On vertex irregular total labelings*

Ali Ahmad[1] and Martin Bača[1,2]

[1] Abdus Salam School of Mathematical Sciences, GC University
68-B, New Muslim Town, Lahore, Pakistan
ahmadsms@gmail.com

[2] Department of Appl. Mathematics, Technical University
Letná 9, 042 00 Košice, Slovak Republic
martin.baca@tuke.sk

Abstract

A vertex irregular total labeling σ of a graph G is a labeling of vertices and edges of G with labels from the set $\{1, 2, \dots, k\}$ in such a way that for any two different vertices x and y their weights $wt(x)$ and $wt(y)$ are distinct. The weight $wt(x)$ of a vertex x in G is the sum of its label and the labels of all edges incident with a given vertex x. The minimum k for which the graph G has a vertex irregular total labeling is called the *total vertex irregularity strength* of G. In this paper, we study the total vertex irregularity strength for two families of graphs, namely Jahangir graphs and circulant graphs.

Keywords : vertex irregular total labeling, total vertex irregularity strength, Jahangir graph, circulant graph.

1 Introduction

As a standard notation, assume that $G = G(V, E)$ is a finite, simple and undirected graph with p vertices and q edges. A labeling of a graph is any mapping that sends some set of graph elements to a set of numbers

*The work was supported by Higher Education Commission Pakistan Grant HEC(FD)/2007/555 and Slovak VEGA Grant 1/4005/07.

(usually positive integers). If the domain is the vertex-set or the edge-set, the labelings are called respectively vertex-labelings or edge-labelings. If the domain is $V \cup E$ then we call the labeling a total labeling. In many cases it is interesting to consider the sum of all labels associated with a graph element. This will be called the *weight* of element.

Chartrand *et al.* in [4] introduced edge-labelings of a graph G with positive integers such that the sum of the labels of edges incident with a vertex is different for all the vertices in G. Such labelings were called *irregular assignments.* What is the minimum value of the largest label over all such irregular assignments? This parameter of a graph G is well known as the *irregularity strength* of the graph G, $s(G)$.

The irregularity strength $s(G)$ can be interpreted as the smallest integer k for which G can be turned into a multigraph G' by replacing each edge by a set of at most k parallel edges, such that the degrees of the vertices in G' are all different.

Finding the irregularity strength of a graph seems to be hard even for simple graphs, see [3, 5, 6, 8, 9].

Motivated by this research and by total labelings mentioned in a book of Wallis [13], Bača *et al.* in [1] recently defined a vertex irregular total labelings of graphs. For a simple graph $G = (V, E)$ with vertex set V and edge set E, a labeling $\sigma : V \cup E \rightarrow \{1, 2, \dots, k\}$ is called *total $k-$labeling.* The associated vertex weight of a vertex $x \in V(G)$ under a total $k-$labeling σ is defined as

$$wt(x) = \sigma(x) + \sum_{y \in N(x)} \sigma(xy),$$

where $N(x)$ is the set of neighbors of x. A total k-labeling σ is defined to be a *vertex irregular total labeling* of a graph G if for every two different vertices x and y of G,

$$wt(x) \neq wt(y).$$

The minimum k for which a graph G has a vertex irregular total k-labeling is called the *total vertex irregularity strength* of G, $tvs(G)$.

It is easy to see that irregularity strength $s(G)$ of a graph G is defined only for graphs containing at most one isolated vertex and no connected component of order 2. On the other hand, the total vertex irregularity strength $tvs(G)$ is defined for every graph G.

If an edge labeling $\zeta : E \rightarrow \{1, 2, \dots, s(G)\}$ provide the irregularity strength $s(G)$, then we extend this labeling to total labeling σ in such a way

$$\begin{array}{ll} \sigma(xy) = \zeta(xy) & \text{for every } xy \in E(G), \\ \sigma(x) = 1 & \text{for every } x \in V(G). \end{array}$$

Thus, the total labeling σ is a vertex irregular total labeling and for graphs with no component of order ≤ 2 is $tvs(G) \leq s(G)$.

Nierhoff [11] proved that for all graphs G with no component of order at most 2 and $G \neq K_3$, the irregularity strength $s(G) \leq p-1$. From this result it follows that

$$tvs(G) \leq p - 1. \tag{1}$$

In this paper, we study properties of the vertex irregular total labelings and determine a value of the total vertex irregularity strength for Jahangir graphs and circulant graphs.

2 Known Results

The following theorem proved in [1], establishes lower and upper bound for the total vertex irregularity strength of a (p, q)-graph.

Theorem 1 [1] *Let G be a (p,q)-graph with minimum degree $\delta = \delta(G)$ and maximum degree $\Delta = \Delta(G)$. Then*

$$\left\lceil \frac{p+\delta}{\Delta+1} \right\rceil \leq tvs(G) \leq p + \Delta - 2\delta + 1. \tag{2}$$

If G is an r-regular (p, q)-graph then from Theorem 1 it follows:

$$\left\lceil \frac{p+r}{r+1} \right\rceil \leq tvs(G) \leq p - r + 1.$$

For a regular hamiltonian (p, q)-graph G, it was showed in [1] that $tvs(G) \leq \lceil \frac{p+2}{3} \rceil$. Thus for cycle C_p we have that $tvs(C_p) = \lceil \frac{p+2}{3} \rceil$.

In [1] is determined an exact value of the total vertex irregularity strength for the complete graph of order p ($tvs(K_p) = 2$), for the prism D_n, $n \geq 3$, ($tvs(D_n) = \lceil \frac{2n+3}{4} \rceil$) and the star $K_{1,n}$ with n pendant vertices ($tvs(K_{1,n}) = \lceil \frac{n+1}{2} \rceil$).

3 Main Results

In this part, we study the parameter *tvs* for two families of graphs i.e. Jahangir graphs and circulant graphs.

The Jahangir graph $J_{n,m}$, $n \geq 3$, $m \geq 1$, consists of a cycle C_{nm} and one additional vertex which is adjacent to n vertices of C_{nm} at distance m to each other on C_{nm}. The Jahangir graph was introduced by Tomescu in [12]. The Jahangir graph $J_{n,2}$ is also known as the *gear graph*, see Ma and Feng [10], and also Gallian [7], page 7. For $m = 1$, the Jahangir graph is wheel W_n. It was shown in [14] that $tvs(W_n) = \lceil \frac{n+3}{4} \rceil$.

Lemma 1 *Let $J_{n,m}$, $n \geq 3$, $m \geq 2$, be the Jahangir graph. Then*

$$\max\left\{\left\lceil\frac{n(m-1)+2}{3}\right\rceil, \left\lceil\frac{nm+2}{4}\right\rceil, \left\lceil\frac{nm+3}{n+1}\right\rceil\right\} \leq tvs(J_{n,m}) \leq nm.$$

Proof. The Jahangir graph $J_{n,m}$ has n vertices of degree 3, $n(m-1)$ vertices of degree 2 and one vertex of degree n. The upper bound of *tvs* follows from (1). To prove the lower bound consider the weights of the vertices. The smallest weight among all vertices of $J_{n,m}$ is at least 3, so the largest weight of vertex of degree 2 is at least $n(m-1)+2$. Since the weight of any vertex of degree 2 is the sum of three positive integers, so at least one label is at least $\left\lceil\frac{n(m-1)+2}{3}\right\rceil$.

The largest value among the weights of vertices of degree 2 and 3 is at least $nm+2$ and this weight is the sum of at most four integers. Hence the largest label contributing to this weight must be at least $\lceil\frac{nm+2}{4}\rceil$.

If we consider all vertices of the Jahangir graph $J_{n,m}$ then the lower bound $\left\lceil\frac{nm+3}{n+1}\right\rceil$ follows from (2).

This gives $\max\left\{\left\lceil\frac{n(m-1)+2}{3}\right\rceil, \lceil\frac{nm+2}{4}\rceil, \left\lceil\frac{nm+3}{n+1}\right\rceil\right\} \leq tvs(J_{n,m})$ and we are done. □

Lemma 2 $tvs(J_{3,2}) = 3$.

Proof. From Lemma 1 it follows that $tvs(J_{3,2}) \geq \max\{\lceil\frac{5}{3}\rceil, 2, \lceil\frac{9}{4}\rceil\} = 3$. For the converse, we define a suitable vertex irregular total labeling by Figure 1.

□

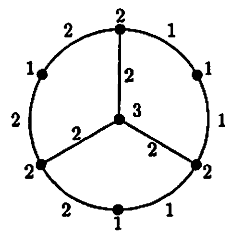

Figure 1: Jahangir graph $J_{3,2}$.

Theorem 2 *For $n \geq 4$, $tvs(J_{n,2}) = \lceil \frac{n+1}{2} \rceil$.*

Proof. According to Lemma 1, we have that $tvs(J_{n,2}) \geq \lceil \frac{n+1}{2} \rceil$. Put $k = \lceil \frac{n+1}{2} \rceil$. It is enough to describe a suitable vertex irregular total k-labeling. Let v be the central vertex of degree n and $u_1, u_2, \ldots, u_n$ be the vertices of degree 2 and $v_1, v_2, \ldots, v_n$ be the vertices of degree 3. Let $E(J_{n,2}) = \{u_iv_i : 1 \leq i \leq n\} \cup \{v_iu_{i+1} : 1 \leq i \leq n-1\} \cup \{vv_i : 1 \leq i \leq n\} \cup \{v_nu_1\}$ be the edge set of $J_{n,2}$.

We define a labeling $\sigma : V(J_{n,2}) \cup E(J_{n,2}) \to \{1, 2, \ldots, k\}$ in the following way

$$\sigma(v_nu_1) = 1, \quad \sigma(v) = k,$$

$$\sigma(u_iv_i) = \begin{cases} i & \text{for } 1 \leq i \leq k \\ n+2-i & \text{for } k+1 \leq i \leq n \end{cases}$$

$$\sigma(v_iu_{i+1}) = \begin{cases} i & \text{for } 1 \leq i \leq k-1 \\ k & \text{for } i = k \text{ if } n \text{ is odd} \\ n+1-k & \text{for } i = k \text{ if } n \text{ is even} \\ n+1-i & \text{for } k+1 \leq i \leq n-1. \end{cases}$$

For $1 \leq i \leq n$ we put

$$\sigma(u_i) = 1, \quad \sigma(v_i) = k,$$

$$\sigma(vv_i) = \begin{cases} k & \text{if } n \text{ is odd} \\ k-1 & \text{if } n \text{ is even.} \end{cases}$$

The weights of vertices of $J_{n,2}$ are as follows:

$$wt(u_i) = \begin{cases} 3 & \text{for } i = 1 \\ 2i & \text{for } 2 \leq i \leq k \\ 2(n+2-i)+1 & \text{for } k+1 \leq i \leq n \end{cases}$$

$$wt(v) = \begin{cases} k(n+1) - n & \text{if } n \text{ is even} \\ k(n+1) & \text{if } n \text{ is odd.} \end{cases}$$

If n is even then

$$wt(v_i) = \begin{cases} 2(k+i) - 1 & \text{for } 1 \le i \le k-1 \\ 2(n+1+k-i) & \text{for } k \le i \le n. \end{cases}$$

If n is odd then

$$wt(v_i) = \begin{cases} 2(k+i) & \text{for } 1 \le i \le k \\ 2(n+1+k-i)+1 & \text{for } k+1 \le i \le n. \end{cases}$$

The weights of vertices u_i, $1 \le i \le n$, successively attain values $3, 4, \ldots, n+2$ and the weights of vertices v_i, $1 \le i \le n$, receive distinct values from $n+3$ up to $2n+2$.

Thus the labeling σ is the desired vertex irregular total k-labeling. □

For $n \ge 3$ and $m \ge 3$ $\left\lceil \frac{n(m-1)+2}{3} \right\rceil \ge \left\lceil \frac{nm+3}{n+1} \right\rceil$, and for $n \ge 4$ and $m \ge 3$ $\left\lceil \frac{nm+2}{4} \right\rceil > \left\lceil \frac{nm+3}{n+1} \right\rceil$. Thus we believe that the following conjecture is true.

Conjecture 1 *Let $J_{n,m}$ be a Jahangir graph for $n \ge 3$, $m \ge 3$. Then*

$$tvs(J_{n,m}) = \max\left\{ \left\lceil \frac{n(m-1)+2}{3} \right\rceil, \left\lceil \frac{nm+2}{4} \right\rceil \right\}.$$

The total vertex irregularity strengths for cycle C_p and complete graph K_p are known. Our aim is to study the parameter tvs for circulant graphs. The circulant graphs are an important class of graphs, which can be used in the design of local area networks [2]. Let n, m and $a_1, \ldots, a_m$ be positive integers, $1 \le a_i \le \lfloor \frac{n}{2} \rfloor$ and $a_i \ne a_j$ for all $1 \le i < j \le m$. An undirected graph with the set of vertices $V = \{v_1, \ldots, v_n\}$ and the set of edges $E = \{v_i v_{i+a_j} : 1 \le i \le n, 1 \le j \le m\}$, the indices being taken modulo n, is called a *circulant graph* and it is denoted by $C_n(a_1, \ldots, a_m)$. The numbers $a_1, \ldots, a_m$ are called the generators and we say that the edge $v_i v_{i+a_j}$ is of type a_j.

It is easy to see that the circulant graph $C_n(a_1, \ldots, a_m)$ is a regular graph of degree r, where

$$r = \begin{cases} 2m-1 & \text{if } \frac{n}{2} \in \{a_1, \ldots, a_m\} \\ 2m & \text{otherwise.} \end{cases}$$

From Theorem 1, it follows that for r-regular circulant graph $C_n(a_1, \ldots, a_m)$, $n \geq 3$, $1 \leq a_i \leq \lfloor \frac{n}{2} \rfloor$,

$$\left\lceil \frac{n+r}{r+1} \right\rceil \leq tvs(C_n(a_1, \ldots, a_m)) \leq n - r + 1.$$

The following theorem gives the exact value of the total vertex irregularity strength for circulant graphs $C_n(1,2)$.

Theorem 3 *For the circulant graph $C_n(1,2)$, $n \geq 5$, we have*

$$tvs(C_n(1,2)) = \left\lceil \frac{n+4}{5} \right\rceil.$$

Proof. According to Theorem 1 we have $tvs(C_n(1,2)) \geq \lceil \frac{n+4}{5} \rceil$. In order to show the converse inequality, it only remains to describe a vertex irregular total $\lceil \frac{n+4}{5} \rceil$-labeling. Let us distinguish four cases:

Case 1. $n \equiv 0, 1 \ (mod\ 5)$

Define the function ψ_1 as follows:

For $1 \leq i \leq \lceil \frac{n}{2} \rceil$ put

$$\psi_1(v_i) = \begin{cases} 1 + 2\lfloor \frac{i}{5} \rfloor & \text{if } i \equiv 1, 2 \ (mod\ 5) \\ 2 + 2\lfloor \frac{i}{5} \rfloor & \text{if } i \equiv 3, 4 \ (mod\ 5) \\ \frac{2i}{5} & \text{if } i \equiv 0 \ (mod\ 5) \end{cases}$$

$$\psi_1(v_i v_{i+2}) = \begin{cases} 1 + 2\lfloor \frac{i}{5} \rfloor & \text{if } i \equiv 1 \ (mod\ 5) \\ 2 + 2\lfloor \frac{i}{5} \rfloor & \text{if } i \equiv 2, 3, 4 \ (mod\ 5) \\ 1 + \frac{2i}{5} & \text{if } i \equiv 0 \ (mod\ 5) \end{cases}$$

$$\psi_1(v_i v_{i+1}) = \begin{cases} 1 + 2\lfloor \frac{i}{5} \rfloor & \text{if } i \equiv 1, 2 \ (mod\ 5) \\ 2 + 2\lfloor \frac{i}{5} \rfloor & \text{if } i \equiv 3, 4 \ (mod\ 5) \\ 1 + \frac{2i}{5} & \text{if } i \equiv 0 \ (mod\ 5). \end{cases}$$

For $\lceil \frac{n}{2} \rceil + 1 \leq i \leq n - 1$ put

$$\psi_1(v_i) = \begin{cases} 2 + 2\lfloor \frac{n-i}{5} \rfloor & \text{if } n - i \equiv 1, 2 \ (mod\ 5) \\ 3 + 2\lfloor \frac{n-i}{5} \rfloor & \text{if } n - i \equiv 3, 4 \ (mod\ 5) \\ 1 + 2\frac{n-i}{5} & \text{if } n - i \equiv 0 \ (mod\ 5) \end{cases}$$

$$\psi_1(v_iv_{i+1}) = \begin{cases} 2+2\lfloor\frac{n-i}{5}\rfloor & \text{if } n-i \equiv 1,2,3 \ (mod\ 5) \\ 3+2\lfloor\frac{n-i}{5}\rfloor & \text{if } n-i \equiv 4 \ (mod\ 5) \\ 1+2\frac{n-i}{5} & \text{if } n-i \equiv 0 \ (mod\ 5). \end{cases}$$

For $\lceil\frac{n}{2}\rceil + 1 \leq i \leq n-2$ put

$$\psi_1(v_iv_{i+2}) = \begin{cases} 1+2\lfloor\frac{n-i}{5}\rfloor & \text{if } n-i \equiv 1 \ (mod\ 5) \\ 2+2\lfloor\frac{n-i}{5}\rfloor & \text{if } n-i \equiv 2,3 \ (mod\ 5) \\ 3+2\lfloor\frac{n-i}{5}\rfloor & \text{if } n-i \equiv 4 \ (mod\ 5) \\ 1+2\frac{n-i}{5} & \text{if } n-i \equiv 0 \ (mod\ 5). \end{cases}$$

Moreover put

$$\psi_1(v_n) = \psi_1(v_nv_1) = \psi_1(v_{n-1}v_1) = \psi_1(v_nv_2) = 1.$$

Observe that

$$wt(v_i) = \psi_1(v_i) + \psi_1(v_iv_{i+1}) + \psi_1(v_{i-1}v_i) + \psi_1(v_iv_{i+2}) + \psi_1(v_{i-2}v_i)$$

$$= \begin{cases} 5 & \text{for } i=1 \\ 2+2i & \text{for } 2 \leq i \leq \lfloor\frac{n}{2}\rfloor + 1 \\ 2n+7-2i & \text{for } \lfloor\frac{n}{2}\rfloor + 2 \leq i \leq n \end{cases}$$

with indices taken modulo n.

The function ψ_1 is a map from $V(C_n(1,2)) \cup E(C_n(1,2))$ into $\{1, 2, \ldots, \lceil\frac{n+4}{5}\rceil\}$ and the weights of the vertices under the labeling ψ_1 constitute the set $\{5, 6, \ldots, n+4\}$.

Case 2. $n \equiv 2 \ (mod\ 5)$

Define the function $\psi_2 : V(C_n(1,2)) \cup E(C_n(1,2)) \to \{1, 2, \ldots, \lceil\frac{n+4}{5}\rceil\}$ such that:

$$\psi_2(v_i) = \begin{cases} \psi_1(v_i) & \text{for } 1 \leq i \leq \lceil\frac{n}{2}\rceil \\ \lceil\frac{n+4}{5}\rceil & \text{for } i = \lceil\frac{n}{2}\rceil + 1 \\ \psi_1(v_i) & \text{for } \lceil\frac{n}{2}\rceil + 2 \leq i \leq n-1 \\ 1 & \text{for } i = n \end{cases}$$

$$\psi_2(v_iv_{i+1}) = \psi_1(v_iv_{i+1}) \quad \text{for } 1 \leq i \leq n-1$$

$$\psi_2(v_iv_{i+2}) = \psi_1(v_iv_{i+2}) \text{ for } 1 \le i \le n-2$$
$$\psi_2(v_nv_1) = \psi_2(v_{n-1}v_1) = \psi_2(v_nv_2) = 1.$$

One can see that the weights of vertices under the function ψ_2 receive distinct labels from the set $\{5, 6, \ldots, n+4\}$ and that ψ_2 is vertex irregular total labeling having the required property.

Case 3. $n \equiv 3$ $(mod\ 5)$

Define the function $\psi_3 : V(C_n(1,2)) \cup E(C_n(1,2)) \to \{1, 2, \ldots, \lceil\frac{n+4}{5}\rceil\}$ in the following way:

$$\psi_3(v_i) = \psi_2(v_i) \text{ for } 1 \le i \le n$$

$$\psi_3(v_iv_{i+1}) = \begin{cases} \psi_1(v_iv_{i+1}) & \text{for } 1 \le i \le \lceil\frac{n}{2}\rceil - 1 \\ \lceil\frac{n+4}{5}\rceil & \text{for } i = \lceil\frac{n}{2}\rceil \\ \psi_1(v_iv_{i+1}) & \text{for } \lceil\frac{n}{2}\rceil + 1 \le i \le n-1 \end{cases}$$

$$\psi_3(v_iv_{i+2}) = \psi_1(v_iv_{i+2}) \text{ for } 1 \le i \le n-2$$
$$\psi_3(v_nv_1) = \psi_3(v_{n-1}v_1) = \psi_3(v_nv_2) = 1.$$

Case 4. $n \equiv 4$ $(mod\ 5)$

Define the function $\psi_4 : V(C_n(1,2)) \cup E(C_n(1,2)) \to \{1, 2, \ldots, \lceil\frac{n+4}{5}\rceil\}$ as follow:

$$\psi_4(v_iv_{i+1}) = \psi_1(v_iv_{i+1}) \text{ for } 1 \le i \le n-1$$
$$\psi_4(v_iv_{i+2}) = \psi_1(v_iv_{i+2}) \text{ for } 1 \le i \le n-2$$
$$\psi_4(v_nv_1) = \psi_4(v_{n-1}v_1) = \psi_4(v_nv_2) = 1.$$

If $n \equiv 9$ $(mod\ 10)$ we put

$$\psi_4(v_i) = \begin{cases} \psi_1(v_i) & \text{for } 1 \le i \le \lceil\frac{n}{2}\rceil - 1 \\ \lceil\frac{n+4}{5}\rceil & \text{for } i = \lceil\frac{n}{2}\rceil \\ \psi_1(v_i) & \text{for } \lceil\frac{n}{2}\rceil + 1 \le i \le n-1 \\ 1 & \text{for } i = n. \end{cases}$$

If $n \equiv 4$ $(mod\ 10)$ we put

$$\psi_4(v_i) = \begin{cases} \psi_1(v_i) & \text{for } 1 \le i \le \lceil\frac{n}{2}\rceil + 1 \\ \lceil\frac{n+4}{5}\rceil & \text{for } i = \lceil\frac{n}{2}\rceil + 2 \\ \psi_1(v_i) & \text{for } \lceil\frac{n}{2}\rceil + 3 \le i \le n-1 \\ 1 & \text{for } i = n. \end{cases}$$

For the labelings ψ_3 and ψ_4, we use a similar observation applied in *Case 1* that the weights of vertices successively attain values $5, 6, \ldots, n+4$. Thus the labelings ψ_3 and ψ_4 are the desired vertex irregular total $\lceil \frac{n+4}{5} \rceil$-labelings.

□

Although we have not yet found the general formulas for vertex irregular total labeling of circulant graph $C_n(a_1, \ldots, a_n)$, $n \geq 3$, $1 \leq a_i \leq \lfloor \frac{n}{2} \rfloor$, that will determine the lower bound in Theorem 1 as exact value of the parameter tvs, the result from Theorem 3 leads us to suggest the following

Conjecture 2 *Let $C_n(a_1, \ldots, a_m)$ be a circulant graph of degree $r \geq 5$, $n \geq 6$ and $1 \leq a_i \leq \lfloor \frac{n}{2} \rfloor$. Then*

$$tvs(C_n(a_1, \ldots, a_m)) = \left\lceil \frac{n+r}{1+r} \right\rceil.$$

Acknowledgement
The authors wish to thank the anonymous referee for his/her valuable comments.

References

[1] M. Bača, S. Jendroľ, M. Miller and J. Ryan, On irregular total labellings, *Discrete Math.* **307** (2007), 1378–1388.

[2] J.C. Bermond, F. Comellas and D.F. Hsu, Distributed loop computer networks: a survey, *J. Parallel Distrib. Comput.* **24** (1995), 2–10.

[3] T. Bohman and D. Kravitz, On the irregularity strength of trees, *J. Graph Theory* **45** (2004), 241–254.

[4] G. Chartrand, M.S. Jacobson, J. Lehel, O.R. Oellermann, S. Ruiz and F. Saba, Irregular networks, *Congr. Numer.* **64** (1988), 187–192.

[5] R.J. Faudree, M.S. Jacobson, J. Lehel and R.H. Schlep, Irregular networks, regular graphs and integer matrices with distinct row and column sums, *Discrete Math.* **76** (1988), 223–240.

[6] A. Frieze, R.J. Gould, M. Karonski, and F. Pfender, On graph irregularity strength, *J. Graph Theory* **41** (2002), 120–137.

[7] J.A. Gallian, A dynamic survey of graph labeling, *Electron. J. Combin.* **14** (2007) #DS6, http://www.combinatorics.org/surveys/ds6.pdf

[8] A. Gyárfás, The irregularity strength of $K_{m,m}$ is 4 for odd m, *Discrete Math.* **71** (1988), 273–274.

[9] S. Jendroľ, M. Tkáč and Z. Tuza, The irregularity strength and cost of the union of cliques, *Discrete Math.* **150** (1996), 179–186.

[10] K.J. Ma and C.J. Feng, On the gracefulness of gear graphs, *Math. Practice Theory* (1984), 72–73.

[11] T. Nierhoff, A tight bound on the irregularity strength of graphs, *SIAM J. Discrete Math.* **13** (2000), 313–323.

[12] I. Tomescu and I. Javaid, On the metric dimension of the Jahangir graph, *Bull. Math. Soc. Sci. Math. Roumanie* **50** (2007), 371–376.

[13] W.D. Wallis, Magic Graphs, Birkhäuser, Boston, 2001.

[14] K. Wijaya and Slamin, Total vertex irregular labeling of wheels, fans, suns and friendship graphs, *JCMCC* **65** (2008), 103–112.