Monophonic and Geodetic Dominations in the Join, Corona and Composition of Graphs

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Abstract

A geodetic (resp. monophonic) dominating set in a connected graph G is any set of vertices of G which is both a geodetic (resp. monophonic) set and a dominating set in G. This paper establishes some relationships between geodetic domination and monophonic domination in a graph. It also investigates the geodetic domination and monophonic domination in the join, corona and composition of connected graphs.

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1 Introduction

Throughout this paper we consider only finite graphs with no loops or multiple edges. For any two vertices u and v in a graph G, a *u-v geodesic* refers to any shortest path in G joining u and v. The length of a u-v geodesic is called the *distance* between u and v, and is denoted by $d_G(u,v)$. The *closed geodetic interval* $I_G[u,v]$ is the set of all vertices lying on any u-v geodesic. For a subset S of the vertex set $V(G)$ of G, the *geodetic closure* of S is the set $I_G[S] = \cup_{u,v\in S} I_G[u,v]$. Various concepts inspired by geodetic closures are introduced in [15, 8]. A *geodetic set* in G is any set S of vertices in G satisfying $I_G[S] = V(G)$. The minimum cardinality $g(G)$ of a geodetic set is the *geodetic number* of G. Any geodetic set of cardinality $g(G)$ is refered to as a *minimum geodetic set*. Geodetic sets and geodetic numbers are studied in [1, 2, 3, 4, 6, 7]. We also define $I_G(u,v) = I_G[u,v] \setminus \{u,v\}$ and $I_G(S) = \cup_{u,v\in S} I_G(u,v)$. We call S a *2-path closure absorbing set* if

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for each $x \in V(G) \setminus S$, there exist $u, v \in S$ such that $d_G(u,v) = 2$ and $x \in I_G(u,v)$. Clearly, a 2-path closure absorbing set is always a geodetic set. The minimum cardinality of a 2-path closure absorbing set in G is denoted by $\rho_2(G)$. In [7], the geodetic numbers of the join of graphs are described in terms of 2-path closure absorbing sets.

By a *chord* of a path $[u_1, u_2, \ldots, u_n]$ in a graph G we mean any edge u_iu_j with $j > i+1$. A *u-v monophonic path* is a chordless u-v path. A longest monophonic path is referred to as a *monophonic diametral* path, and its length is denoted by $diam_m(G)$. The *closed monophonic interval* $J_G[u,v]$ consists of all vertices lying in any u-v monophonic path, and for $S \subseteq V(G)$, the *monophonic closure* of S is the set $J_G[S] = \cup_{u,v \in S} J_G[u,v]$. We also define $J_G(u,v) = J_G[u,v] \setminus \{u,v\}$ and $J_G(S) = \cup_{u,v \in S} J_G(u,v)$. In case $J_G[S] = V(G)$, S is called a *monophonic set.* The minimum cardinality $m(G)$ of a monophonic set is called the *monophonic number* of G. Any monophonic set of cardinality $m(G)$ is referred to as a *minimum monophonic set.* Monophonic sets and monophonic numbers are investigated in [10, 11].

The *open neighborhood* of a vertex v in G is the set $N_G(v) = \{u \in V(G) : uv \in E(G)\}$. Any vertex v is an *extreme vertex* if the induced subgraph $\langle N_G(v) \rangle$ is a complete graph. The symbol $Ext(G)$ denotes the set of all extreme vertices in G. The *degree* $deg_G(v)$ of a vertex v refers to the value $|N_G(v)|$, and we define $\Delta(G) = \max\{deg_G(v) : v \in V(G)\}$. The *closed neighborhood* of v is the set $N_G[v] = N_G(v) \cup \{v\}$. For $S \subseteq V(G)$, we define $N_G(S) = \cup_{v \in S} N_G(v)$ and $N_G[S] = N_G(S) \cup S$. If $N_G[S] = V(G)$, then S is a *dominating set* in G. The minimum cardinality among dominating sets in G is called the *domination number* of G, and is denoted by $\gamma(G)$. The concept of domination in graphs has historical roots in a chessboard problem - to find the minimum number of queens needed on a 8×8 chessboard such that all squares are occupied or attacked by a queen. A considerable number of studies have been dedicated in obtaining variations of the concept (see [16, 17, 19]). The authors in [13] cited over 75 variations of domination and listed over 1,200 papers related to domination in graphs.

In [9], the geodetic domination in a graph is introduced. A geodetic set which is at the same time a dominating set is called a *geodetic dominating set.* A geodetic dominating set is also called a *γ_g-set.* The minimum cardinality $\gamma_g(G)$ of a γ_g-set set is called a *geodetic domination number* of G. Any γ_g-set of cardinality $\gamma_g(G)$ is called a *minimum γ_g-set.* The following theorem is found in [9].

Theorem 1.1 [9] *Let G be a connected graph of order $n \geq 2$. Then*

(i) $\gamma_g(G) = 2$ *if and only if there exists a geodetic set* $S = \{u, v\}$ *such*

that $d_G(u,v) \leq 3$.

(ii) *$\gamma_g(G) = n$ if and only if G is the complete graph on n vertices.*

(iii) *$\gamma_g(G) = n-1$ if and only if there is a vertex v in G such that v is adjacent to every other vertex of G and $G - v$ is the union of at least two complete graphs.*

Inspired by the work of H. Escuardo et. al. [9], the present paper introduces the monophonic domination in graphs. Some interesting relationships between the geodetic domination and monophonic domination in graphs are known. The geodetic domination and monophonic domination in the join, corona and composition of graphs are also characterized.

2 Monophonic domination

Any set S of vertices of a connected graph G is a *monophonic dominating set* or a *γ_m-set* in G if S is both a monophonic set and a dominating set in G. The minimum cardinality among all monophonic dominating sets in G is called the *monophonic domination number* of G, and is denoted by $\gamma_m(G)$. Any γ_m-set of cardinality $\gamma_m(G)$ is called *minimum γ_m-set.* Clearly, $\max\{m(G), \gamma(G)\} \leq \gamma_m(G)$ for all connected graphs G.

Let d denote the monophonic diameter of a connected graph G and $P = [u_1, u_2, \ldots, u_{d+1}]$ a monophonic diametral path, and let $k = \lceil \frac{d+3}{3} \rceil$. Put $A = \{u_1, u_4, \ldots, u_{3k-2}\}$ if d is a multiple of 3, and put $A = \{u_1, u_4, \ldots, u_{3(k-1)-2}, u_{d+1}\}$, otherwise. Then $S = (V(G) \setminus V(P)) \cup A$ is a monophonic dominating set in G. Thus

$$\gamma_m(G) \leq |V(G)| - |V(P)| + k = |V(G)| - \left\lfloor \frac{2d}{3} \right\rfloor.$$

In particular, $\gamma_m(P_n) = \lceil \frac{n+2}{3} \rceil = n - \lfloor \frac{2d}{3} \rfloor$.

Since geodetic dominating sets are themselves monophonic dominating sets, $\gamma_m(G) \leq \gamma_g(G)$. Strict inequality in the preceding statement can be verified by considering the cycle $G = C_5$ on 5 vertices. In particular, since every monophonic path in any tree T is actually a geodesic, $\gamma_m(T) = \gamma_g(T)$. See [9] for interesting results on geodetic domination in trees.

Lemma 2.1 *Let G be a connected graph. Then $Ext(G) \subseteq S$ for all monophonic dominating sets S in G.*

It follows from Lemma 2.1 that $\gamma_m(K_n) = n$. If $n \geq 3$ and $G = K_1 + \cup_j K_{n_j}$, where $j \geq 2$ and $\sum_j n_j = n - 1$, then $\gamma_m(G) = n - 1$.

It is worth noting that statement *(ii)* in Theorem 1.1 actually holds when $n = 1$, and it is necessary for statement *(iii)* that $n \geq 3$. The following theorem follows from the preceding results and Theorem 1.1.

Theorem 2.2 *Let G be a connected graph of order $n \geq 2$. Then*

(i) $\gamma_m(G) = 2$ *if and only if there exists a monophonic set* $S = \{u, v\}$ *such that* $d_G(u, v) \leq 3$.

(ii) $\gamma_m(G) = n$ *if and only if G is the complete graph K_n.*

(iii) *If $n \geq 3$, then* $\gamma_m(G) = n - 1$ *if and only if* $G = K_1 + \cup_j K_{n_j}$, *where* $j \geq 2$ *and* $\sum_j n_j = n - 1$

Corollary 2.3 *Let G be a connected graph of order $n \geq 1$. Then*

(i) $\gamma_m(G) = n$ *if and only if* $\gamma_g(G) = n$.

(ii) *If $n \geq 3$, then* $\gamma_m(G) = n - 1$ *if and only if* $\gamma_g(G) = n - 1$.

3 Realization Problems

Theorem 3.1 *Let a and b be positive integers with $2 \leq a < b$. Then*

(i) *a and b are the domination and monophonic domination numbers, respectively, of a connected graph G;*

(ii) *a and b are the monophonic number and monophonic domination number, respectively, of a connected graph G.*

Proof: To prove *(i)*, put $n = 3a - 2$, and let the path P_n be given by $P_n = [x_1, x_2, \ldots, x_n]$. Let G be the graph formed by adding to P_n $(b - a)$ pendant edges x_2u_j, $j = 1, 2, \ldots, b - a$. The set $\{x_2, x_5, \ldots, x_{3k-1}, \ldots, x_{n-2}, x_n\}$ is a minimum dominating set in G. Thus $\gamma(G) = a$. On the other hand, the set $\{u_j, x_1, x_4 \ldots, x_{3k-2}, \ldots, x_n : j = 1, 2, \ldots, (b - a)\}$ is a minimum γ_m-set in G. Thus, $\gamma_m(G) = (b - a) + a = b$.

To prove *(ii)*, obtain a similar graph G with $n = 3(b-a)+4$ having $(a-2)$ pendant edges x_2u_j, $j = 1, 2, \ldots, a-2$. Then $\gamma_m(G) = (a-2)+(b-a+2) = b$. Since the set $\{u_j, x_1, x_n : j = 1, 2, \ldots, (a - 2)\}$ is a minimum monophonic set, $m(G) = (a - 2) + 2 = a$. ■

Theorem 3.2 *For all positive integers d, k and n with $3 \leq d \leq k$ and $n \geq k + 1 + \lfloor \frac{2d}{3} \rfloor$, there exists a connected graph G such that $|V(G)| = n$, $diam_m(G) = d$ and $\gamma_m(G) = k$.*

Proof: We consider three cases.

Case 1: Suppose that $3 = d \leq k$. Then $n \geq k+3$. Let $t = n-k-1$. Let $\{x, y\}$ and $\{u_1, u_2, \ldots, u_t\}$ be the partite sets of the complete bipartite $K_{2,t}$. Take G as the graph G_1 in Figure 1 obtained by adding to $K_{2,t}$ $(k-1)$ pendant edges yv_j, $j = 1, 2, \ldots, k-1$. By Lemma 2.1, $A = \{v_j : j = 1, 2, \ldots, k-1\} \subseteq S$ for all γ_m-sets S in G. Since A is not a γ_m-set in G, $\gamma_m(G) \geq k$. One can easily verify that $A \cup \{x\}$ is a γ_m-set in G. Thus $\gamma_m(G) = k$, $diam_m(G) = 3$ and $|V(G)| = n$.

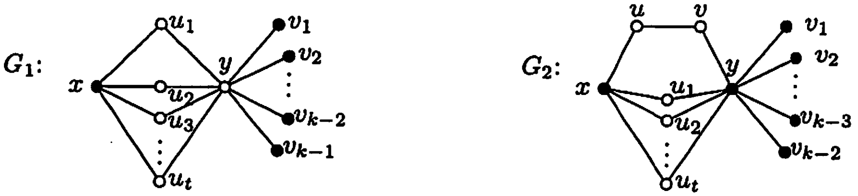

Figure 1

Case 2: Suppose that $4 = d \leq k$. Let $t = n-k-2$, and let $\{x, y\}$ and $\{u_1, u_2, \ldots, u_t\}$ denote the partite sets of $K_{2,t}$. Form G as the graph G_2 in Figure 1 by adding to $K_{2,t}$ the path $[x, u, v, y]$ and $(k-2)$ pendant edges yv_j, $j = 1, 2, \ldots, k-2$. Since $\{x, y, v_j : j = 1, 2, \ldots, k-2\}$ is a γ_m-set in G, $\gamma_m(G) \leq k$. Let S be a γ_m-set in G. Then $|S \cap \{x, u, v, y, u_j : j = 1, 2, \ldots, t\} \geq 2$ so that with Lemma 2.1, $|S| \geq k$. Therefore, $\gamma_m(G) = k$, $diam_m(G) = 4$ and $|V(G)| = n$.

Case 3: Now suppose that $5 \leq d \leq k$. Let $r = k - \lceil \frac{d}{3} \rceil$, and put $t = n-d-r \geq 1$. Obtain the graph G as in Figure 2 from G_2 in Figure 1 by considering monophonic diametral path $P_{d+1} = [x_1, x_2, \ldots, x_d, x_{d+1} = v_j]$ $(j = 1, 2, \ldots, r)$. Put $A = \{x_1, x_4, \ldots, x_{(3\lceil \frac{d}{3} \rceil - 2)}\}$. Then $A \cup \{v_j :$

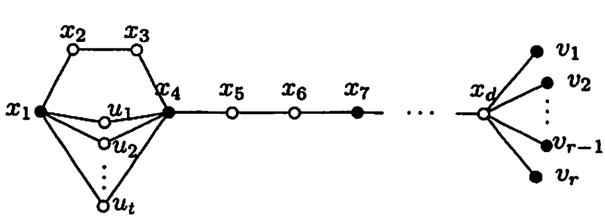

Figure 2

$j = 1, 2, \ldots, v_r\}$ is a γ_m-set. Thus $\gamma_m(G) \leq k$. Let S be a minimum

γ_m-set in G. Then we have $|S \cap \{x_1, x_2, x_3, x_4, u_j : j = 1, 2, \ldots, t\}| \geq 2$. Further, if $P^* = [x_4, x_5, \ldots, x_d, v_j]$ ($j = 1, 2, \ldots, r$), then, by previous remark, $|S \cap V(P^*)| \geq \lceil \frac{d}{3} \rceil$. These results together with Lemma 2.1 yield $\gamma_m(G) \geq 1 + \lceil \frac{d}{3} \rceil + (r-1) = k$. Therefore, $\gamma_m(G) = k$, $diam_m(G) = d$ and $|V(G)| = (d+1) + (r-1) + t = n$. ■

Theorem 3.3 *For all positive integers d, k and n satisfying $1 + \lceil \frac{d}{3} \rceil \leq k \leq d$ and $n \geq k + \lfloor \frac{2d}{3} \rfloor$, there exists a connected graph G such that $|V(G)| = n$, $diam_m(G) = d$ and $\gamma_m(G) = k$.*

Proof: If $d = 2$, then $k = 2$. In this case we may take $G = K_{2,(n-2)}$. Suppose that $d \geq 3$. Let $t = k - \lceil \frac{d}{3} \rceil$, and put $r = n - d - t$. Form the graph G as the graph in Figure 3 by adding to path $P_{d+1} = [x_1, x_2, \ldots, x_{d+1}]$ r paths $[x_2, u_j, x_4]$, $j = 1, 2, \ldots, r$, and $(t-1)$ pendant edges $x_2 v_j$, $j = 1, 2, \ldots, (t-1)$. Following the arguments used in Case 3 of the proof

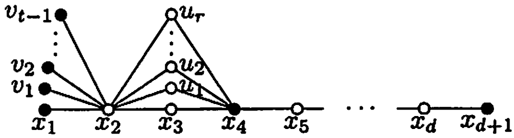

Figure 3

of Theorem 3.2, $\gamma_m(G) = k$, $diam_m(G) = d$ and $|V(G)| = n$. ■

Theorem 3.4 *For every pair of positive integers k and n with $2 \leq k \leq n$, there exists a connected graph G such that $\gamma_m(G) = k$ and $\gamma_g(G) = n$.*

Proof: By Theorem 2.2 and Corollary 2.3, we may take $G = K_n$ when $k = n$. Suppose that $k < n$. We consider two cases:

Case 1: Suppose that $k = 2$. If $n = 3$, we take $G = C_5$. Then $\gamma_m(G) = 2$ and $\gamma_g(G) = 3$. Suppose that $n = 2 + r$ for some positive integer $r \geq 2$. Let G be the graph G_1 as in Figure 4. The set $\{u, v\}$ is the unique γ_m-set, and thus $\gamma_m(G) = 2$. Since the set $\{x, u_1, v_2, u_3, v_4, \ldots, u_{r+1}\}$ (when r is even) or $\{x, u_1, v_2, u_3, \ldots, v_{r+1}\}$ (when r is odd) is a γ_g-set in G, $\gamma_g(G) \leq r + 2$. Now, let $S \subseteq V(G)$ be a γ_g-set in G. Being a geodetic set, for all $j = 1, 2, \ldots, r+1$, $u_j \in S$ or $v_j \in S$. Since $\{u_j, v_j : j = 1, 2, \ldots, r+1\}$ is not a dominating set in G, S contains one of the vertices u, v and x. Thus $|S| \geq r + 2$. Therefore, $\gamma_g(G) \geq r + 2$.

Case 2. Suppose that $k \geq 3$, and $n = k + r$ for some positive integer r. Suppose that $r = 1$, and let x denote a vertex in the cycle graph C_5. Take

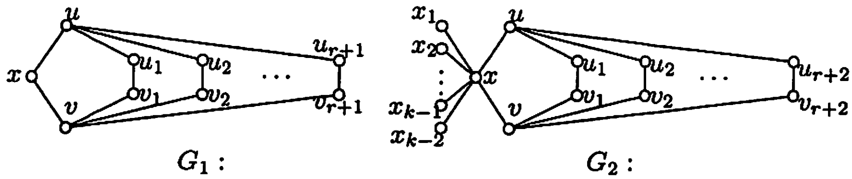

Figure 4

G being the resulting graph after adding to C_5 $(k-2)$ pendant edges xx_j, $j = 1, 2, \ldots, k-2$. On the other hand, if $r \geq 2$, form the graph G as the graph G_2 in Figure 4 by adding to the graph G_1 $(k-2)$ pendant edges x_jx, $j = 1, 2, \ldots, k-2$, and the path $[u, u_{r+2}, v_{r+2}, v]$. In any case, Lemma 2.1 implies that S contains all vertices x_j for all monophonic dominating sets S in G. In view of Case 1, $\gamma_m(G) = (k-2) + 2 = k$. Similarly, $\gamma_g(G) = (k-2) + r + 2 = n$. ■

4 Join of graphs

The *join* of two graphs G and H is the graph $G + H$ with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{uv : u \in V(G), u \in V(H)\}$. In [7], it is known that for noncomplete graphs G and H, the value of $g(G+H)$ is either 2, 3 or 4. In the same paper, characterization of graphs $G + H$ are given for each of these values. It can be readily verified that the same characterizations hold if $g(G + H)$ is replaced by $\gamma_g(G + H)$. Since $\gamma_m(G+H) \leq \gamma_g(G+H)$, we also have $2 \leq \gamma_m(G+H) \leq 4$ for noncomplete graphs G and H. In what immediately follows, we give characterizations for graphs $G + H$ with monophonic domination numbers 2, 3 or 4.

Recall that for connected graphs G and H, the monophonic paths in $G + H$ consist of the monophonic paths in G, the monophonic paths in H, the edges $[u, v]$ where $u \in V(G)$ and $v \in V(H)$, the paths $[u, w, v]$ with $w \in V(H)$, $u, v \in V(G)$ and $uv \notin E(G)$ (in case G is not complete), and the paths $[u, w, v]$ with $w \in V(G)$, $u, v \in V(H)$ and $uv \notin E(H)$ (in case H is not complete) [11].

Lemma 4.1 *If G is a noncomplete graph and $S \subseteq V(G)$ is a γ_m-set in G, then S is a γ_m-set in $G + H$ for any connected graph H.*

Lemma 4.2 *Let G and H be connected graphs. If G is noncomplete and $S \subseteq V(G)$ is a monophonic set in G, then $S \cup \{v\}$ is a γ_m-set in $G + H$*

for all $v \in V(H)$.

Proof: Let $S \subseteq V(G)$ be a monophonic set in G. Then $V(G) \subseteq J_{G+H}[S]$. Further, if G is noncomplete, then S contains vertices x and y with $xy \notin E(G)$. Hence $V(H) \subseteq J_{G+H}[x,y] \subseteq J_{G+H}[S]$. This means that S is a monophonic set in $G+H$. That $S \cup \{v\}$, $v \in V(H)$, is dominating in $G+H$ follows from the fact that $V(H) \subseteq N_{G+H}[S]$ and $V(G) \subseteq N_{G+H}[v]$. ■

Theorem 4.3 *Let G and H be noncomplete connected graphs. Then*

(*i*) $\gamma_m(G+H) = 2$ *if and only* $\gamma_m(G) = 2$ *or* $\gamma_m(H) = 2$.

(*ii*) *If* $m(G) \neq 2$ *and* $m(H) \neq 2$, *then* $\gamma_m(G+H) = 3$ *if and only if* $\gamma_m(G) = 3$ *or* $\gamma_m(H) = 3$.

(*iii*) *If* $m(G) = 2$, *then* $\gamma_m(G+H) = 3$ *if and only if* $\gamma_m(G) > 2$ *and* $\gamma_m(H) > 2$.

(*iv*) $\gamma_m(G+H) = 4$ *if and only if* $m(G) \geq 3$, $m(H) \geq 3$, $\gamma_m(G) > 3$ *and* $\gamma_m(H) > 3$.

Proof: (*i*) is clear. To prove (*ii*), assume $m(G) \neq 2$ and $m(H) \neq 2$. In view of Lemma 4.1, if $\gamma_m(G) = 3$ or $\gamma_m(H) = 3$, then $\gamma_m(G+H) \leq 3$. If $\gamma_m(G+H) = 2$, then $\gamma_m(G) = 2$ or $\gamma_m(H) = 2$, by (*i*). Consequently, $m(G) = 2$ or $m(H) = 2$, a contradiction. Thus $\gamma_m(G+H) = 3$. Conversely, suppose that $\gamma_m(G+H) = 3$, and $\{u,v,w\}$ a γ_m-set in $G+H$. Suppose, further, that $u, v \in V(G)$ and $w \in V(H)$. Then $uv \notin E(G)$ and, as pointed out earlier, $V(G) \setminus \{u,v\} \subseteq J_G[u,v]$. This means that $\{u,v\}$ is a monophonic set in G. Thus, $m(G) = 2$, a contradiction. Similarly, $|V(H) \cap \{u,v,w\}| \neq 2$. Therefore, either $\{u,v,w\} \subseteq V(G)$ or $\{u,v,w\} \subseteq V(H)$. Since the vertices u, v and w are arbitrary, $\gamma_m(G) = 3$ or $\gamma_m(H) = 3$.

To verify (*iii*), assume $m(G) = 2$. By Lemma 4.2, $\gamma_m(G+H) \leq 3$. In view of result (*i*), if $\gamma_m(G) > 2$ and $\gamma_m(H) > 2$, then $\gamma_m(G+H) = 3$. The converse is clear.

Lastly, we will prove (*iv*). Suppose that $\gamma_m(G+H) = 4$. By Lemma 4.1, $\gamma_m(G) > 3$ and $\gamma_m(H) > 3$. Further, by result (*iii*), $m(G) > 2$ and $m(H) > 2$. Conversely, by result (*ii*), $\gamma_m(G+H) \neq 3$. Result (*i*) also implies that $\gamma_m(G+H) \neq 2$. The conclusion follows. ■

Now, we turn to the geodetic domination number and monophonic domination number of the join $G + K_p$.

Theorem 4.4 *Let G be a noncomplete graph and* $p \geq 1$. *Then every geodetic set in* $G + K_p$ *is dominating in* $G + K_p$.

Proof: Let $S \subseteq V(G+K_p)$ be a geodetic set in $G+K_p$. Then $S \cap V(G) \neq \emptyset$, and $V(K_p) \subseteq N_{G+K_p}[S \cap V(G)] \subseteq N_{G+K_p}[S]$. Let $x \in V(G) \setminus S$. There exist $u, v \in S$ such that $x \in I_{G+K_p}[u, v]$. Necessarily, $u, v \in V(G)$. Since $d_{G+K_p}(u, v) = 2$, $[u, x, v]$ is a u-v geodesic in $G + K_p$, consequently in G. In particular, $ux \in E(G)$. Thus, $x \in N_{G+K_p}[S]$. Since x is arbitrary, $V(G) \subseteq N_{G+K_p}[S]$. Therefore, S is a dominating set in $G + K_p$. ■

Corollary 4.5 *For any noncomplete graph G and $p \geq 1$,*

$$\gamma_g(G + K_p) = g(G + K_p).$$

Lemma 4.6 *Let G be a noncomplete graph and $p \geq 1$. If $S \subseteq V(G + K_p)$ is a γ_m-set in $G + K_p$, then $S \cap V(G)$ is a monophonic set in G.*

Proof: Let $S \subseteq V(G+K_p)$ be a γ_m-set in $G+K_p$. Let $x \in V(G) \setminus S$. There exist $u, v \in S$ such that x lies in a u-v monophonic path P in $G+K_p$. This implies that $V(P) \subseteq V(G)$. Thus $x \in J_G[S \cap V(G)]$. Since x is arbitrary, $S \cap V(G)$ is a monophonic set in G. ■

Theorem 4.7 *If G is a noncomplete graph and $p \geq 1$, then*

$$m(G) \leq \gamma_m(G + K_p) \leq m(G) + 1.$$

More precisely,

(*i*) *$\gamma_m(G + K_p) = m(G)$ if and only if G has a minimum monophonic set which is dominating in G;*

(*ii*) *$\gamma_m(G + K_p) = m(G) + 1$ if and only if every minimum monophonic set in G is not dominating in G.*

Proof: The right-hand inequality follows from Lemma 4.2, while the left-hand inequality follows from Lemma 4.6. Suppose that $\gamma_m(G+K_p) = m(G)$. Suppose, further, that G has no minimum monophonic set which is dominating in G. Let S be a minimum γ_m-set in $G + K_p$. By Lemma 4.6, $S \cap V(G)$ is a monophonic set in G. If $S \subseteq V(G)$, then S is not a minimum monophonic set in G. Hence $m(G) < |S| = \gamma_m(G + K_p)$, contrary to our assumption. Therefore, $S \cap V(G) \neq S$ and $m(G) \leq |S \cap V(G)| < |S| = \gamma_m(G+K_p)$. Again, this gives a contradiction. Accordingly, G has a minimum monophonic set which is dominating in G. The converse of (*i*) is clear. Statement in (*ii*) follows immediately from (*i*) and the above inequality. ■

5 Corona of graphs

The *corona* $G \circ H$ of graphs G and H is the graph obtained by taking one copy of G and $|V(G)|$ copies of H, and then joining the ith vertex of G to every vertex in the ith copy of H. It is customary to denote by H_v that copy of H whose vertices are adjoined with the vertex v of G. In effect, $G \circ H$ is composed of the subgraphs $H_v + v$ joined together by the edges of G. If G is the trivial graph, then $G \circ H = H + K_1$.

Throughout this section, G and H are connected graphs.

Theorem 5.1 *Let G be nontrivial. Then $S \subseteq V(G \circ H)$ is a minimum γ_g-set in $G \circ H$ if and only if for every $v \in V(G)$, $S \cap V(H_v)$ is a minimum 2-path closure absorbing set in H_v and $S = \cup_{v \in V(G)}(S \cap V(H_v))$.*

Proof: First, we show that if $S_v \subseteq V(H_v)$ is a 2-path closure absorbing set in H_v for each $v \in V(G)$, then $S = \cup_{v \in V(G)} S_v$ is a γ_g-set in $G \circ H$. For each $v \in V(G)$, $v \in I_{G \circ H}[S_v \cup S_u]$, for all $u \in V(G) \setminus \{v\}$. Thus S is a geodetic set in $G \circ H$. Let $u \in V(G \circ H) \setminus S$. If $u \in V(G)$, then $u \in N_{G \circ H}[S_u] \subseteq N_{G \circ H}[S]$. Suppose that $u \in V(H_v)$. Since S_v is a 2-path closure absorbing set in H_v, there exist $x, y \in S_v$ such that $d_{H_v}(x, y) = 2$ and $u \in I_{H_v}[x, y]$. Thus, $u \in N_{G \circ H}[x] \subseteq N_{G \circ H}[S_v] \subseteq N_{G \circ H}[S]$. Therefore, $S = \cup_{v \in V(G)} S_v$ is a γ_g-set in $G \circ H$.

Next, we show that if $S \subseteq V(G \circ H)$ is a γ_g-set in $G \circ H$, then $S \cap V(H_v)$ is a 2-path closure absorbing set in H_v for all $v \in V(G)$. Let $v \in V(G)$, and let $u \in V(H_v) \setminus (S \cap V(H_v))$. Since S is a geodetic set in $G \circ H$, there exist $x, y \in S$ such that $u \in I_{G \circ H}[x, y]$. Necessarily, $x, y \in S \cap V(H_v)$. Since $xv, yv \in E(G \circ H)$, $d_{H_v}(x, y) = d_{G \circ H}(x, y) = 2$. Thus, $S \cap V(H_v)$ is a 2-path closure absorbing set in H_v.

Now, suppose that S is a minimum γ_g-set in $G \circ H$. For each $v \in V(G)$, let $S_v = S \cap V(H_v)$. By the above results, $S^* = \cup_{v \in V(G)} S_v$ is a γ_g-set in $G \circ H$. If S is a minimum γ_g-set in $G \circ H$, then $S = S^*$. Suppose that for some $v_0 \in V(G)$, $T_{v_0} \subseteq V(H_{v_0})$ is a minimum 2-path closure absorbing set in H_v. By the first result, $(S \setminus S_{v_0}) \cup T_{v_0}$ is a γ_g-set in $G \circ H$. The definition of S implies that $|T_{v_0}| = |S_{v_0}|$. This proves that S_v is a minimum 2-path closure absorbing set in H_v for all $v \in V(G)$. Conversely, suppose that $S = \cup_{v \in V(G)}(S \cap V(H_v))$, where $S \cap V(H_v)$ is a minimum 2-path closure absorbing set in H_v. Then S is a γ_g-set in $G \circ H$. Let $T \subseteq V(G \circ H)$ be a γ_g-set in $G \circ H$. By the second result above, $|S \cap V(H_v)| \leq |T \cap V(H_v)|$ for all $v \in V(G)$. Since T is arbitray, $|S| = \gamma_g(G \circ H)$. This completes the proof of the theorem. ■

Corollary 5.2 *If G is nontrivial, then $\gamma_g(G \circ H) = \rho_2(H) \cdot |V(G)|$. In* *particular,* $\gamma_g(G \circ K_p) = p|V(G)|$.

Theorem 5.3 *Let G be a nontrivial graph and H a noncomplete graph, and let $S \subseteq V(G \circ H)$. Then S is a γ_m-set in $G \circ H$ if and only if $S \cap V(H_v + v)$ is a γ_m-set in $H_v + v$ for all $v \in V(G)$.*

Proof: Let S be a γ_m-set in $G \circ H$. Since H is noncomplete, for each $v \in V(G)$, there exist $x_v, y_v \in S \cap V(H_v)$ such that $d_{H_v}(x_v, y_v) \geq 2$. Let $v \in V(G)$, and let $S_v = S \cap V(H_v + v)$. We first claim that S_v is a monophonic set in $H_v + v$. Let $u \in V(H_v + v) \setminus S_v$. If $u = v$, then we take $x_v, y_v \in S_v \cap V(H_v)$ such that $d_{H_v}(x_v, y_v) \geq 2$. Then $u \in J_{H_v+v}[x_v, y_v] \subseteq J_{H_v+v}[S_v]$. Suppose that $u \neq v$. Since S is a monophonic set in $G \circ H$, there exist $x, y \in S$ such that $u \in J_{G \circ H}[x, y]$. Necessarily $x, y \in S_v$ so that $u \in J_{H_v+v}[S_v]$. This proves that S_v is a monophonic set in $H_v + v$. Second, we show that S_v is dominating in $H_v + v$. Suppose that $v \in S_v$. Then $V(H_v) \subseteq N_{H_v+v}[v] \subseteq N_{H_v+v}[S_v]$. Suppose that $v \notin S_v$. Clearly, $v \in N_{H_v+v}[S_v]$. Let $u \in V(H_v) \setminus S_v$. There exists $w \in S$ such that $u \in N_{G \circ H}[w]$. Since $ux \notin E(G \circ H)$ for all $x \in V(G) \setminus \{v\}$, $w \in V(H_v)$. Thus $u \in N_{H_v}[w] \subseteq N_{H_v}[S_v]$. This means that S_v is a dominating set in $H_v + v$.

Conversely, let $u \in V(G \circ H) \setminus S$. Then for some $v \in V(G)$, $u \in V(H_v + v) \setminus S_v$. There exist $x, y \in S_v = S \cap V(H_v + v)$ such that $u \in J_{H_v+v}[x, y] \subseteq J_{G \circ H}[S]$. This proves that S is a geodetic set in $G \circ H$. Note also that the domination of S_v in $H_v + v$ implies the existence of $x \in S_v$ such that $u \in N_{H_v+v}[x]$. Since $N_{H_v+v}[x] = N_{G \circ H}[x]$, S is a dominating set in $G \circ H$. ■

Theorem 5.3 and Theorem 4.7 yield

Corollary 5.4 *If G is nontrivial and H is noncomplete, then*

$$m(H) \cdot |V(G)| \leq \gamma_m(G \circ H) \leq (1 + m(H)) \cdot |V(G)|.$$

Corollary 5.5 *If G is nontrivial and H is noncomplete, then*

(*i*) $\gamma_m(G \circ H) = (1 + m(H))\,|V(G)|$ *if and only if H has no minimum monophonic set that is dominating.*

(*ii*) $\gamma_m(G \circ H) = m(H)\,|V(G)|$ *if and only if H has a minimum monophonic set that is dominating.*

6 Composition of graphs

The *composition* $G[H]$ of two connected graphs G and H is the graph with $V(G[H]) = V(G) \times V(H)$ and $(u, v)(u', v') \in E(G[H])$ if and only if either $uu' \in E(G)$ or $u = u'$ and $vv' \in E(H)$. Given $S \subseteq V(G[H])$, the *G-projection* S_G of S is the set of all first components of S. That is,

$$S_G = \{x \in V(G) : (x, y) \in S \text{ for some } y \in V(H)\}.$$

Similar definition is given for S_H. For any $A \subseteq V(G)$, we define $A^\circ = A \cap N_G(A)$, $A^g = A \cap I_G(A)$ and $A^m = A \cap J_G(A)$.

Theorem 14 in [10] can be re-stated as follows:

Theorem 6.1 [10] *In the composition $G[K_p]$, where G is connected and $p \geq 2$, $S \subseteq V(G[K_p])$ is a monophonic set in $G[K_p]$ if and only if*

$$S = [(A \setminus A^m) \times V(K_p)] \cup T, \tag{1}$$

where $A \subseteq V(G)$ is a monophonic set in G and $T_G = A^m$.

It should be understood that, in Theorem 6.1, $T \neq \emptyset$ if and only if $A^m \neq \emptyset$. Theorem 6.1 remains true if "monophonic" and A^m are simultaneously replaced by "geodetic" and A^g, respectively (see [2]).

Theorem 6.2 *Let G be a connected noncomplete graph and $p \geq 2$. Then S in Theorem 6.1 is dominating in $G[K_p]$ if and only if A is dominating in G.*

Proof: Suppose that S is dominating in $G[K_p]$. Let $x \in V(G) \setminus A$. Take any $y \in V(K_p)$. There exists $(u, v) \in S$ such that $(x, y) \in N_{G[K_p]}[(u, v)]$. This means that $xu \in E(G)$. Since $u \in S_G = A$, $x \in N_G[A]$, and so A is dominating in G. Conversely, suppose that A is dominating in G. Let $(x, y) \in V(G[K_p]) \setminus S$. Suppose that $x \notin A$. There exists $u \in A$ such that $x \in N_G[u]$. If $u \notin A^m$, then $(u, y) \in S$ and $(x, y) \in N_{G[K_p]}[(u, y)]$. If $u \in A^m$ and $(u, v) \in T$ for some $v \in V(K_p)$, then $(u, v) \in S$ and $(x, y) \in N_{G[K_p]}[(u, v)]$. Suppose that $x \in A$. Then $x \in A^m$ but $(x, y) \notin T$. Pick $w \in V(K_p)$ such that $(x, w) \in T$. Then $(x, y) \in N_{G[K_p]}[(x, w)]$. ∎

Corollary 6.3 *For any connected noncomplete graph G and $p \geq 2$,*

$$\gamma_g(G[K_p]) = \min\{p\,|A| - (p-1)\,|A^g| : A \text{ is a } \gamma_g\text{-set in } G\}, \text{ and}$$

$$\gamma_m(G[K_p]) = \min\{p\,|A| - (p-1)\,|A^m| : A \text{ is a } \gamma_m\text{-set in } G\}.$$

Proof: If $A^g \neq \emptyset$, choose any $v \in V(K_p)$ and put $T = A^g \times \{v\}$. This establishes the formula for $\gamma_g(G[K_p])$. Similar arguments will establish the second result. ∎

Now we turn to the composition $K_p[G]$.

Theorem 6.4 *If G is a connected noncomplete graph and $p \geq 2$, then*

$$2 \leq \gamma_g(K_p[G]) \leq 4.$$

Proof: Let u_1 and u_2 be distinct vertices in $V(K_p)$, and let $v_1, v_2 \in V(G)$ such that $d_G(v_1, v_2) = 2$. Put $S = \{(u_1, v_1), (u_1, v_2), (u_2, v_1), (u_2, v_2)\}$, and suppose that $(x, y) \notin S$. If $x \neq u_1$, then $(x, y)(u_1, v_1)$, $(x, y)(u_1, v_2) \in E(K_p[G])$ so that $(x, y) \in N_{K_p[G]}[S]$ and $(x, y) \in I_{K_p[G]}[(u_1, v_1), (u_1, v_2)] \subseteq I_{K_p[G]}[S]$. Suppose that $x = u_1$. Then $x \neq u_2$, and we have $(x, y) \in N_{K_p[G]}[(u_2, v_1)] \subseteq N_{K_p[G]}[S]$ and $(x, y) \in I_{K_p[G]}[(u_2, v_1), (u_2, v_2)] \subseteq I_{K_p[G]}[S]$. Thus, S is a γ_g-set in $K_p[G]$, and the conclusion follows immediately. ■

Theorem 6.5 *Let G be a connected noncomplete graph and $p \geq 2$. Then*

(*i*) $\gamma_g(K_p[G]) = 2$ *if and only if G has a minimum geodetic set $\{v_1, v_2\}$ with $d_G(v_1, v_2) = 2$.*

(*ii*) $\gamma_g(K_p[G]) = 3$ *if and only if there exists $A \subseteq V(G)$ with $|A| = 3$ and A is a 2-path closure absorbing in G.*

Proof: Let $S = \{(u_1, v_1), (u_2, v_2)\}$ be a γ_g-set in $K_p[G]$. Then $u_1 = u_2$ and $v_1 v_2 \notin E(G)$. Let $u = u_1$, and let $y \in V(G) \setminus \{v_1, v_2\}$. Then $(u, y) \in I_{K_p[G]}[(u, v_1), (u, v_2)]$. Since $diam(K_p[G]) = 2$, $v_1 y, v_2 y \in E(G)$. Thus, $d_G(v_1, v_2) = 2$ and $y \in I_G[v_1, v_2]$. Therefore, $\{v_1, v_2\}$ is a geodetic set in G, and so $g(G) = 2$. Conversely, suppose that $\{v_1, v_2\}$ is a geodetic set in G with $d_G(v_1, v_2) = 2$. Pick $u \in V(K_p)$, and let $S = \{(u, v_1), (u, v_2)\}$. We claim that S is a γ_g-set in $K_p[G]$. Let $(x, y) \in V(K_p[G]) \setminus S$. If $x \neq u$, then $xu \in E(K_p)$ so that $(x, y) \in I_{K_p[G]}[S]$ and $(x, y) \in N_{K_p[G]}[S]$. Suppose that $x = u$. Then $y \neq v_1$ and $y \neq v_2$. Since $\{v_1, v_2\}$ is a geodetic set in G, $y \in I_G[v_1, v_2]$. Since $d_G(v_1, v_2) = 2$, $yv_1, yv_2 \in E(G)$. Consequently, $(x, y)(u, v_1)$, $(x, y)(u, v_2) \in E(K_p[G])$. Hence, $(x, y) \in I_{K_p[G]}[S]$ and $(x, y) \in N_{K_p[G]}[S]$. Therefore, S is a γ_g-set in $K_p[G]$ and $\gamma_g(K_p[G]) = 2$. This proves (*i*).

Suppose that $\gamma_g(K_p[G]) = 3$ and $S = \{(u_1, v_1), (u_2, v_2), (u_3, v_3)\}$ is a γ_g-set in $K_p[G]$. First, we claim that $u_1 = u_2 = u_3$. Suppose that $u_1 \neq u_2$. Since $diam(K_p[G]) = 2$, $V(K_p[G]) = N_{K_p[G]}[(u_1, v_1), (u_3, v_3)]$ or $V(K_p[G]) = N_{K_p[G]}[(u_2, v_2), (u_3, v_3)]$. This means that $\{(u_1, v_1), (u_3, v_3)\}$ or $\{(u_2, v_2), (u_3, v_3)\}$ is a γ_g-set in $K_p[G]$, a contradiction. Denote by u the vertex $u_1 = u_2 = u_3$, and let $A = \{v_1, v_2, v_3\}$. Let $y \in V(G) \setminus A$. In particular, $(u, y) \notin S$. Assume that $(u, y) \in I_{K_p[G]}[(u, v_1), (u, v_2)]$. Since $diam(K_p[G]) = 2$, $(u, v_1)(u, y)$, $(u, v_2)(u, y) \in E(K_p[G])$. This means that $v_1 y, v_2 y \in E(G)$. Consequently, $d_G(v_1, v_2) = 2$ and $y \in I_G[v_1, v_2] \subseteq I_G[T]$.

Since y is arbitrary, A is a 2-path closure absorbing set in G. Conversely, suppose that G has a geodetic set $A = \{v_1, v_2, v_3\}$ which is 2-path closure absorbing in G. Pick $u \in V(K_p)$, and let $S = \{(u, v_1), (u, v_2), (u, v_3)\}$. Let $(x, y) \in V(K_p[G]) \setminus S$. Suppose that $x \neq u$. Since G is noncomplete, we may assume that $v_1v_2 \notin E(G)$. Then $(x, y)(u, v_1)$, $(x, y)(u, v_2) \in E(K_p[G])$. Thus $(x, y) \in I_{K_p[G]}[(u, v_1), (u, v_2)] \subseteq I_{K_p[G]}[S]$ and, in particular, $(x, y) \in N_{K_p[G]}[(u, v_1)] \subseteq N_{K_p[G]}[S]$. Now suppose that $x = u$. Then $y \in V(G) \setminus A$. Since A is 2-path absorbing in G, we may assume that $d_G(v_1, v_2) = 2$ and $y \in I_G[v_1, v_2]$. This means that $v_1y, v_2y \in E(G)$ so that $(u, v_1)(x, y), (u, v_2)(x, y) \in E(K_p[G])$. Thus $(x, y) \in I_{K_p[G]}[(u, v_1), (u, v_2)] \subseteq I_{K_p[G]}[S]$ and, in particular, $(x, y) \in N_{K_p[G]}[(u, v_1)] \subseteq N_{K_p[G]}[S]$. Therefore, S is a γ_g-set in $K_p[G]$. In view of Theorem 6.5(*i*), $\gamma_g(K_p[G]) = 3$. This establishes (*ii*) ■

In view of Theorem 6.4, we also have $2 \leq \gamma_m(K_p[G]) \leq 4$ for any connected noncomplete graph G and $p \geq 2$.

Theorem 6.6 *Let G be a connected noncomplete graph and $p \geq 2$. Then*

(*i*) $\gamma_m(K_p[G]) = 2$ *if and only if* $\gamma_m(G) = 2$.

(*ii*) $\gamma_m(K_p[G]) = 3$ *if and only if* $\gamma_m(G) = 3$ *or* $m(G) = 2$ *but* $\gamma_m(G) \neq 2$.

Proof: Let us prove (*i*). If $\gamma_m(K_p[G]) = 2$, then a γ_m-set in $K_p[G]$ is of the form $S = \{(u, v_1), (u, v_2)\}$, where $v_1v_2 \notin E(G)$. Let $y \in V(G) \setminus \{v_1, v_2\}$. Then $(u, y) \in J_{K_p[G]}[(u, v_1), (u, v_2)]$. Note that if $[(x_1, y_1), (x_2, y_2), \ldots, (x_{n-1}, y_{n-1}), (x_n, y_n)]$ is a (u, v_1)-(u, v_2) monophonic path in $K_p[G]$, then $u = x_k$ for all $k = 1, 2, \ldots, n$. Consequently, $[y_1, y_2, \ldots, y_n]$ is a v_1-v_2 monophonic path. Thus $y \in J_G[v_1, v_2]$. Since $(u, y) \in N_{K_p[G]}[(u, v_1), (u, v_2)]$, we may assume that $(u, y)(u, v_1) \in E(K_p[G])$. This means that $yv_1 \in E(G)$ and so $y \in N_G[v_1]$. This means that $\{v_1, v_2\}$ is a γ_m-set in G. Therefore, $\gamma_m(G) = 2$. The converse follows from the fact that if $\{v_1, v_2\} \subseteq V(G)$ is a γ_m-set in G, then $\{(u, v_1), (u, v_2)\}$ is a γ_m-set in $K_p[G]$ for all $u \in V(K_m)$.

Now, we will prove (*ii*). Suppose that $S = \{(u_1, v_2), (u_2, v_2), (u_3, v_3)\}$ is a minimum γ_m-set in $K_p[G]$. If the vertices u_1, u_2 and u_3 are distinct in K_p, then $J_{K_p[G]}[S] = S$, a contradiction. Let $u_1 = u_2 = u$. Suppose that $u_3 \neq u$. If $v_1v_2 \in E(G)$, then $J_{K_p[G]}[S] = S$, a contradiction. Thus $d_G(v_1, v_2) \geq 2$. We claim that $J_{K_p[G]}[(u, v_1), (u, v_2)] = V(K_p[G])$. Clearly, if $x \neq u$, then $(x, y) \in J_{K_p[G]}[(u, v_1), (u, v_2)]$ for all $y \in V(G)$. On the other hand, since $(u, v_1)(u_3, v_3), (u, v_2)(u_3, v_3) \in E(K_p[G])$ and S is a monophonic set in $K_p[G]$, $(u, y) \in J_{K_p[G]}[(u, v_1), (u, v_2)]$ for all $y \in V(G)$. This establishes the claim, and $\{(u, v_1), (u, v_2)\}$ is a monophonic set in $K_p[G]$. Following previous arguments, $\{v_1, v_2\}$ is a monophonic set in G. Thus $m(G) = 2$. That

$\gamma_m(G) \neq 2$ follows from Theorem 6.6(i). Now, suppose that $u_3 = u$. Let $y \in V(G) \setminus \{v_1, v_2, v_3\}$. Since $(u, y) \in J_{K_p[G]}[S]$, $y \in J_G[\{v_1, v_2, v_3\}]$. Thus $\{v_1, v_2, v_3\}$ is a monophonic set in G. Moreover, since $(u, y) \in N_{K_p[G]}[S]$, $y \in N_G[\{v_1, v_2, v_3\}]$. This implies that $\{v_1, v_2, v_3\}$ is a γ_m-set in G. In view of Theorem 6.6(i), $\gamma_m(G) = 3$.

Conversely, if $\{v_1, v_2, v_3\}$ is a γ_m-set in G, then $\{(u, v_1), (u, v_2), (u, v_3)\}$ is a γ_m-set in $K_p[G]$ for any $u \in V(K_p)$. In view of Theorem 6.6(i), if $\gamma_m(G) = 3$, then $\gamma_m(K_p[G]) = 3$. Suppose that $m(G) = 2$, and let $\{v_1, v_2\}$ be a monophonic set in G. Pick distinct vertices $u, w \in V(K_p)$, and put $S = \{(u, v_1), (u, v_2), (w, v_1)\}$. Then $V(K_p[G]) \subseteq J_{K_p[G]}[(u, v_1), (u, v_2)]$. Note also that $V(K_p[G]) \subseteq N_{K_p[G]}[(u, v_1), (w, v_1)]$. Thus S is a γ_m-set in $K_p[G]$. Because $\gamma_m(G) \neq 2$, Theorem 6.6(i) implies that $\gamma_m(K_p[G]) \neq 2$. Therefore, $\gamma_m(K_p[G]) = 3$. ∎

Finally, we consider the composition $G[H]$, where G is any graph and H is noncomplete. It is easy to verify that if $S \subseteq V(G[H])$ is a dominating set in $G[H]$, then S_G is a dominating set in G. Moreover, for each $u \in A \setminus A^\circ$, $T_u = \{y \in V(H) : (u, y) \in S\}$ is dominating in H. However, it does not follow that if S is a geodetic set in $G[H]$, then S_G is a geodetic set in G. Consider, for example, the composition $P_3[P_3]$ as in Figure 4, with w_{ij} denoting the vertex (v_i, u_j) in $P_3[P_3]$. The set $S = \{(v_2, u_1), (v_2, u_3)\}$ is

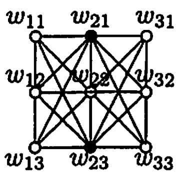

Figure 4: $P_3[P_3]$

both a geodetic set and a dominating set in $P_3[P_3]$, but $S_{P_3} = \{v_2\}$ is not a geodetic set in P_3.

The following lemma is usefull for the next results.

Lemma 6.7 [10] *Let G be any graph and H a noncomplete graph. If $P = [(u_1, v_1), (u_2, v_2), \ldots, (u_n, v_n)]$, where $n \geq 2$, is a monophonic path in $G[H]$, then we have the following possibilities:*

(i) *If the u_i's are dinctinct, then $[u_1, u_2, \ldots, u_n]$ is a monophonic path in G.*

(ii) *If the u_i's are equal, then $[v_1, v_2, \ldots, v_n]$ is a monophonic path in H.*

(iii) *If the u_i's are not distinct and not all equal, then $n = 3$.*

If P in Lemma 6.7 is a geodesic in $G[H]$ and the u_i's are distinct, then $[u_1, u_2, \ldots, u_n]$ is a geodesic in G. On the other hand, if the u_i's are all equal, then $n = 3$ and $[v_1, v_2, v_3]$ is a geodesic in H.

Lemma 6.8 *Let G be a nontrivial graph and H a noncomplete graph, and $S \subseteq G[H]$. If S is a geodetic (resp. monophonic) set in $G[H]$, then $S = \cup_{u \in A} (\{u\} \times T_u)$, where $A \subseteq V(G)$ and $T_u = \{y \in V(H) : (u, y) \in S\}$ satisfying the following:*

(*i*) *If $x \in V(G) \setminus I_G[A]$ (resp. $x \in V(G) \setminus J_G[A]$), then there exists $u \in A$ such that $xu \in E(G)$ and $\{x\} \times V(H) \subseteq I_{G[H]}[\{u\} \times T_u]$ (resp. $\{x\} \times V(H) \subseteq J_{G[H]}[\{u\} \times T_u]$).*

(*ii*) *For each $u \in A \setminus A^g$ (resp. $u \in A \setminus A^m$), T_u is a 2-path closure absorbing (resp. monophonic dominating) set in H.*

Proof: Let $A = S_G$. Let $x \in V(G) \setminus I_G[A]$. Let $y \in V(H)$. Then (x, y) lies on a geodesic $P=[(u_1, y_1), (u_2, y_2), \ldots, (u_n, y_n)]$, where $(u_1, y_1), (u_n, y_n) \in S$. Since $x \notin I_G[A]$, the vertices $u_1, u_2, \ldots, u_n$ should not be distinct in G. Consequently, $n = 3$, $u_1 = u_3 = u$ and $x = u_2$. This means that $ux \in E(G)$ and $(x, y) \in I_{G[H]}[(u, y_1), (u, y_3)] \subseteq I_{G[H]}[\{u\} \times T_u]$. The conclusion in (*i*) follows from the arbitrary nature of y.

Let $u \in A \setminus A^g$, and let $y \in V(H) \setminus T_u$. Then (u, y) lies in a (u_1, v_1)-(u_n, v_n) geodesic $P=[(u_1, v_1), (u_2, v_2), \ldots, (u_n, v_n)]$ in $G[H]$ with $(u_1, v_1), (u_n, v_n) \in S$. Since $u \notin A^g$, $n = 3$, $d_H(v_1, v_2) = 2$ and $y \in I_H[v_1, v_2]$. This means that T_u is a 2-path closure absorbing set in H.

The monophonic counterpart is done similarly. ■

In Lemma 6.8(*i*), it is necessary that $ux \in E(G)$. Under the same condition, for each $x \in V(G) \setminus I_G[A]$ (resp. for each $x \in V(G) \setminus J_G[A]$), pick exactly one $u \in A$ such that $ux \in E(G)$ and denote by $\hat{A}^g$ (resp. $\hat{A}^m$) the set of all such u's.

Theorem 6.9 *Let H be a noncomplete graph. Then for any nontrivial graph G,*

$$\gamma_g(G[H]) \geq \min\{\lambda_1 \cdot \rho_2(H) + \alpha_1 + 2|\hat{A}^g| : A \text{ is dominating in } G\}, \text{and}$$

$$\gamma_m(G[H]) \geq \min\{\lambda_2 \cdot \gamma_m(H) + \alpha_2 + 2|\hat{A}^m| : A \text{ is dominating in } G\},$$

where $\lambda_1 = |A| - |A^\circ \cup A^g \cup \hat{A}^g|$, $\lambda_2 = |A| - |A^\circ \cup A^m \cup \hat{A}^m|$, $\alpha_1 = |(A^\circ \cup A^g) \setminus \hat{A}^g|$ and $\alpha_2 = |(A^\circ \cup A^m) \setminus \hat{A}^m|$.

Proof: Let $S \subseteq V(G[H])$ be a $\gamma_g(G[H])$. Then $S = \cup_{u \in A} (\{u\} \times T_u)$, where $A = S_G$ and $T_u = \{y \in V(H) : (u, y) \in S\}$ satisfying the the conditions in Lemma 6.8. Property (*ii*) of Lemma 6.8 yields the inequality

$$|\cup_{u \in A \setminus (A^\circ \cup A^g \cup \hat{A}^g)} (\{u\} \times T_u)| \geq \lambda_1 \cdot \rho_2(H).$$

By property (*i*) in Lemma 6.8, we may assume that in choosing $u \in \hat{A}^g$, $\{x\} \times V(H) \subseteq I_{G[H]}[\{u\} \times T_u]$. This means that

$$|\cup_{u \in A^\circ \cup A^g \cup \hat{A}^g} (\{u\} \times T_u)| \geq \alpha_1 + 2|\hat{A}^g|.$$

This establishes the first inequality in the theorem.

The second inequality can be done using parallel arguments. ∎

Let us revisit the graph $P_3[P_3]$ in Figure 5. Since $\{w_{21}, w_{23}\}$ is a γ_g-set in $P_3[P_3]$, $\gamma_g(P_3[P_3]) = 2$. On the other hand, the estimate given in Theorem 6.9 has value 2 for this particular graph and is attained when $A^\circ = A^g = \emptyset$ and $\hat{A}^g = A$. This shows that such lower bound is sharp.

Theorem 6.10 *Let G be a nontrivial graph and H a noncomplete graph, and let $A \subseteq V(G)$ be a dominating set in G. For each $u \in A$, let $T_u \subseteq V(H)$ be such that the following hold:*

(*i*) *For each $u \in A \setminus A^\circ$, T_u is a 2-path closure absorbing set in H;*

(*ii*) *For each $u \in A^\circ$, there exist $w, z \in T_u$ such that $d_H(w, z) \geq 2$.*

Then $S = \cup_{u \in A} (\{u\} \times T_u)$ is a γ_g-set in $G[H]$.

Proof: Let $(x, y) \in V(G[H]) \setminus S$. Suppose that $x \notin A$. Either $x \in I_G[A]$ or $x \notin I_G[A]$. If $x \in I_G[A]$ and x lies in a x_1-x_n geodesic $[x_1, x_2, \ldots, x_n]$ with $x_1, x_n \in A$, then (x, y) lies in the geodesic $[(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)]$ for any $y_1, y_2, \ldots, y_n \in V(H)$ with $(x_1, y_1), (x_n, y_n) \in S$. This means that $(x, y) \in I_{G[H]}[S]$. Suppose that $x \notin I_G[A]$. Since A is dominating in G, there exists $v \in A$ such that $xv \in E(G)$. Conditions (*i*) and (*ii*) guarantee the existence of $z_1, z_2 \in T_v$ such that $d_H(z_1, z_2) \geq 2$. (Note in here that since H is noncomplete, if $v \notin A^\circ$, then T_v, being a geodetic set, contains nonadjacent vertices.) Then $(x, y) \in I_{G[H]}[(v, z_1), (v, z_2)]$. Now, suppose that $x \in A$. Then $y \notin T_x$. If $x \notin A^\circ$, then condition (*i*) implies the existence of vertices $z_1, z_2 \in T_x$ such that $d_H(z_1, z_2) = 2$ and $y \in I_H[z_1, z_2]$. Clearly, $(x, z_1), (x, z_2) \in S$ and $(x, y) \in I_{G[H]}[(x, z_1), (x, z_2)] \subseteq I_{G[H]}[S]$. On the other hand, if $x \in A^\circ$ and $u \in A$ such that $xu \in E(G)$, and if $z_1, z_2 \in T_u$ are such that $d_H(z_1, z_2) \geq 2$, then $(x, y) \in I_{G[H]}[(u, z_1), (u, z_2)] \subseteq I_{G[H]}[S]$. Indeed, S is a geodetic set in $G[H]$.

To show that S is dominating in $G[H]$, let $(x,y) \in V(G[H]) \setminus S$. Since A is dominating in G, if $x \notin A$, then there exists $u \in A$ such that $xu \in E(G)$. Pick any $v \in T_u$. Then $(u,v) \in S$ and $(x,y)(u,v) \in E(G[H])$ so that $(x,y) \in N_{G[H]}[S]$. Suppose that $x \in A$. Then $y \notin T_x$. Similarly as above, if $x \notin A^\circ$, then there exist $z_1, z_2 \in T_x$ such that $d_H(z_1,z_2) = 2$ and $y \in I_H[z_1,z_2]$. In particular, $(x,z_1) \in S$ and $z_1 y \in E(H)$ so that $(x,y) \in N_{G[H]}[(x,z_1)] \subseteq N_{G[H]}[S]$. On the other hand, if $x \in A^\circ$ and $u \in A$ such that $xu \in E(G)$, and if $z \in T_u$, then $(x,y) \in N_{G[H]}[(u,z)] \subseteq N_{G[H]}[S]$. This shows that S is dominating in $G[H]$. ■

Corollary 6.11 *Let H be noncomplete and G a nontrivial graph. Then*

$$\gamma_g(G[H]) \leq \min\{|A|\,\rho_2(H) - |A^\circ|\,(\rho_2(H) - 2) : A \text{ is dominating in } G\}.$$

By revisiting again the composition $P_3[P_3]$ or considering the graph $P_4[P_n]$ for $n \geq 3$, we can easily verify that the upperbound given in Corollary 6.11 is sharp.

Parallel arguments will also establish the following results:

Theorem 6.12 *Let G be a nontrivial graph and H a noncomplete graph, and let $A \subseteq V(G)$ be a dominating set in G. For each $u \in A$, let $T_u \subseteq V(H)$ be such that the following hold:*

(*i*) *For each $u \in A \setminus A^\circ$, T_u is a γ_m-set in H;*

(*ii*) *For each $u \in A^\circ$, there exist $w, z \in T_u$ such that $d_H(w,z) \geq 2$.*

Then $S = \bigcup_{u \in A} (\{u\} \times T_u)$ is a γ_m-set in $G[H]$.

Corollary 6.13 *Let H be noncomplete and G a nontrivial graph. Then*

$$\gamma_m(G[H]) \leq \min\{|A|\,\gamma_m(H) - |A^\circ|\,(\gamma_m(H) - 2) : A \text{ is dominating in } G\}.$$

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