Hamilton Cycles in Cubic Polyhex Graphs on the Klein Bottle

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Abstract

Let G be a connected cubic graph embedded on a surface Σ such that every face is bounded by a cycle of length 6. By Euler formula, Σ is either the torus or the Klein bottle. The corresponding graphs are called toroidal polyhex graphs and Klein-bottle polyhex graphs, respectively. It was proved that every toroidal polyhex graph is hamiltonian. In this paper, we prove that every Klein-bottle polyhex graph is hamiltonian. Furthermore, lower bounds for the number of Hamilton cycles in Klein-bottle polyhex graphs are obtained.

1 Introduction

A graph G is *hamiltonian* if G has a cycle through all vertices of G and the cycle is called a *Hamilton cycle* of G. It is NP-complete to determine if a graph is hamiltonian, even for a 3-connected cubic graph [10]. For a plane cubic graph, Barnette [4] made the following conjecture.

Conjecture 1.1 (Barnette, [4]). *Let G be a 3-connected cubic plane graph. If every face of G is bounded by a cycle of length at most* 6, *then G contains a Hamilton cycle.* □

Goodey [6] proved that every cubic plane graph with only square and hexagonal faces is hamiltonian, which implies that Barnette's Conjecture holds for bipartite graphs. Aldred et al. [1] confirmed Conjecture 1.1 for graphs with less than 178 vertices. A *fullerene graph* is a 3-connected cubic plane graph with only (exactly 12) pentagonal and hexagonal faces. The following is a week version of Barnette's conjecture for fullerene graphs.

Conjecture 1.2 (Myrvold, [9]). *Every fullerene graph is hamiltonian.* □

Recently, Conjecture 1.2 has been verified for partial fullerene graphs such as partial leapfrog fullerene graphs [8] by Marušič, and fullerene graphs with a non-trivial cyclic-5-edge cut [7] by Kutnar and Marušič. Deza et al. [5] considered the extension of fullerenes on other surfaces and found that torus and Klein bottle are only two surfaces which can be tiled entirely with hexagons.

A cubic graph G embedded on the Klein bottle is called a *Klein-bottle polyhex graph* if every face of G is bounded by a hexagon. Analogically, a *toroidal polyhex graph* is a cubic graph embedded on the torus with only hexagonal face. They have been discussed as hexagon tessellations of the Klein bottle and the torus [14], respectively.

For toroidal graphs, Thomas and Yu [12] showed that every 5-connected toroidal graph is hamiltonian, and Thomas, Yu and Zang [13] proved that every 4-connected toroidal graph contains a Hamilton path. However, it is not necessary for a cubic toroidal graph to be hamlitonian. Hamiltonian properties of toroidal polyhex graphs have been an attractive topic of research in the last decades [2, 3, 15] whereas Klein-bottle polyhex graphs have been less attractive in this respect. It has been shown that every toroidal polyhex graph is hamiltonian. In this paper, we show that every Klein-bottle polyhex graph is hamiltonian.

Theorem 1.3. *Every Klein-bottle polyhex graph is hamiltonian.* □

Combining with earlier results in [5] for surface embedding and results in [2, 3, 15] for toroidal polyhex graphs, we have the following conclusion.

Theorem 1.4. *If G is a cubic graph embedded in a surface such that every face is bounded by a cycle of length* 6*, then G contains a Hamilton cycle.* □

2 Klein-bottle polyhex graphs

In this section, we will give a construction of Klein-bottle polyhex graphs according to [14].

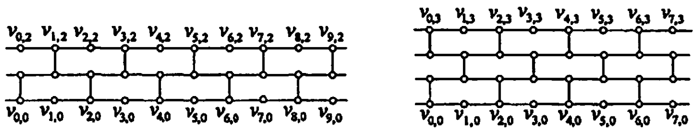

Figure 1: Hexagonal cylinder $(6,2)$ *(left): $v_{11,j}$ is adjacent to $v_{0,j}$ for $j \in \mathbb{Z}_3$; and hexagonal cylinder* $(5,3)$ *(right): $v_{9,j}$ is adjacent to $v_{0,j}$ for $j \in \mathbb{Z}_4$.*

Let $C = x_0x_1 \cdots x_{2k-1}$ be an even cycle and let $P = y_0y_1 \cdots y_m$ be a path. A hexagonal cylinder (k, m) is the "brick" product $C \sharp P$ of C with

P as: using $v_{i,j}$ to denote the vertex (x_i, y_j) in $C \sharp P$ where $i \in \mathbb{Z}_{2k}$ and $j \in \mathbb{Z}_{m+1}$, the edge set consists of all pairs of $v_{i,j}v_{i',j}$ where $i' \equiv i+1 \pmod{2k}$, together with $v_{i,j}v_{i,j+1}$ where $i \equiv j \pmod 2$ and $j \neq m$. For examples, the hexagonal cylinders $(6,2)$ and $(5,3)$ are shown in *Figure 1.* Note that the hexagonal cylinder $(k,0)$ is the cycle C.

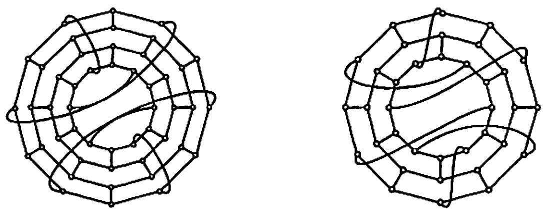

Figure 2: Representations of $H_{6,4,a}$ (left) and $H_{6,3,b}$ (right) given by Thomassen.

Following Thomassen [14], the graphs $H_{k,m,a}$, $H_{k,m,b}$, $H_{k,m,c}$ and $H_{k,m,f}$ are defined as graphs obtained from the hexagonal cylinder (k,m) by adding edges or vertices in the following way:

- $H_{k,m,a}$: add edges $v_{1,0}v_{2,m}, v_{3,0}v_{0,m}, v_{5,0}v_{2k-2,m}, \ldots, v_{2k-1,0}v_{4,m}$ for even m;
- $H_{k,m,b}$: add edges $v_{1,0}v_{1,m}, v_{3,0}v_{2k-1,m}, v_{5,0}v_{2k-3,m}, \ldots, v_{2k-1,0}v_{3,m}$ for odd m;
- $H_{k,m,c}$: for even k, add edges $v_{2i+1,0}v_{2i+k+1,0}$ (where $i \in \mathbb{Z}_k$), $v_{j,m}v_{j+k,m}$ (where $j \equiv m \pmod 2$);
- $H_{k,m,f}$: for odd k, add new vertices $w_0, w_1, \ldots, w_{k-1}$ and $u_0, u_1, \ldots, u_{k-1}$, and the edges $v_{2i+1,0}w_i$, $v_{j,m}u_i$ (where $j = 2i$ if m is even and $j = 2i+1$, otherwise), and the cycles $w_i w_{i+\frac{k+1}{2}} w_{i+1} w_{i+\frac{k+3}{2}} \cdots w_i$ and $u_i u_{i+\frac{k+1}{2}} u_{i+1} u_{i+\frac{k+3}{2}} \cdots u_i$.

Theorem 2.1 (Thomassen, [14]). *Let G be a Klein-bottle polyhex graph with girth 6. Then G is isomorphic to one of $H_{k,m,a}$, $H_{k,m,b}$, $H_{k,m,c}$, $H_{k,m,f}$ for some non-negative integers k and m.* □

Note that, in [14], Klein-bottle polyhex graphs were originally classified into five classes: $H_{k,m,a}$, $H_{k,m,b}$, $H_{k,m,c}$, $H_{k,m,f}$ and $H_{k,d}$. The class $H_{k,d}$ with no definition in this paper was further identified as a sub-family of $H_{k,m,f}$ in [14].

Let $P = v_0 v_1 \cdots v_{2k}$ and $P \times K_2$ be the Cartesian product of P and K_2 (the complete graph with two vertices). Then $v_{1,0}, v_{2,0}, v_{1,2k}$ and $v_{2,2k}$

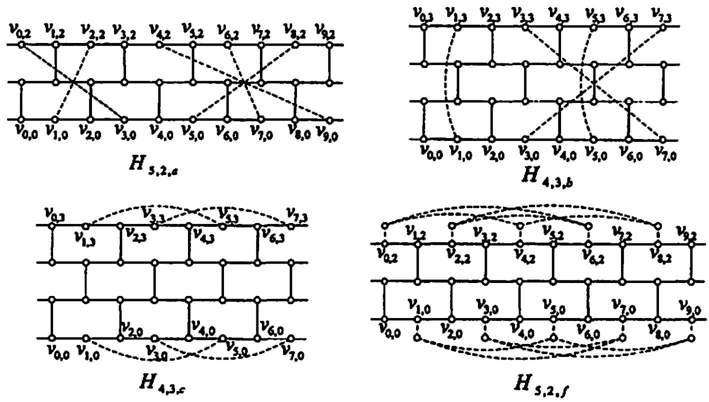

Figure 3: Examples: $H_{k,m,a}, H_{k,m,b}, H_{k,m,c}$ and $H_{k,m,f}$: dashed lines are the new edges added in the construction of Klein-bottle polyhex graphs.

are the four vertices of degree 2 in $P \times K_2$. Let G be the cubic graph obtained from $P \times K_2$ by identifying $v_{1,2k}$ with $v_{2,0}$, and identifying $v_{2,2k}$ with $v_{1,0}$. According to the proof of Theorem 2.1 in [14], a cubic polyhex graph with girth less then six is isomorphic to G for some k, which is hamiltonian. Hence it suffices to show the four classes of Klein-bottle polyhexes in Theorem 2.1 are hamiltonian.

An equivalent representation for $H_{k,m,a}$ and $H_{k,m,b}$ was given by Shiu and Zhang [11] in notation $K(p,q)$, which implies that both $H_{k,m,a}$ and $H_{k,m,b}$ are bipartite [11]. However, for other two cases $H_{k,m,c}$ and $H_{k,m,f}$, it is easy to see that they are non-bipartite since each of them contains odd-length cycles.

3 Hamilton cycles

In this section, we are to prove Theorem 1.3 which says that every Klein-bottle polyhex is hamiltonian.

Lemma 3.1. *Each $H_{k,m,a}$ and each $H_{k,m,b}$ has at least $2^{\lfloor \frac{m+1}{2} \rfloor}$ Hamilton cycles.*

Proof: We prove a slightly stronger result that each Hamilton cycle mentioned in the lemma contains exactly one edge from $\{v_{\alpha,j}v_{\alpha,j+1} | \ \alpha \equiv j \pmod 2\}$ for each $j \in \{0, \cdots, m-1\}$ and the edge $v_{1,0}v_{\beta,m}$, where $\beta = 2$ for $H_{k,m,a}$ and $\beta = 1$ for $H_{k,m,b}$. We use induction on m to prove this statement. Note that m is even in $H_{k,m,a}$ and m is odd in $H_{k,m,b}$.

For $m = 0$, $H_{k,0,a}$ has a required Hamilton cycle:

$$v_{1,0}v_{2,0}v_{3,0}\cdots v_{2k-1,0}v_{0,0}v_{1,0},$$

which satisfies the statement (see *Figure 4*).

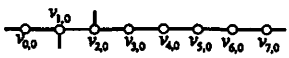

Figure 4: A Hamilton cycle in $H_{4,0,a}$.

For $m = 1$, $H_{k,1,b}$ has two Hamilton cycles (see *Figure 5*):

$$v_{1,0}v_{1,1}v_{2,1}\cdots v_{2k-1,1}v_{0,1}v_{0,0}v_{2k-1,0}\cdots v_{2,0}v_{1,0}$$

and

$$v_{1,0}v_{1,1}v_{0,1}v_{2k-1,1}\cdots v_{2,1}v_{2,0}v_{3,0}\cdots v_{2k-1,0}v_{0,0}v_{1,0}.$$

Clearly, both of them satisfy the statement. Hence, the statement holds for $m = 0, 1$. Now suppose $m \geq 2$ and the statement is true for smaller m.

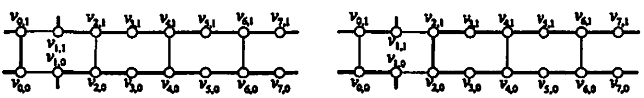

Figure 5: Two Hamilton cycles in $H_{4,1,b}$.

In the following, we only prove the lemma for graphs $H_{k,m,a}$. A similar discussion will prove the lemma for graphs $H_{k,m,b}$.

Delete all vertices in $\{v_{i,j} | i \in \mathbb{Z}_{2k}; j = m-1, m\}$ and the edges incident with them from $H_{k,m,a}$. Add the edges $v_{i,0}v_{j,m-2}$ to the remaining graph if $v_{i,0}v_{j,m}$ are edges of $H_{k,m,a}$. So we obtain a graph $H_{k,m-2,a}$. By inductive hypothesis, $H_{k,m-2,a}$ has $2^{\lfloor \frac{m-1}{2} \rfloor}$ Hamilton cycles and each of them contains the edge $e = v_{1,0}v_{2,m-2}$. Let C be one of such Hamilton cycles. Note that $v_{1,m-1}$ and $v_{3,m-1}$ are adjacent to $v_{1,m}$ and $v_{3,m}$, respectively.

In the graph $H_{k,m,a}$ there exist two paths joining $v_{1,0}$ and $v_{2,m-2}$ that contains all the vertices that were deleted from $H_{k,m,a}$ to obtain $H_{k,m-2,a}$:

$$P = v_{1,0}v_{2,m}v_{1,m}v_{0,m}\cdots v_{3,m}v_{3,m-1}v_{4,m-1}\cdots v_{1,m-1}v_{2,m-1}v_{2,m-2}$$

and

$$P' = v_{1,0}v_{2,m}v_{3,m}\cdots v_{0,m}v_{1,m}v_{1,m-1}v_{0,m-1}\cdots v_{3,m-1}v_{2,m-1}v_{2,m-2}.$$

Since C is a Hamilton cycle and $C - e$ is a Hamilton path joining $v_{1,0}$ and $v_{2,m-2}$ in the smaller graph $H_{k,m-2,a}$, we have that $(C - e) \cup P$ and

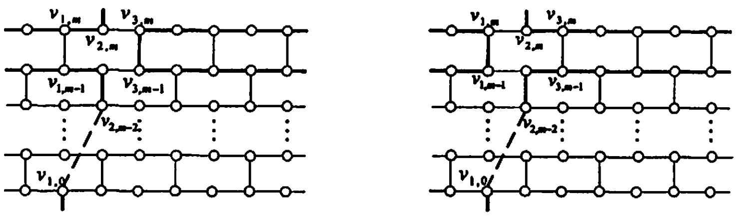

Figure 6: Two paths joining $v_{1,0}$ and $v_{2,m-2}$ in $H_{7,m,a}$.

$(C-e)\cup P'$ are two distinct Hamilton cycles of $H_{k,m,a}$. So $H_{k,m,a}$ has $2^{\lfloor\frac{m+1}{2}\rfloor}$ required Hamilton cycles, which completes the proof of the lemma. □

Lemma 3.2. *Each $H_{k,m,c}$ and each $H_{k,m,f}$ has at least $k2^{m+1}$ Hamilton cycles.*

Proof: We prove a slightly stronger result: for $H_{k,m,f}$, each of these $k2^{m+1}$ Hamilton cycles contains exactly two edges $w_i v_{2i+1,0}, w_{i+\frac{k+1}{2}} v_{2i+k+2,0}$ for some $i \in \mathbb{Z}_k$; for $H_{k,m,c}$, each of them contains exactly one edge $v_{i,0}v_{i+k,0}$ for some odd $i \in \mathbb{Z}_{2k}$.

We first use induction on m to prove the lemma for $H_{k,m,f}$. Notice that k is odd. Let

$$P_{i,i+\frac{k+1}{2}} := v_{2i+1,0}w_i w_{i+\frac{k+1}{2}} w_{i+1} \cdots w_{i-\frac{k+1}{2}} v_{2i+k,0}$$

be the path containing all vertices of $\{w_i | i \in \mathbb{Z}_k\}$ and let

$$P'_{i,i+\frac{k+1}{2}} := v_{j,m}u_i u_{i+\frac{k+1}{2}} u_{i+1} \cdots u_{i-\frac{k+1}{2}} v_{j',m}$$

be the path containing all vertices of $\{u_i | i \in \mathbb{Z}_k\}$, where $j = 2i, j' = 2i-k-1 \equiv 2i+k-1 \pmod{2k}$ if m is even, and $j = 2i+1, j' = 2i-k$, otherwise. Then

$$P_{i,i+\frac{k+1}{2}} v_{2i+k,0}v_{2i+k+1,0} \cdots v_{2i,0}P'_{i,i+\frac{k+1}{2}} v_{2i+k-1,0}v_{2i+k-2,0} \cdots v_{2i+1,0}$$

and

$$P_{i,i+\frac{k+1}{2}} v_{2i+k,0}v_{2i+k-1,0} \cdots v_{2i+2,0}P'_{i+1,i+\frac{k+3}{2}} v_{2i+k+1,0}v_{2i+k+2,0} \cdots v_{2i+1,0}$$

are two required Hamilton cycles of $H_{k,0,f}$ (see *Figure 7*). Since i can be any one in $\mathbb{Z}_k$, we obtain $2k$ Hamilton cycles of $H_{k,0,f}$.

Now suppose $m \geq 1$ and the statement holds for smaller m. Delete all vertices in $\{v_{i,0} | i \in \mathbb{Z}_{2k}\}$ together with all edges incident with them. Add edges $w_i v_{2i,1}$ to the remaining graph and obtain a graph $H_{k,m-1,f}$.

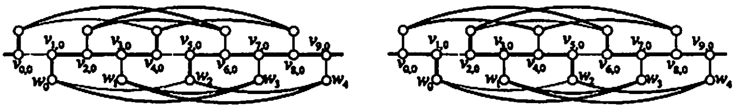

Figure 7: Two Hamilton cycles of $H_{5,0,f}$.

By inductive hypothesis, $H_{k,m-1,f}$ has at least $k2^m$ Hamilton cycles and each of them contains two edges $e_1 = w_i v_{2i,1}$ and $e_2 = w_{i+\frac{k+1}{2}} v_{2i+k+1,1}$ for some $i \in \mathbb{Z}_k$. Let C be one of such Hamilton cycles containing e_1 and e_2. Clearly $e_1, e_2 \in S = \{w_i v_{2i,1} | i \in \mathbb{Z}_k\}$ and S is an edge-cut set of $H_{k,m-1,f}$. So deleting e_1 and e_2 from C separates C into two paths and let P be the one joining $v_{2i,1}$ and $v_{2i+k+1,1}$. Then P covers all vertices in $\{v_{i,j} | i \in \mathbb{Z}_{2k}; j = 1, 2, ..., m-1\} \cup \{u_i | i \in \mathbb{Z}_k\}$.

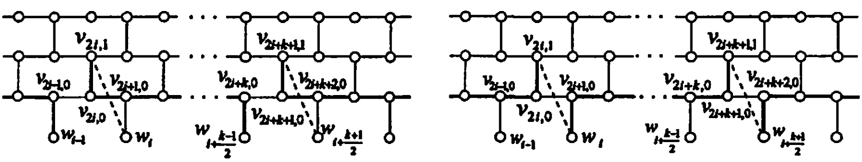

Figure 8: Extending Hamilton cycles of $H_{k,m-1,f}$ to Hamilton cycles of $H_{k,m,f}$.

In the following (see *Figure 8*), we extend C to Hamilton cycles of $H_{k,m,f}$ in two different ways. Let $P_1 = v_{2i,1}v_{2i,0}v_{2i+1,0} \cdots v_{2i+k,0}$ and $P_2 = v_{2i+k+1,1}v_{2i+k+1,0}v_{2i+k+2,0} \cdots v_{2i-1,0}$ and $P_1' = v_{2i,1}v_{2i,0}v_{2i-1,0} \cdots v_{2i+k+2,0}$ and $P_2' = v_{2i+k+1,1}v_{2i+k+1,0}v_{2i+k,0} \cdots v_{2i+1,0}$. Then $P \cup P_1 \cup P_2 \cup P_{i-1,i+\frac{k-1}{2}}$ and $P \cup P_1' \cup P_2' \cup P_{i,i+\frac{k+1}{2}}$ are two Hamilton cycles of $H_{k,m,f}$. Since $H_{k,m-1,f}$ has at least $k2^m$ Hamilton cycles and each of them can be extended to two distinct Hamilton cycles of $H_{k,m,f}$, $H_{k,m,f}$ has $k2^{m+1}$ Hamilton cycles. So the lemma is true for $H_{k,m,f}$.

For $H_{k,m,c}$, using $v_{i,0}v_{i+k,0}$ and $v_{j,m}v_{j',m}$ (i is odd; $j = 2t, j' = 2t + k$ if m is even, and $j = 2t+1, j' = 2t+k+1$, otherwise) instead of $P_{i,i+\frac{k+1}{2}}$ and $P'_{i,i+\frac{k+1}{2}}$, a similar discussion shows that the lemma holds for $H_{k,m,c}$. □

Theorem 1.3 is an immediate corollary of Lemmas 3.1 and 3.2 and can be, therefore, further stated as follows,

Theorem 3.3. *Every Klein-bottle polyhex graph is hamiltonian. In particular, $H_{k,m,a}$ and $H_{k,m,b}$ have at least $2^{\lfloor \frac{m+1}{2} \rfloor}$ Hamilton cycles, and $H_{k,m,c}$ and $H_{k,m,f}$ have at least $k2^{m+1}$ Hamilton cycles.* □

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