A new family of pooling designs

Haixia Guo[1,2,*], Jizhu Nan [1]

1.Dept.of Applied Math.,Dalian University of Technology,Dalian,116024,P.R.China

2.College of Science,Tianjin University of Technology and Education,Tianjin,300222,P.R.China

Abstract

In [J. Guo, K. Wang, A construction of pooling designs with high degree of error correction, J. Combin. Theory Ser. A 118 (2011) 2056–2058], Guo and Wang proposed a new model for disjunct matrices. As a generalization of Guo-Wang's designs, we obtain a new family of pooling designs. Our designs and Guo-Wang's designs have the same numbers of items and pools, but the error-tolerance property of our design is better than that of Guo-Wang's designs under some conditions.

Key words: Pooling design; Finite set; Disjunct matrix

The basic problem of non-adaptive group testing is to identify the defective parts as the subset of objects being tested. Pooling design is a mathematical tool to non-adaptive group testing and it can reduce the number of tests in many areas such as DNA library screening ([1, 2]). A mathematical model of pooling designs is an s^e-disjunct matrix. A binary matrix M is called s^e*-disjunct* if given any $s+1$ columns of M with one designated, there are $e+1$ rows with a 1 in the designated column and 0 in each of the other s columns ([3, 4]). An s^0-disjunct matrix is called s-*disjunct*.

For positive integers $k < n$, let $[n] = \{1, 2, \ldots, n\}$ and $\binom{[n]}{k}$ be the set of all k-subsets of $[n]$.

Macula ([4]) proposed a novel way of constructing d-disjunct matrices which uses the containment relation in a structure. Guo and Wang ([5]) proposed a new model for pooling designs.

*E-mail address: ghx626@126.com.

Definition 1. ([5]) *Given integers* $1 \le d < k < n$ *and* $0 \le i \le d$. *Let* $M(i; n, d, k)$ *be the binary matrix with rows indexed with* $\binom{[n]}{d}$ *and columns indexed with* $\binom{[n]}{k}$ *such that* $M(A, B) = 1$ *if and only if* $|A \cap B| = i$.

Theorem 2. ([5]) *Let* $1 \le s \le i$, $\lfloor (d+1)/2 \rfloor \le i \le d < k$ *and* $n - k - s(k + d - 2i) \ge d - i$. *Then* $M(i; n, d, k)$ *is an* s^{e_1}*-disjunct matrix, where* $e_1 = \binom{k-s}{i-s}\binom{n-k-s(k+d-2i)}{d-i} - 1$.

In [5], Guo and Wang provided the following construction, and proposed the following problem.

Problem. For positive integers $1 \le d < k < n$, let I be a nonempty proper subset of $\{0, 1, \dots, d\}$, and let $M(I; d, k, n)$ be the binary matrix with rows indexed with $\binom{[n]}{d}$ and columns indexed with $\binom{[n]}{k}$ such that $M(A, B) = 1$ if and only if $|A \cap B| \in I$. How about the error-tolerant property of this matrix?

In this note we solve this problem when $I = \{i, i+1, \dots, d-1\}$.

Theorem 3. *Let* $1 \le s < i$ *and* $n - sk + (s-1)\max\{2(k-i+1) - n + d, 0\} \ge d$. *Then* $M(I; d, k, n)$ *is an* s^{e_2}*-disjunct matrix, where*

$$e_2 = \binom{k-s}{k-i}\binom{n - i - sk + (s-1)\max\{2(k-i+1) - n + d, 0\}}{d-i} - 1.$$

Proof. Let $B_0, B_1, \cdots, B_s \in \binom{[n]}{k}$ be any $s+1$ distinct columns of $M(I; d, k, n)$ Then, for each $j \in [s]$, there exists an x_j such that $x_j \in B_0 \backslash B_j$. Suppose $X_0 = \{x_j \mid 1 \le j \le s\}$. Then $X_0 \subseteq B_0$, and $X_0 \not\subseteq B_j$ for each $j \in [s]$. Note that the number of i-subsets of B_0 containing X_0 is $\binom{k-|X_0|}{i-|X_0|} = \binom{k-|X_0|}{k-i}$. Since $\binom{k-|X_0|}{k-i}$ is decreasing for $1 \le |X_0| \le s$ and gets its minimum at $|X_0| = s$, the number of i-subsets of B_0 containing X_0 is at least $\binom{k-s}{k-i}$. Let A_0 be an i-subset of B_0 containing X_0 and D be a d-subset of $[n]$ containing A_0. Then $|D \cap B_0| \ge i$, and $|D \cap B_j| < i$ if and only if $|\bar{D} \cap B_j| \ge k - i + 1$ for each $j \in [s]$, where $\bar{D} = [n] \backslash D$. Therefore, for each $j \in \{2, \dots, s\}$, we have $|B_1 \cap B_j| \ge |\bar{D} \cap B_1 \cap B_j| \ge \max\{2(k-i+1) - n + d, 0\}$, which implies that $|\bigcup_{j=1}^{s} B_j| \le sk - (s-1)\max\{2(k-i+1) - n + d, 0\}$. Since $|A_0 \cup (\bigcup_{j=1}^{s} B_j)| \le i + sk - (s-1)\max\{2(k-i+1) - n + d, 0\}$, the number of d-subsets D of $[n]$ containing A_0 such that $|D \cap B_0| \ge i$, and $|D \cap B_j| < i$

for each $j \in [s]$ is at least $\binom{n-i-sk+(s-1)\max\{2(k-i+1)-n+d,0\}}{d-i}$. Hence the number of d-subsets D of $[n]$ satisfying $|D \cap B_0| \geq i$, and $|D \cap B_j| < i$ for each $j \in [s]$ is at least $\binom{k-s}{i-s}\binom{n-i-sk+(s-1)\max\{2(k-i+1)-n+d,0\}}{d-i}$, as desired. □

If $n-k-s(k+d-2i) \geq d-i$, $n-sk+(s-1)\max\{2(k-i+1)-n+d,0\} > (s-1)n+1 > n-k-s(k+d-2i)$, then $e_2 > e_1$.

Example 1. $M(\{3,4\};5,7,50)$ is 1^{11699}, 2^{2639} and 3^{324}-disjunct, but $M(3;5,7,50)$ is 1^{9989}, 2^{2324} and 3^{299}-disjunct.

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