbor of u_2 in V_0^f. Obviously, the ordered partition $(V(G) - \{v_0, v_2\}, \emptyset, \{v_0, v_2\})$ defines a RDF for $\overline{G}$ of weight 4, a contradiction. This completes the proof. □

Theorem 4. Let G be a graph of order $n \geq 4$ such that G and $\overline{G}$ have connected components of order at least 3. Then

$$\mathrm{sd}_{\gamma_R}(G) + \mathrm{sd}_{\gamma_R}(\overline{G}) \leq \lfloor\frac{n}{2}\rfloor + 3.$$

Proof. First let G be disconnected (the case $\overline{G}$ is disconnected is similar). Then obviously $\overline{G}$ is connected and $2 \leq \gamma_R(\overline{G}) \leq 4$. By Theorems C and 1, $\mathrm{sd}_{\gamma_R}(\overline{G}) \leq 2$. Suppose that G_1 is a connected component of G of order more than 2. Then by Theorem E

$$\mathrm{sd}_{\gamma_R}(G) + \mathrm{sd}_{\gamma_R}(\overline{G}) \leq 1 + \lceil\frac{n-1}{2}\rceil \leq \lfloor\frac{n}{2}\rfloor + 2.$$

Now let G and $\overline{G}$ be connected. If $\gamma_R(G) \leq 4$ or $\gamma_R(\overline{G}) \leq 4$, then the result follows as before. Suppose now that $\gamma_R(G), \gamma_R(\overline{G}) \geq 5$. Then G and $\overline{G}$ are connected and $n \geq 5$. Therefore $G, \overline{G} \notin \{C_5, \frac{n}{2}K_2\}$. Hence, we have $\gamma_R(G) + \gamma_R(\overline{G}) \leq n + 2$ by Theorem A. Now by Theorems 3 and D

$$\begin{aligned}\mathrm{sd}_{\gamma_R}(G) + \mathrm{sd}_{\gamma_R}(\overline{G}) &\leq \lfloor\frac{\gamma_R(G)}{2}\rfloor + \lfloor\frac{\gamma_R(\overline{G})}{2}\rfloor + 2\\ &\leq \lfloor\frac{\gamma_R(G) + \gamma_R(\overline{G})}{2}\rfloor + 2\\ &\leq \lfloor\frac{n+2}{2}\rfloor + 2 \leq \lfloor\frac{n}{2}\rfloor + 3.\end{aligned}$$

This completes the proof. □

We conclude this paper with a result on the sum of the Roman domination subdivision number of the components of G of order at least 3.

Theorem 5. Let G be a simple disconnected graph of order $n \geq 4$ such that each of G and $\overline{G}$ has at least one component of order at least 3. Then

$$\sum_{i=1}^{k} \mathrm{sd}_{\gamma_R}(G_i) + \mathrm{sd}_{\gamma_R}(\overline{G}) \leq \begin{cases} \frac{n-k-r}{2} + 1 & \text{if } \delta = 1 \\ \frac{n-k-r}{2} + 2 & \text{if } \delta \geq 2, \end{cases}$$

where $G_1, \ldots, G_k$ are the connected components of G of order at least 3 and r is the number of even connected components of G of order at least 3. Furthermore, the bound is sharp when $\delta = 1, 2$.

Proof. Since G is disconnected, $\gamma_R(\overline{G}) \leq 3$ if $\delta(G) = 1$ and $\gamma_R(\overline{G}) \leq 4$ if $\delta(G) \geq 2$. By Theorems C and 1, $\mathrm{sd}_{\gamma_R}(\overline{G}) = 1$ if $\delta(G) = 1$ and $\mathrm{sd}_{\gamma_R}(\overline{G}) \leq 2$ if $\delta(G) \geq 2$. By Theorem E, each connected component G_i of G of order at least 3 satisfies $\mathrm{sd}_{\gamma_R}(G_i) \leq \lceil \frac{|V(G_i)|}{2} \rceil - 1$. Hence, $\mathrm{sd}_{\gamma_R}(G_i) \leq \frac{|V(G_i)| - 2}{2}$ if G_i is an even connected component and $\mathrm{sd}_{\gamma_R}(G_i) \leq \frac{|V(G_i)| - 1}{2}$ if G_i is an odd connected component. Thus,

$$\sum_{i=1}^{k} \mathrm{sd}_{\gamma_R}(G_i) + \mathrm{sd}_{\gamma_R}(\overline{G}) \leq \begin{cases} \frac{n-k-r}{2} + 1 & \text{if } \delta = 1 \\ \frac{n-k-r}{2} + 2 & \text{if } \delta \geq 2. \end{cases}$$

If G is the disjoint union of paths (respectively, cycles) of order 5, then the upper bound is achieved when $\delta = 1$ (respectively, $\delta = 2$). □

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