Super $(a, 1)$-cycle-antimagic labeling of the grid

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Abstract

Let G and F be graphs. If every edge of G belongs to a subgraph of G isomorphic to F, and there exists a bijection $\lambda : V(G) \bigcup E(G) \rightarrow \{1, 2, ..., |V(G)| + |E(G)|\}$ such that the set $\{ \sum_{v \in V(F')} \lambda(v) + \sum_{e \in E(F')} \lambda(e) : F' \cong F, F' \subseteq G\}$ form an arithmetic progression starting from a and having common difference d, then we say that G is (a, d)-F-antimagic. If, in addition, $\lambda(V(G)) = \{1, 2, ..., |V(G)|\}$, then G is super (a, d)-F-antimagic. In this paper, we prove that the grid (i.e. the Cartesian product of two nontrivial paths) is super $(a, 1)$-C_4-antimagic.

1 Introduction

We consider finite simple graphs. Let G be a graph. An *edge labeling* of G is a bijection from $E(G)$ to $\{1, 2, ..., |E(G)|\}$. A *vertex labeling* of G is a bijection from $V(G)$ to $\{1, 2, ..., |V(G)|\}$. A *total labeling* of G is a bijection from $V(G) \bigcup E(G)$ to $\{1, 2, ..., |V(G)| + |E(G)|\}$. Suppose λ is a total labeling of G, and H is a subgraph of G, we let $\sum \lambda(H) = \sum_{v \in V(H)} \lambda(v) + \sum_{e \in E(H)} \lambda(e)$.

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Let G and F be graphs. If every edge of G belongs to a subgraph of G isomorphic to F and there exists a total labeling λ of G such that for every subgraph F' of G which is isomorphic to F, the set $\{\sum \lambda(F') : F' \cong F, F' \subseteq G\}$ form an arithmetic progression starting from a and having common difference d, then we say that G is *(a, d)-F-antimagic.* Furthermore if $\lambda(V(G)) = \{1, 2, ..., |V(G)|\}$, then we say that G is super *(a, d)-F-antimagic*, and λ is a super *(a, d)-F-antimagic* labeling of G. Gutiérrez and Lladó [2] introduced the concept of F-magic labeling (that is, $(a, 0)$-F-antimagic labeling). Then in [3], the authors introduced the notion of (a, d)-F-antimagic labeling and studied some basic properties.
Let C_n denote a cycle on n vertices. The Cartesian product of graphs G_1 and G_2, denoted by $G_1 \square G_2$, is the graph with vertex set $V(G_1) \times V(G_2)$ such that a vertex (u, v) is adjacent to a vertex (u', v') if and only if either $u = u'$ and $vv' \in E(G_2)$ or $v = v'$ and $uu' \in E(G_1)$. We call $P_m \square P_n$ $(m, n \geq 2)$ a grid. In [7], A.A.G. Ngurah et al. proved that $P_m \square P_n$ is super $(a, 0)$-C_4-antimagic for $m \geq 2$, $n = 2, 3, 4, 5$. Furthermore, in [8], the authors proved that every grid $P_m \square P_n$ $(m, n \geq 2)$ is super $(a, 0)$-C_4-antimagic. The purpose of this paper is to show that every grid $P_m \square P_n$ $(m, n \geq 2)$ is super $(a, 1)$-C_4-antimagic.

2 Main Result

Suppose that $m, n \geq 2$ are integers. Let P_m be a path $u_1 \varepsilon_1 u_2 \varepsilon_2 u_3 ... u_{m-1} \varepsilon_{m-1} u_m$ where $u_1, u_2, u_3, ..., u_m \in V(P_m)$, and $\varepsilon_i = u_i u_{i+1} \in E(P_m)$ for $i = 1, 2, ..., m-1$, and P_n be a path $v_1 e_1 v_2 e_2 v_3 ... v_{n-1} e_{n-1} v_n$ where $v_1, v_2, v_3, ..., v_n \in V(P_n)$, and $e_j = v_j v_{j+1} \in E(P_n)$ for $j = 1, 2, ..., n-1$. Then in the grid $P_m \square P_n$ the following notations are used. The vertex (u_i, v_j) is denoted by $w_{i,j}$. The edge joining $w_{i,j}$ and $w_{i+1,j}$ is denoted by $\varepsilon_{i,j}$. Thus $\varepsilon_{i,j}$ is the edge corresponding to the edge ε_i in P_m and the vertex v_j in P_n. The edge joining $w_{i,j}$ and $w_{i,j+1}$ is denoted by $e_{i,j}$. Thus $e_{i,j}$ is the edge corresponding to the vertex u_i in P_m and the edge e_j in P_n. For $i = 1, 2, ..., m-1$, $j = 1, 2, ..., n-1$, the cycle $w_{i,j} w_{i+1,j} w_{i+1,j+1} w_{i,j+1} w_{i,j}$ is denoted by $C_{i,j}$. Thus $C_{i,j}$ is the cycle consisting of the edges $\varepsilon_{i,j}$, $\varepsilon_{i,j+1}$, $e_{i,j}$ and $e_{i+1,j}$.
We begin with some definitions, remarks and lemmas. Suppose that g is an edge labeling of a graph G, and H is a subgraph of G. We let $\sum g(H) = \sum_{e \in E(H)} g(e)$. The following remark is trivial.

Remark 2.1 *Let g be an edge labeling of $P_m \square P_n$ $(m, n \geq 2)$. Then*

(1) $\sum g(C_{i,j}) - \sum g(C_{i+1,j}) = (g(\varepsilon_{i,j}) - g(\varepsilon_{i+1,j})) + (g(\varepsilon_{i,j+1}) - g(\varepsilon_{i+1,j+1})) + (g(e_{i,j}) - g(e_{i+2,j}))$, *for* $1 \leq i \leq m-2$, $1 \leq j \leq n-1$, *and*

(2) $\sum g(C_{i,j}) - \sum g(C_{i,j+1}) = (g(\varepsilon_{i,j}) - g(\varepsilon_{i,j+2})) + (g(e_{i,j}) - g(e_{i,j+1})) + (g(e_{i+1,j}) - g(e_{i+1,j+1}))$, *for* $1 \le i \le m-1$, $1 \le j \le n-2$.

Lemma 2.2 *There exists an edge labeling* g *of* $P_m \square P_n$ $(m, n \ge 2)$ *such that*
(1) $\sum g(C_{i,j}) = \sum g(C_{i+1,j}) - 1$ *for* $1 \le i \le m-2$, $1 \le j \le n-1$, *and*
(2) $\sum g(C_{i,j}) = \sum g(C_{i,j+1}) + 1 - m$ *for* $1 \le i \le m-1$, $1 \le j \le n-2$.

Proof. We distinguish three cases according to the parity of m and n.
Case 1. m and n are odd.
A required edge labeling of $P_5 \square P_7$ is given in Fig. 1.

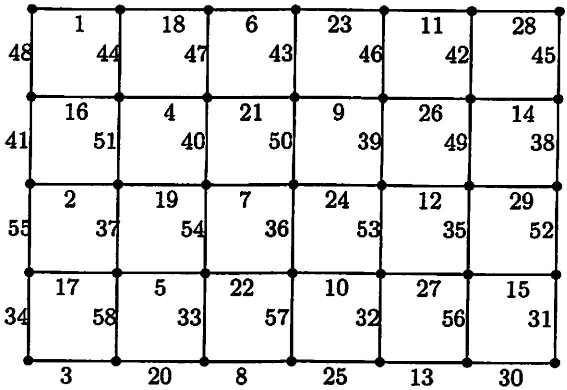

Fig. 1. An edge labeling of $P_5 \square P_7$

In general, we consider the edge labeling of $P_m \square P_n$ defined as follows:
For $1 \le i \le m$, $1 \le j \le n-1$, let

$$g(e_{i,j}) = \begin{cases} \frac{j-1}{2}m + \frac{i+1}{2}, & \text{if } i+j \equiv 0 (\text{mod } 2); \\ \frac{n+j-2}{2}m + \frac{i}{2}, & \text{if } i+j \equiv 1 (\text{mod } 2). \end{cases}$$

For $1 \le i \le m-1$, $1 \le j \le n$, let

$$g(\varepsilon_{i,j}) = \begin{cases} (n-1)m + \frac{m+i-1}{2}n - \frac{j-2}{2}, & \text{if } i+j \equiv 0 (\text{mod } 2); \\ (n-1)m + \frac{m-i}{2}n - \frac{j-2}{2}, & \text{if } i+j \equiv 1 (\text{mod } 2). \end{cases}$$

We first show that $\sum g(C_{i,j}) = \sum g(C_{i+1,j}) - 1$ for $1 \le i \le m-2$, $1 \le j \le n-1$. It is not difficult to see that $g(e_{i,j}) - g(e_{i+2,j}) = -1$ and

$$g(\varepsilon_{i,j}) - g(\varepsilon_{i+1,j}) = (-1)^{i+j} in.$$

Then

$$\begin{aligned} & \sum g(C_{i,j}) - \sum g(C_{i+1,j}) \\ = \; & (g(\varepsilon_{i,j}) - g(\varepsilon_{i+1,j})) + (g(\varepsilon_{i,j+1}) - g(\varepsilon_{i+1,j+1})) + (g(e_{i,j}) - g(e_{i+2,j})) \\ = \; & (-1)^{i+j} in + (-1)^{i+j+1} in + (-1) \\ = \; & -1. \end{aligned}$$

Thus, condition (1) holds for the edge labeling g.

Next, we show that $\sum g(C_{i,j}) = \sum g(C_{i,j+1}) + 1 - m$ for $1 \le i \le m-1$, $1 \le j \le n-2$. It is not difficult to see that $g(\varepsilon_{i,j}) - g(\varepsilon_{i,j+2}) = 1$ and

$$g(e_{i,j}) - g(e_{i,j+1}) = ((-1)^{i+j} - 1)m/2 + (-1)^{i+j+1}(mn-1)/2.$$

Then

$$\begin{aligned}
& \sum g(C_{i,j}) - \sum g(C_{i,j+1}) \\
= \; & (g(\varepsilon_{i,j}) - g(\varepsilon_{i,j+2})) + (g(e_{i,j}) - g(e_{i,j+1})) + (g(e_{i+1,j}) - g(e_{i+1,j+1})) \\
= \; & 1 + ((-1)^{i+j} - 1)m/2 + (-1)^{i+j+1}(mn-1)/2 \\
& + ((-1)^{i+j+1} - 1)m/2 + (-1)^{i+j+2}(mn-1)/2 \\
= \; & 1 - m.
\end{aligned}$$

Thus, condition (2) also holds for the edge labeling g.

Case 2. m is odd, n is even or m is even, n is odd.

Without loss of generality, assume that n is even. A required edge labeling of $P_5 \square P_6$ is given in Fig. 2.

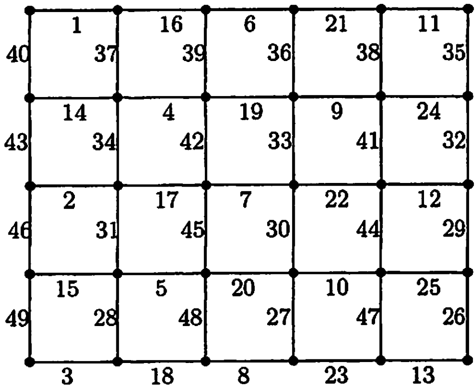

Fig. 2. An edge labeling of $P_5 \square P_6$

In general, we consider the edge labeling of $P_m \square P_n$ defined as follows:

For $1 \le i \le m$, $1 \le j \le n-1$, let

$$g(e_{i,j}) = \begin{cases} \frac{j-1}{2}m + \frac{i+1}{2}, & \text{if } i+j \equiv 0 (\text{mod } 2); \\ \frac{n+j-2}{2}m + \frac{i+1}{2}, & \text{if } i+j \equiv 1 (\text{mod } 2). \end{cases}$$

For $1 \le i \le m-1$, $1 \le j \le n$, let

$$g(\varepsilon_{i,j}) = \begin{cases} (n-1)m + (\frac{m-i}{2})n + \frac{2-j}{2}, & \text{if } j \text{ is even}; \\ (n-1)m + (\frac{m+i-1}{2})n + \frac{1-j}{2}, & \text{if } j \text{ is odd}. \end{cases}$$

First, we show that $\sum g(C_{i,j}) = \sum g(C_{i+1,j}) - 1$ for $1 \le i \le m-2$, $1 \le j \le n-1$. It is not difficult to see that $g(e_{i,j}) - g(e_{i+2,j}) = -1$ and

$$g(\varepsilon_{i,j}) - g(\varepsilon_{i+1,j}) = (-1)^j n/2.$$

Then

$$
\begin{aligned}
& \sum g(C_{i,j}) - \sum g(C_{i+1,j}) \\
= \ & (g(\varepsilon_{i,j}) - g(\varepsilon_{i+1,j})) + (g(\varepsilon_{i,j+1}) - g(\varepsilon_{i+1,j+1})) + (g(e_{i,j}) - g(e_{i+2,j})) \\
= \ & (-1)^j n/2 + (-1)^{j+1} n/2 + (-1) \\
= \ & -1.
\end{aligned}
$$

Thus, condition (1) holds for the edge labeling g.
Next, we show that $\sum g(C_{i,j}) = \sum g(C_{i,j+1}) + 1 - m$ for $1 \le i \le m-1$, $1 \le j \le n-2$. It is not difficult to see that $g(\varepsilon_{i,j}) - g(\varepsilon_{i,j+2}) = 1$ and $g(e_{i,j}) - g(e_{i,j+1}) = ((-1)^{i+j} - 1)m/2 + (-1)^{i+j+1} mn/2.$
Then

$$
\begin{aligned}
& \sum g(C_{i,j}) - \sum g(C_{i,j+1}) \\
= \ & (g(\varepsilon_{i,j}) - g(\varepsilon_{i,j+2})) + (g(e_{i,j}) - g(e_{i,j+1})) + (g(e_{i+1,j}) - g(e_{i+1,j+1})) \\
= \ & 1 + ((-1)^{i+j} - 1)m/2 + (-1)^{i+j+1} mn/2 \\
& +((-1)^{i+j+1} - 1)m/2 + (-1)^{i+j+2} mn/2 \\
= \ & 1 - m.
\end{aligned}
$$

Thus, condition (2) also holds for the edge labeling g.

Case 3. m and n are even.
A required edge labeling of $P_4 \square P_6$ is given in Fig. 3.

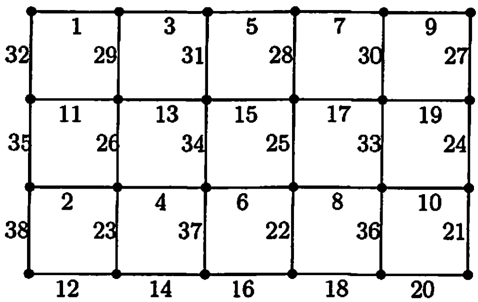

Fig. 3. An edge labeling of $P_4 \square P_6$

In general, we consider the edge labeling of $P_m \square P_n$ defined as follows:
For $1 \le i \le m$, $1 \le j \le n-1$, let

$$
g(e_{i,j}) = \begin{cases} \frac{n+j-2}{2} m + \frac{i}{2}, & \text{if } i \text{ is even;} \\ \frac{j-1}{2} m + \frac{i+1}{2}, & \text{if } i \text{ is odd.} \end{cases}
$$

For $1 \le i \le m-1$, $1 \le j \le n$, let

$$
g(\varepsilon_{i,j}) = \begin{cases} (n-1)m + (\frac{m-i}{2})n + \frac{2-j}{2}, & \text{if } j \text{ is even;} \\ (n-1)m + (\frac{m+i-1}{2})n + \frac{1-j}{2}, & \text{if } j \text{ is odd.} \end{cases}
$$

First, we show that $\sum g(C_{i,j}) = \sum g(C_{i+1,j}) - 1$ for $1 \le i \le m-2$, $1 \le j \le n-1$. It is not difficult to see that $g(e_{i,j}) - g(e_{i+2,j}) = -1$ and

$$g(\varepsilon_{i,j}) - g(\varepsilon_{i+1,j}) = (-1)^j n/2.$$

Then

$$\begin{aligned}
& \sum g(C_{i,j}) - \sum g(C_{i+1,j}) \\
= \ & (g(\varepsilon_{i,j}) - g(\varepsilon_{i+1,j})) + (g(\varepsilon_{i,j+1}) - g(\varepsilon_{i+1,j+1})) + (g(e_{i,j}) - g(e_{i+2,j})) \\
= \ & (-1)^j n/2 + (-1)^{j+1} n/2 + (-1) \\
= \ & -1.
\end{aligned}$$

Thus, condition (1) holds for the edge labeling g.
Next, we show that $\sum g(C_{i,j}) = \sum g(C_{i,j+1}) + 1 - m$ for $1 \le i \le m-1$, $1 \le j \le n-2$. It is not difficult to see that $g(e_{i,j}) - g(e_{i,j+1}) = -m/2$ and $g(\varepsilon_{i,j}) - g(\varepsilon_{i,j+2}) = 1$.
Then

$$\begin{aligned}
& \sum g(C_{i,j}) - \sum g(C_{i,j+1}) \\
= \ & (g(\varepsilon_{i,j}) - g(\varepsilon_{i,j+2})) + (g(e_{i,j}) - g(e_{i,j+1})) + (g(e_{i+1,j}) - g(e_{i+1,j+1})) \\
= \ & 1 - m/2 - m/2 \\
= \ & 1 - m.
\end{aligned}$$

Thus, condition (2) also holds for the edge labeling g. This completes the proof. □

Suppose that f is a vertex labeling of a graph G, and H is a subgraph of G. We let $\sum f(H) = \sum_{v \in V(H)} f(v)$. The following remark is trivial.

Remark 2.3 *Let f be a vertex labeling of $P_m \square P_n$ $(m, n \ge 2)$. Then*

(1) $\sum f(C_{i,j}) - \sum f(C_{i+1,j}) = (f(w_{i,j}) - f(w_{i+2,j})) + (f(w_{i,j+1}) - f(w_{i+2,j+1}))$ *for* $1 \le i \le m-2$, $1 \le j \le n-1$, *and*

(2) $\sum f(C_{i,j}) - \sum f(C_{i,j+1}) = (f(w_{i,j}) - f(w_{i,j+2})) + (f(w_{i+1,j}) - f(w_{i+1,j+2}))$ *for* $1 \le i \le m-1$, $1 \le j \le n-2$.

Lemma 2.4 *There exists a vertex labeling f of $P_m \square P_n$ $(m, n \ge 2)$ such that*
(1) $\sum f(C_{i,j}) = \sum f(C_{i+1,j}) + 2$ *for* $1 \le i \le m-2$, $1 \le j \le n-1$, *and*
(2) $\sum f(C_{i,j}) = \sum f(C_{i,j+1})$ *for* $1 \le i \le m-1$, $1 \le j \le n-2$.

Proof. We distinguish two cases according to the parity of m and n.

Case 1. m and n are even.
A required vertex labeling of $P_4 \square P_6$ is given in Fig. 4.

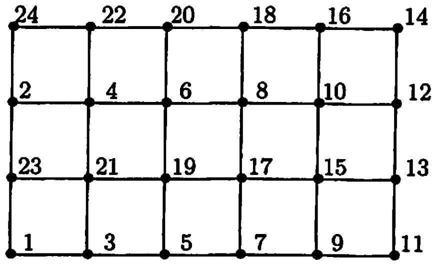

Fig. 4. A vertex labeling of $P_4 \Box P_6$

In general, we consider the vertex labeling of $P_m \Box P_n$ defined as follows: For $1 \le i \le m,\ 1 \le j \le n$, let

$$f(w_{i,j}) = \begin{cases} \frac{j}{2}m + \frac{2-i}{2}, & \text{if } i \text{ is even;} \\ mn + (\frac{1-j}{2})m + \frac{1-i}{2}, & \text{if } i \text{ is odd.} \end{cases}$$

First we show that $\sum f(C_{i,j}) = \sum f(C_{i+1,j}) + 2$ for $1 \le i \le m-2$, $1 \le j \le n-1$. Notice that $f(w_{i,j}) - f(w_{i+2,j}) = 1$.

Then

$$\begin{aligned} & \sum f(C_{i,j}) - \sum f(C_{i+1,j}) \\ = & (f(w_{i,j}) - f(w_{i+2,j})) + (f(w_{i,j+1}) - f(w_{i+2,j+1})) \\ = & 2. \end{aligned}$$

Thus, condition (1) holds for the vertex labeling f.

Next, we show that $\sum f(C_{i,j}) = \sum f(C_{i,j+1})$ for $1 \le i \le m-1$, $1 \le j \le n-2$. Notice that $f(w_{i,j}) - f(w_{i,j+2}) = (-1)^{i+1}m$.

Then

$$\begin{aligned} & \sum f(C_{i,j}) - \sum f(C_{i,j+1}) \\ = & (f(w_{i,j}) - f(w_{i,j+2})) + (f(w_{i+1,j}) - f(w_{i+1,j+2})) \\ = & (-1)^{i+1}m + (-1)^{i+2}m \\ = & 0. \end{aligned}$$

Thus, condition (2) also holds for the vertex labeling f.

Case 2. m or n is odd.

Without loss of generality, assume that m is odd. A required vertex labeling of $P_5 \Box P_7$ is given in Fig. 5.

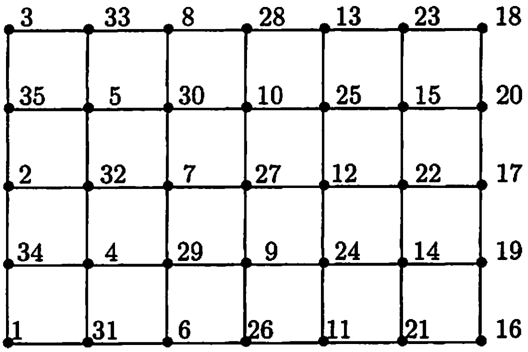

Fig. 5. A vertex labeling of $P_5 \Box P_7$

In general, we consider the vertex labeling of $P_m \Box P_n$ defined as follows: For $1 \le i \le m$, $1 \le j \le n$, let

$$f(w_{i,j}) = \begin{cases} \frac{j}{2}m + \frac{2-i}{2}, & \text{if } i+j \equiv 0 (\text{mod } 2); \\ mn + (\frac{1-j}{2})m + \frac{2-i}{2}, & \text{if } i+j \equiv 1 (\text{mod } 2). \end{cases}$$

First we show that $\sum f(C_{i,j}) = \sum f(C_{i+1,j}) + 2$ for $1 \le i \le m-2$, $1 \le j \le n-1$. Notice that $f(w_{i,j}) - f(w_{i+2,j}) = 1$.
Then

$$\begin{aligned} & \sum f(C_{i,j}) - \sum f(C_{i+1,j}) \\ = & (f(w_{i,j}) - f(w_{i+2,j})) + (f(w_{i,j+1}) - f(w_{i+2,j+1})) \\ = & 2. \end{aligned}$$

Thus, condition (1) holds for the vertex labeling f.
Next, we show that $\sum f(C_{i,j}) = \sum f(C_{i,j+1})$ for $1 \le i \le m-1$, $1 \le j \le n-2$. Notice that $f(w_{i,j}) - f(w_{i,j+2}) = (-1)^{i+j+1}m$.
Then

$$\begin{aligned} & \sum f(C_{i,j}) - \sum f(C_{i,j+1}) \\ = & (f(w_{i,j}) - f(w_{i,j+2})) + (f(w_{i+1,j}) - f(w_{i+1,j+2})) \\ = & (-1)^{i+j+1}m + (-1)^{i+j+2}m \\ = & 0. \end{aligned}$$

Thus, condition (2) also holds for the vertex labeling f. This completes the proof. □

The main result of this section follows from Lemma 2.2 and Lemma 2.4.

Theorem 2.5 *The grid $P_m \Box P_n$ $(m, n \ge 2)$ is super $(a, 1)$-C_4-antimagic.*

Proof. By Lemma 2.2, there exists an edge labeling g of $P_m \Box P_n$ such that

$$\sum g(C_{i,j}) = \sum g(C_{i+1,j}) - 1 \quad \text{for } 1 \leq i \leq m-2,\, 1 \leq j \leq n-1, \text{ and}$$
$$\sum g(C_{i,j}) = \sum g(C_{i,j+1}) + 1 - m \quad \text{for } 1 \leq i \leq m-1,\, 1 \leq j \leq n-2.$$

By Lemma 2.4, there exists a vertex labeling f of $P_m \Box P_n$ such that

$$\sum f(C_{i,j}) = \sum f(C_{i+1,j}) + 2 \quad \text{for } 1 \leq i \leq m-2,\, 1 \leq j \leq n-1, \text{ and}$$
$$\sum f(C_{i,j}) = \sum f(C_{i,j+1}) \quad \text{for } 1 \leq i \leq m-1,\, 1 \leq j \leq n-2.$$

We consider the total labeling of $P_m \Box P_n$ defined by

$$\lambda(x) = \begin{cases} f(x), & \text{if } x \in V(P_m \Box P_n); \\ mn + g(x), & \text{if } x \in E(P_m \Box P_n). \end{cases}$$

Notice that $\sum \lambda(C_{i,j}) = \sum f(C_{i,j}) + 4mn + \sum g(C_{i,j})$.
Thus, by the first part of Lemmas 2.2 and 2.4, we get
$\sum \lambda(C_{i,j}) = \sum f(C_{i+1,j}) + 4mn + \sum g(C_{i+1,j}) + 1$, that is

$$\sum \lambda(C_{i,j}) = \sum \lambda(C_{i+1,j}) + 1. \tag{1}$$

Similarly, using the second part of Lemmas 2.2 and 2.4, we get

$$\sum \lambda(C_{i,j}) = \sum \lambda(C_{i,j+1}) + 1 - m. \tag{2}$$

Thus, combining equalities (1) and (2), it follows that,

$$\begin{aligned} & \sum \lambda(C_{1,j}) \\ = & \sum \lambda(C_{2,j}) + 1 \\ = & \cdots \\ = & \sum \lambda(C_{i,j}) + i - 1 \\ = & \cdots \\ = & \sum \lambda(C_{m-1,j}) + m - 2 \\ = & [\sum \lambda(C_{m-1,j+1}) + 1 - m] + m - 2 \\ = & \sum \lambda(C_{m-1,j+1}) - 1 \quad \text{for } 1 \leq j \leq n-2. \end{aligned}$$

Hence the set $\{\sum \lambda(C_{i,j}) : 1 \leq i \leq m-1, 1 \leq j \leq n-1\} = \{\sum \lambda(C_{m-1,1}),$ $\sum \lambda(C_{m-2,1}), \cdots, \sum \lambda(C_{1,1}), \sum \lambda(C_{m-1,2}), \sum \lambda(C_{m-2,2}), \cdots, \sum \lambda(C_{1,2}),$ $\sum \lambda(C_{m-1,3}), \sum \lambda(C_{m-2,3}), \cdots, \sum \lambda(C_{1,3}), \cdots\cdots\cdots\cdots, \sum \lambda(C_{m-1,n-1}),$ $\sum \lambda(C_{m-2,n-1}), \cdots, \sum \lambda(C_{1,n-1})\}$ constitutes an arithmetic progression of difference 1. Therefore, the graph $P_m \Box P_n$ $(m, n \geq 2)$ is super $(a, 1)$-C_4-antimagic. □

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