MATROIDS WITH EVERY TWO ELEMENTS IN A 4-CIRCUIT

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ABSTRACT. This paper proves that the graphic matroids of the title with at least two edges and no isolated vertices coincide with the class of complete k-partite graphs where, when $k \leq 3$, no partition class has size one. It also shows that a simple rank-r binary matroid M has every two elements in a 4-circuit if $|E(M)| \geq 2^{r-1} + 2$.

1. INTRODUCTION

This paper considers the problem of characterizing those matroids in which every two elements lie in a 4-element circuit. In particular, we solve the problem for graphic matroids. We were unable to solve the problem for cographic matroids, and this proved the major impediment to settling the problem for regular matroids. A complete characterization of the binary matroids with the specified property may be unlikely as these matroids include every simple, rank-r, binary matroid having at least two more than half the elements in a complete projective geometry.

The graph and matroid terminology used here will follow Diestel [1] and Oxley [3]. A *Sylvester matroid* is one in which every pair of elements is contained in a circuit of length three, or a 3-circuit. Murty showed that a Sylvester matroid of rank r must have at least $2^r - 1$ elements [2]. Thus a 3-cycle is the only regular Sylvester matroid, and projective geometries are the only binary examples.

2. GRAPHIC MATROIDS

In this section, we prove the following theorem, the main result of the paper.

Theorem 2.1. *Let G be a graph with no isolated vertices containing at least two edges. Every two edges of G are contained in a cycle of size four if and only if G is a complete k-partite graph, where, when $k \leq 3$, no partition class has size one.*

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Proof. Let G be a complete k-partite graph, so that if $k \leq 3$, then no partition class has size one. It is straightforward to check that every two edges of G are contained in a 4-cycle, and we omit the details.

Now we show the inverse. Let G be a graph with at least two edges and no isolated vertices containing every two edges in a 4-cycle. Notice that the resulting graph must be 2-connected, otherwise G has two components, each with an edge which do not share a 4-cycle. A loop or double edge clearly cannot be contained in a 4-cycle, so G is simple.

Let $\chi(G)$ be the chromatic number of G. Let $\lambda : V(G) \mapsto \{1, 2, ..., \chi(G)\}$ be a proper coloring of $V(G)$, chosen to minimize the weight

$$\sum_{v \in V(G)} \lambda(v).$$

We intend to show these color classes are the partition classes of a complete χ-partite graph.

Assume not. No edge is contained in a color class, so there must be two non-adjacent vertices, u and v, of different colors. Assume $\lambda(u) < \lambda(v)$.

Let w be a neighbor of u. Let x be a neighbor of v other than w. The edge uw is contained in a 4-cycle with edge vx. Since u and v are non-adjacent, the edges ux and vw must exist. Hence every neighbor of u is a neighbor of v. By symmetry, the neighborhoods are equal, and we may recolor vertex v with the color $\lambda(u)$, producing a lower-weight, proper coloring. We conclude that G is a complete $\chi(G)$-partite graph. □

3. $GF(q)$-Representable Matroids

The result for binary matroids stated in the abstract is a special case of the following.

Theorem 3.1. *Let M be a simple, $GF(q)$-representable matroid of rank $r \geq 2$. If $|E(M)| > q + \frac{q^{r-1}-1}{q-1}$, then every two elements of M are in a 4-circuit.*

Proof. Embed M into the projective space $PG(r-1, q)$. Suppose that there are elements x and y of M that are not contained in a 4-circuit. Let L be the set of points on the line of $PG(r-1, q)$ containing these elements. Because every line in a projective geometry has at least three points, there is an element $z \in L - \{x, y\}$. There are $c = \frac{q^{r-1}-1}{q-1}$ lines of the projective geometry containing z [3, pp. 164-177], one of which is L. Note that these lines cover all points in the projective geometry. Of the $c-1$ lines other than L, at most one point, other than possibly z, is contained in M. Therefore, M has at most $q+1$ elements in L and $c-1$ other elements, and $|E(M)| \leq q + 1 + \frac{q^{r-1}-1}{q-1} - 1$. □

It is easy to see that this is a sharp bound. Let M be the parallel connection, with basepoint x, of $PG(r-2,q)$ and $PG(1,q)$. Then $|E(M)| = \frac{q^{r-1}-1}{q-1} + q$. But if $y \in E(PG(1,q)) - x$, then there is no 4-circuit containing $\{x,y\}$.

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