Maximum Hexagon Packing of $K_v - L$ Where L is a 2-regular Subgraph

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Abstract

In this paper, we extend the study on packing complete graph K_v with 6-cycles. Mainly, the maximum packing of $K_v - L$ and a leave are obtained where L is a vertex-disjoint union of cycles in K_v.
Keywords: 6-cycle, Complete graph, Maximum packing.

1 Introduction

An H-decomposition of the graph G is a partition of $E(G)$ such that each element of the partition induces a subgraph isomorphic to H. In the case where H is an m-cycle, such a decomposition is referred to as m-cycle decomposition.

A packing of a graph G with 6-cycles (hexagons) is a partition of the edge set of a subgraph P of G, each element of which induces a 6-cycle; the remainder graph of this packing, also known as the leave, is the subgraph $G-P$ formed from G by removing the edges in P. If the remainder graph is empty, we can get 6-cycle decomposition of the graph G. If the remainder graph is minimum in size (that is, has the least number of edges among all possible leaves of G), then the packing is called a maximum packing.

Hanani [7] showed the remainder graphs P for any maximum packing

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of K_v with triangles are as follows:

v(mod 6)	0	1	2	3	4	5
P	F	$\emptyset$	F	$\emptyset$	F_1	C_4

F is a 1-factor, F_1 is an odd spanning forest with $\frac{v}{2}+1$ edges (tripole), and C_4 is a cycle of length 4.

Results in H-decomposition of graph G date back to the nineteenth century [9], but have received a lot of attention over the past 40 years. There have been many results found on H-decompositions of G for various graphs H and G, but mainly on H-decompositions of K_v. The graphs H that have been of most interest are path [17], m-stars [18], m-cycles [13, 8, 11], m-wheels [5] and m-nestings [5, 14]. Recently a paper by Alspach and Gavlas [2] and another by Šajna [15] settled the problem of finding the values of v for which there exists an m-cycle system of K_v and of $K_v - I$, where I is a 1-factor. This can alternatively be viewed as a partial m-cycle system in which the set of edges not in any m-cycles is either $\emptyset$ or induces a 1-factor respectively. These edges not in any m-cycle (or the subgraph they induce) are called the leave L.

Continuing with the theme of finding graph decompositions of graphs which are close to complete, one way to extend these results is to assume L, the leave induces a 2-regular graph and find the necessary and sufficient conditions for the existence of an m-cycle system of $K_v - E(L)$. This naturally generalizes the previously stated results where the leave was empty. In 1986, Colbourn and Rosa [4] used difference methods to find necessary and sufficient conditions for the existence of a 3-cycle system of $K_v - E(L)$ for any 2-regular graph of L. In 1966, Buchanan [3] solved this problem for $m = n$, that is, for Hamilton decompositions of $K_v - E(L)$, by using amalgamations. Fu and Rodger [6] using yet a third approach to this problem, namely induction, settled the existence problem for 4-cycle systems of $K_v - E(L)$, for any 2-regular subgraph of K_v. Leach and Rodger [10] have found necessary and sufficient conditions for the existence of a Hamilton decomposition of the complete bipartite graph $K_{a,b}$ with a 2-regular leave. Recently, Ashe, Rodger and Fu found necessary and sufficient conditions for the existence of a 6-cycle system of $K_v - E(L)$ for every 2-regular not necessarily spanning subgraph L of K_v [1].

In this paper, we extend the results of Ashe, Rodger and Fu [1]. We shall consider maximum hexagon packing of $K_v - L$.

2. Small cases

A cycle of length l is denoted by $C_l = (x_1, x_2, ..., x_l)$. Let A be an m-set, B be an n-set and $A \cap B = \emptyset$. Let a complete bipartite graph with two partite sets A and B be $K_{A,B}$ ($K_{m,n}$). In order to prove our main result, we need to solve the following small cases.

Lemma 2.1 Let $G_1 = K_{\{y_6\},\{y_1,y_2,y_3,x_1,x_2,x_3\}} + K_{\{y_1,y_2,y_3,x_1,x_2,x_3\}} - (y_1, y_2, y_3) - (x_1, x_2, x_3)$, $G_2 = K_{\{x_4,x_t,y_6\},\{y_1,y_2,y_3,x_1,x_2,x_3\}} + K_{\{y_1,y_2,y_3,x_1,x_2,x_3\}} - (y_1, y_2, y_3) - (x_1, x_2, x_3) + x_1x_3 + x_4x_t - x_1x_t - x_3x_4$ and $G_3 = K_{\{y_4,y_5,x_4,x_t,y_6\},\{y_1,y_2,y_3,x_1,x_2,x_3\}} + K_{\{y_1,y_2,y_3,x_1,x_2,x_3\}} - (y_1, y_2, y_3) - (x_1, x_2, x_3) + x_1x_3 + x_4x_t - x_1x_t - x_3x_4 + y_1y_3 + y_4y_5 - y_1y_5 - y_3y_4$, then G_1, G_2 and G_3 can be packed by 6-cycles with a 3-cycle as leave.

Proof. We give the proof by direct construction.

$G_1 = \{(y_6, y_1, x_1, y_3, x_2, y_2), (x_2, y_1, x_3, y_2, x_1, y_6), (x_3, y_6, y_3)\}$.

$G_2 = \{(y_1, x_4, y_2, x_t, y_3, x_3), (y_3, x_4, x_1, y_6, y_2, x_2), (x_3, x_t, y_1, y_6, y_3, x_1), (x_3, y_6, x_2, y_1, x_1, y_2), (x_4, x_2, x_t)\}$.

$G_3 = \{(y_2, y_5, y_3, x_4, x_1, x_3), (x_1, y_5, x_2, y_4, x_3, y_1), (x_3, y_5, y_4, x_1, y_2, x_t), (y_2, y_4, y_1, x_4, x_t, x_2), (x_2, x_4, y_2, y_6, y_1, y_3), (y_3, x_t, y_1, x_2, y_6, x_1), (x_3, y_6, y_3)\}$. □

Lemma 2.2 Let $G_1 = (e_1, e_2, e_3, e_4, e_5, e_6, e_7) + K_{\{e_1,e_2,e_4\},\{y_1,y_2,y_3,x_1,x_2,x_3\}} + K_{\{y_1,y_2,y_3,x_1,x_2,x_3\}} - (y_1, y_2, y_3) - (x_1, x_2, x_3)$, $G_2 = (x_4, e_2, e_3, x_t, e_5, e_6, e_7) + K_{\{y_1,y_2,y_3,x_1,x_2,x_3\}} - (y_1, y_2, y_3) - (x_1, x_2, x_3) + K_{\{x_4,e_2,x_t\},\{y_1,y_2,y_3,x_1,x_2,x_3\}} + x_1x_3 + x_4x_t - x_1x_t - x_3x_4$ and $G_3 = (x_4, y_4, x_t, y_5, e_1, e_2, e_3) + K_{\{y_5,y_4,x_4,x_t,e_1\},\{y_1,y_2,y_3,x_1,x_2,x_3\}} + K_{\{y_1,y_2,y_3,x_1,x_2,x_3\}} - (y_1, y_2, y_3) - (x_1, x_2, x_3) + x_1x_3 + x_4x_t - x_1x_t - x_3x_4 + y_1y_3 + y_4y_5 - y_1y_5 - y_3y_4$, then G_1, G_2 and G_3 can be packed respectively by 6-cycles with a 4-cycle as leave.

Proof. We give the proof by direct construction.

$G_1 = \{(e_4, e_5, e_6, e_7, e_1, x_3), (e_1, e_2, e_3, e_4, x_2, y_2), (x_1, e_4, y_3, x_3, y_2, e_2), (x_2, y_3, x_1, e_1, y_1, e_2), (x_3, y_1, x_2, e_1, y_3, e_2), (y_1, x_1, y_2, e_4)\}$.

$G_2 = \{(x_4, e_2, e_3, x_t, x_3, y_2), (x_t, e_5, e_6, e_7, x_4, x_2), (y_1, x_1, x_4, x_t, y_2, e_2), (x_1, y_3, x_2, y_1, x_3, e_2), (y_2, x_1, x_3, y_3, e_2, x_2), (x_t, y_3, x_4, y_1)\}$.

$G_3 = \{(y_5, e_1, e_2, e_3, x_4, x_1), (x_4, y_4, x_t, y_3, e_1, x_2), (y_2, y_4, y_5, x_3, x_1, e_1), (x_2, y_1, x_3, y_2, x_4, x_t), (x_3, y_3, x_1, y_4, y_1, e_1), (x_3, y_4, x_2, y_5, y_2, x_t), (y_3, x_4, y_1, x_1, y_2, x_2), (x_t, y_5, y_3, y_1)\}$. □

Lemma 2.3 Let $G_1 = (e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8) + K_{\{e_1,e_2,e_5\},\{y_1,y_2,y_3,x_1,x_2,x_3\}} + K_{\{y_1,y_2,y_3,x_1,x_2,x_3\}} - (y_1, y_2, y_3) - (x_1, x_2, x_3)$, $G_2 = (e_1, x_4, e_3, e_4, x_t, e_6, e_7, e_8) + K_{\{y_1,y_2,y_3,x_1,x_2,x_3\}} - (y_1, y_2, y_3) - (x_1, x_2, x_3) + K_{\{e_1,x_4,x_t\},\{y_1,y_2,y_3,x_1,x_2,x_3\}} + x_1x_3 + x_4x_t - x_1x_t - x_3x_4$ and $G_3 = (e_1, x_4, e_3, e_4, x_t, x_5, y_5, e_8) + K_{\{y_5,y_4,x_4,x_t,e_1\},\{y_1,y_2,y_3,x_1,x_2,x_3\}} + K_{\{y_1,y_2,y_3,x_1,x_2,x_3\}} - (y_1, y_2, y_3) - (x_1, x_2, x_3) + x_1x_3 + x_4x_t - x_1x_t - x_3x_4 + y_1y_3 + y_4y_5 - y_1y_5 - y_3y_4$, then G_1, G_2 and G_3 can be packed by 6-cycles with a 5-cycle as leave.

Proof. We give the proof by direct construction.

$G_1 = \{(e_5, e_6, e_7, e_8, e_1, y_1), (e_1, e_2, e_3, e_4, e_5, y_2), (y_1, e_2, y_2, x_1, y_3, x_3), (y_3, e_1, x_1, y_1, x_2, e_5), (e_2, x_1, e_5, x_3, e_1, x_2), (y_3, e_2, x_3, y_2, x_2)\}$.

$G_2 = \{(x_t, e_6, e_7, e_8, e_1, y_1), (e_1, x_4, e_3, e_4, x_t, y_2), (y_1, x_4, y_2, x_1, y_3, x_3), (y_3, e_1, x_1, y_1, x_2, x_t), (x_4, y_3, x_2, y_2, x_3, x_t), (x_3, e_1, x_2, x_4, x_1)\}$.

$G_3 = \{(x_t, x_5, y_5, e_8, e_1, y_1), (e_1, x_4, e_3, e_4, x_t, y_2), (y_1, x_4, y_2, x_1, y_3, x_3), (y_3, e_1, x_1, y_1, x_2, x_t), (x_4, y_3, x_2, y_2, x_3, x_t), (y_4, y_5, x_1, x_3, e_1, x_2), (x_3, y_5, x_2, x_4, x_1, y_4), (y_3, y_5, y_2, y_4, y_1)\}$. □

Lemma 2.4 L is a 2-regular subgraph of K_v. For $v = 7, 9$, $K_v - L$ can be packed by hexagons with leave L_i if and only if $|E(K_v - L)| \equiv i \pmod 6$ where for $v = 7$, $i = 2, 3, 4, 5$ and $v = 9$, $i = 1, 2, 3, 4, 5$, $L_1 = C_7$ or $C_3 \cup C_4$, $L_2 = C_8$ or $C_3 \cup C_5$ or $C_4 \cup C_4$ and $L_i = C_i$ for $i = 3, 4, 5$ respectively.

Proof. When $v = 7$, K_7 is defined on $\{x_j | j \in Z_7\}$.

For $i = 2$, $K_7 - (x_0, x_1, x_2) \cup (x_3, x_4, x_5, x_6) = \{(x_3, x_0, x_4, x_1, x_5, x_2)\} \cup \{(x_5, x_0, x_6, x_1, x_3) \cup (x_2, x_4, x_6)\}$.

$K_7 - (x_0, x_1, x_2, x_3, x_4, x_5, x_6) = \{(x_0, x_2, x_4, x_6, x_3, x_5)\} \cup \{(x_3, x_0, x_4, x_1) \cup (x_5, x_1, x_6, x_2)\}$.

For $i = 3$, $K_7 - (x_0, x_1, x_2) \cup (x_3, x_4, x_5) = \{(x_3, x_0, x_4, x_1, x_6, x_2), (x_5, x_1, x_3, x_6, x_4, x_2)\} \cup \{(x_6, x_0, x_5)\}$.

For $i = 4$, $K_7 - (x_0, x_1, x_5, x_2, x_6) = \{(x_0, x_2, x_4, x_6, x_3, x_5), (x_6, x_1, x_2,$

$x_3, x_4, x_5)\} \cup \{(x_3, x_0, x_4, x_1)\}$.

For $i = 5$, $K_7 - (x_3, x_0, x_4, x_1) = \{(x_0, x_2, x_4, x_6, x_3, x_5), (x_6, x_1, x_2, x_3, x_4, x_5)\} \cup \{(x_0, x_1, x_5, x_2, x_6)\}$.

When $v = 9$, K_9 is defined on $\{x_j | j \in Z_9\}$. Let $S = \{(x_3, x_4, x_6, x_8, x_2, x_5), (x_2, x_6, x_1, x_5, x_7, x_0), (x_6, x_3, x_7, x_1, x_8, x_0), (x_3, x_8, x_5, x_0, x_4, x_1)\}$.

For $i = 1$, then $K_9 - (x_4, x_5, x_6, x_7, x_8) = S \cup \{(x_0, x_1, x_2, x_3) \cup (x_4, x_2, x_7)\}$.

For $i = 2$, $K_9 - (x_0, x_1, x_2, x_3) = S \cup \{(x_4, x_5, x_6, x_7, x_8) \cup (x_4, x_2, x_7)\}$.

For $i = 3$, $K_9 - (x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = \{(x_2, x_5, x_8, x_3, x_7, x_4), (x_8, x_2, x_6, x_3, x_1, x_4), (x_5, x_3, x_0, x_6, x_8, x_1), (x_7, x_5, x_0, x_4, x_6, x_1)\} \cup \{(x_2, x_0, x_7)\}$. $K_9 - (x_0, x_1, x_2, x_3) \cup (x_4, x_5, x_6, x_7, x_8) = \{(x_3, x_4, x_6, x_8, x_2, x_5), (x_2, x_6, x_1, x_5, x_7, x_0), (x_6, x_3, x_7, x_1, x_8, x_0), (x_3, x_8, x_5, x_0, x_4, x_1)\} \cup \{(x_4, x_2, x_7)\}$. $K_9 - (x_0, x_1, x_2) \cup (x_3, x_4, x_5, x_6, x_7, x_8) = \{(x_8, x_0, x_4, x_7, x_1, x_6), (x_4, x_6, x_3, x_5, x_1, x_8), (x_5, x_0, x_3, x_1, x_4, x_2), (x_6, x_2, x_8, x_5, x_7, x_0)\} \cup \{(x_2, x_7, x_3)\}$.

For $i = 4$, $K_9 - (x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7) = \{(x_0, x_4, x_8, x_3, x_6, x_2), (x_4, x_7, x_2, x_5, x_8, x_6), (x_7, x_5, x_1, x_6, x_0, x_3), (x_7, x_8, x_0, x_5, x_3, x_1)\} \cup \{(x_8, x_2, x_4, x_1)\}$. $K_9 - (x_0, x_1, x_2) \cup (x_3, x_4, x_5, x_6, x_7) = \{(x_2, x_3, x_1, x_8, x_6, x_4), (x_0, x_4, x_1, x_7, x_5, x_3), (x_7, x_0, x_6, x_3, x_8, x_4), (x_8, x_2, x_6, x_1, x_5, x_0)\} \cup \{(x_2, x_5, x_8, x_7)\}$. $K_9 - (x_0, x_1, x_2, x_3) \cup (x_4, x_5, x_6, x_7) = \{(x_8, x_1, x_3, x_6, x_2, x_4), (x_7, x_8, x_5, x_0, x_6, x_1), (x_5, x_7, x_3, x_8, x_0, x_2), (x_0, x_4, x_6, x_8, x_2, x_7)\} \cup \{(x_4, x_3, x_5, x_1)\}$.

For $i = 5$, $K_9 - (x_0, x_1, x_2) \cup (x_3, x_4, x_5, x_6) = \{(x_5, x_0, x_4, x_7, x_1, x_3), (x_2, x_6, x_1, x_8, x_0, x_3), (x_8, x_6, x_0, x_7, x_2, x_5), (x_8, x_3, x_7, x_6, x_4, x_2)\} \cup \{(x_7, x_8, x_4, x_1, x_5)\}$. $K_9 - (x_0, x_1, x_2, x_3, x_4, x_5, x_6) = \{(x_5, x_0, x_4, x_7, x_1, x_3), (x_4, x_8, x_3, x_7, x_5, x_2), (x_8, x_1, x_4, x_6, x_7, x_2), (x_6, x_1, x_5, x_8, x_0, x_3)\} \cup \{(x_2, x_0, x_7, x_8, x_6)\}$. □

3. The Main Result

A tool that we will need is from a theorem by Sotteau [16]. Sotteau proved a generalization of the following result. It is stated here for 6-cycle only.

Lemma 3.1 [16] There exists a 6-cycle system of $K_{a,b}$ if and only if:

1) a and b are even.
2) 6 divides a or b. and
3) $\min\{a, b\} \geq 4$.

Next we need the following result which is proved by Ashe, Rodger and Fu [1]

Lemma 3.2 [1] Let G be a vertex-disjoint union of cycles in the complete graph K_v. For each odd $v \geq 7$, there exists a 6-cycle system of $K_v - G$ if and only if $|E(K_v - G)| \equiv 0 \pmod 6$.

Lemma 3.3 For each 2-regular subgraph L of K_v and an integer v, $v \geq 6$, $K_v - L$ can be packed by hexagons with leave L_i if and only if v is odd and $|E(K_v - L)| \equiv i \pmod 6$ where $i = 0, 1, 2, 3, 4, 5$. Here, $L_0 = \emptyset$, $L_1 = C_7$ or $C_3 \cup C_4$, $L_2 = C_8$ or $C_3 \cup C_5$ or $C_4 \cup C_4$ and $L_i = C_i$ for $i = 3, 4, 5$ respectively. And these hold if and only if v, $E(L)$ are related as in Table 1.

Table 1

The number of edges required in L for $|E(K_v - L)| \equiv i \pmod 6$ when v is odd.

v	$12k+1$	$12k+3$	$12k+5$	$12k+7$	$12k+9$	$12k+11$
$\lvert E(K_v - L)\rvert \equiv 1 \pmod 6$, $\lvert E(L)\rvert \pmod 6$	5	2	3	2	5	0
$\lvert E(K_v - L)\rvert \equiv 2 \pmod 6$, $\lvert E(L)\rvert \pmod 6$	4	1	2	1	4	5
$\lvert E(K_v - L)\rvert \equiv 3 \pmod 6$, $\lvert E(L)\rvert \pmod 6$	3	0	1	0	3	4
$\lvert E(K_v - L)\rvert \equiv 4 \pmod 6$, $\lvert E(L)\rvert \pmod 6$	2	5	0	5	2	3
$\lvert E(K_v - L)\rvert \equiv 5 \pmod 6$, $\lvert E(L)\rvert \pmod 6$	1	4	5	4	1	2

Proof. Clearly once the edges in L are removed each vertex must have even degree in order for $K_v - L$ to be packed by hexagons with leave L_i. So v is odd and $|E(K_v - L)| \equiv i \pmod 6$.

Suppose v is odd and $|E(K_v - L)| \equiv i \pmod 6$. Then $|E(L)| = \frac{v(v-1)}{2} - i \pmod 6$ and thus the table1 can be given. □

With the above preparation, we are now in a position to prove our main result. For convenience, we denote the vertex set of graph G by $V(G)$, edge set of G by $E(G)$, the number of edges of G by $|E(G)|$. xy is an edge in a graph with vertex x and y. The union of two graphs $G_1 \cup G_2$ is denoted by $G_1 + G_2$.

Theorem 3.1 For each 2-regular subgraph L of K_v and an integer v, $v \geq 6$, $K_v - L$ can be packed by 6-cycles with leave L_i if and only if $|E(K_v - L)| \equiv i \pmod 6$ where $i = 0, 1, 2, 3, 4, 5$. Here, $L_0 = \emptyset$, $L_1 = C_7$ or $C_3 \cup C_4$, $L_2 = C_8$ or $C_3 \cup C_5$ or $C_4 \cup C_4$ and $L_i = C_i$ for $i = 3, 4, 5$ respectively.

Proof. The necessity is obvious. We only need to prove the sufficiency. Note that Lemma 3.2 is a special case when $i = 0$. Now we consider the cases $i = 1, 2, 3, 4, 5$.

Note 1. It suffices to consider $|E(L)| \geq v - 5$. For otherwise, we can add 6-cycles to enlarge the graph G and then find the packing.

Note 2. When $|E(K_v - L)| \equiv i \pmod 6$ for $i = 3, 4, 5$, we may assume $|E(L)| \geq v - i$. For otherwise, we may add an i-cycle to L and then use Lemma 3.2 to obtain the result.

We give the proof by induction on v. When $v = 7, 9$, we give the proof in Lemma 2.4. Assume the assertion is true for smaller v, we shall prove the assertion is true for v. For clearness, we divide the proof into four cases.

Case(1). $|E(K_v - L)| \equiv 1 \pmod 6$

By Table1 and Note 1, we only consider $v - 1 \geq |E(L)| \geq v - 5$. For any $C = (x_1, x_2, ..., x_t) \in L$ and $x_0 \in V(K_v \setminus L)$, let $L = L^1 + C$. Furthermore, $C^* = (x_0, x_1, ..., x_t) = C + x_0x_1 + x_0x_t - x_1x_t$. Let $L^2 = L^1 + C^*$, then $L = L^2 - x_0x_1 - x_0x_t + x_1x_t$.

So $K_v - L$

$= (K_v - L^2) + x_0x_1 + x_0x_t - x_1x_t$.

Since $|E(K_v - L^2)| \equiv 0 \pmod 6$, we can get 6-cycles collection T of $K_v - L^2$ by applying Lemma 3.2.

(i) If there exists a 6-cycle in T which contains the edge x_1x_t but not contains the vertex x_0, without lose generality, suppose the 6-cycle is $C_6 = (x_1, x_t, y_1, y_2, y_3, y_4)$. We can get the leave $(x_1, x_0, x_t, y_1, y_2, y_3, y_4)$ from

$C_6 + x_0x_1 + x_0x_t - x_1x_t$

(ii) If there exists a 6-cycle in T which contains the edge x_1x_t and the vertex x_0, without lose generality, suppose the 6-cycle is $C_6 = (x_1, x_t, y_1, x_0, y_2, y_3)$. We can get the leave $(y_1, x_0, x_t) \cup (x_0, y_2, y_3, x_1)$ from $C_6 + x_0x_1 + x_0x_t - x_1x_t$.

Case(2). $|E(K_v - L)| \equiv 2, 3 \pmod 6$

(i) If L contains two 3-cycles, let $L = L^1 + C_3^1 + C_3^2$ where $C_3^1 = (y_1, y_2, y_3)$ and $C_3^2 = (x_1, x_2, x_3)$, then we have $K_v - L = (K_{v-6} - L^1) + K_{6,v-7} + K_{1,6} + (K_6 - C_3^1 - C_3^2)$ where $K_{v-6} - L^1$ is defined on $V(K_v \setminus \{y_1, y_2, y_3, x_1, x_2, x_3\})$, $K_{6,v-7}$ is defined on $\{y_1, y_2, y_3, x_1, x_2, x_3\} \cup V(K_v \setminus \{y_1, y_2, y_3, x_1, x_2, x_3, e\})$, $K_{1,6}$ is defined on $\{e\} \cup \{y_1, y_2, y_3, x_1, x_2, x_3\}$ and $K_6 - C_3^1 - C_3^2$ is defined on $\{y_1, y_2, y_3, x_1, x_2, x_3\}$ respectively.

When $|E(K_v - L)| \equiv 2 \pmod 6$, $|E(K_{v-6} - L^1)| \equiv 5 \pmod 6$, by induction, we can get a collection T_1 of hexagons and a C_5 as a leave.

When $|E(K_v - L)| \equiv 3 \pmod 6$, $|E(K_{v-6} - L^1)| \equiv 0 \pmod 6$, by induction, we can get a collection T_1^* of hexagons.

By Lemma 3.2, we can get a collection T_2 of hexagons from $K_{6,v-7}$. By Lemma 2.1, we can get a collection of hexagons T_3 and a C_3 from $K_{1,6} + (K_6 - C_3^1 - C_3^2)$.

Thus $K_v - L = T_1 \cup T_2 \cup T_3 \cup C_3 \cup C_5$ when $|E(K_v - L)| \equiv 2 \pmod 6$ and

$K_v - L = T_1^* \cup T_2 \cup T_3 \cup C_3$ when $|E(K_v - L)| \equiv 3 \pmod 6$.

(ii) If L contains one 3-cycle and one t-cycle ($t \geq 4$), let $L = L^* + C_3^1 + C_t$ where $C_3^1 = (y_1, y_2, y_3)$ and $C_t = (x_1, x_2, x_3, ..., x_t)$. Furthermore, let $L = L^1 + C_3^1 + C_3^2 + x_1x_t + x_3x_4 - x_1x_3 - x_4x_t$ where $L^1 = (x_4, x_5, ..., x_t) + L^*$ and $C_3^2 = (x_1, x_2, x_3)$. Then we have $K_v - L = (K_{v-6} - L^1) + K_{6,v-9} + [K_{3,6} + (K_6 - C_3^1 - C_3^2) - x_1x_t - x_3x_4 + x_1x_3 + x_4x_t]$ where $K_{v-6} - L^1$ is defined on $V(K_v \setminus \{y_1, y_2, y_3, x_1, x_2, x_3\})$, $K_{6,v-9}$ is defined on $\{y_1, y_2, y_3, x_1, x_2, x_3\} \cup V(K_v \setminus \{y_1, y_2, y_3, x_1, x_2, x_3, x_4, x_t, e_1\})$, $K_{3,6}$ is defined on $\{x_4, x_t, e_1\} \cup \{1, 2, 3, x_1, x_2, x_3\}$ and $K_6 - C_3^1 - C_3^2$ is defined on $\{1, 2, 3, x_1, x_2, x_3\}$ respectively.

When $|E(K_v - L)| \equiv 2 \pmod 6$, $|E(K_{v-6} - L^1)| \equiv 5 \pmod 6$, by induction, we can get a collection T_1 of hexagons and a C_5 as a leave.

When $|E(K_v - L)| \equiv 3 \pmod 6$, $E(K_{v-6} - L^1) \equiv 0 \pmod 6$, by induction, we can get 6-cycle collection T_1^*.

By Lemma 3.2, we can get a collection T_2 of hexagons from $K_{6,v-7}$. By Lemma 2.1, we can get a collection of hexagons and a C_3 from $K_{3,6}+(K_6-C_3^1-C_3^2)-x_1x_t-x_3x_4+x_1x_3+x_4x_t$.

Thus $K_v-L=T_1\cup T_2\cup T_3\cup C_3\cup C_5$ when $|E(K_v-L)|\equiv 2 \pmod 6$.

$K_v-L=T_1^*\cup T_2\cup T_3\cup C_3$ when $|E(K_v-L)|\equiv 3 \pmod 6$.

(iii) If L contains one l-cycle and one t-cycle ($l,t\geq 4$), let $L=L^*+C_l+C_t$ where $C_l=(y_1,y_2,y_3,...,y_l)$ and $C_t=(x_1,x_2,x_3,...,x_t)$. Furthermore, let $L=L^1+C_3^1+C_3^2+y_1y_l+y_3y_4-y_1y_3-y_4y_l+x_1x_t+x_3x_4-x_1x_3-x_4x_t$ where $L^1=(y_4,y_5,...,y_l)+(x_4,x_5,...,x_t)+L^*$, $C_3^1=(y_1,y_2,y_3)$ and $C_3^2=(x_1,x_2,x_3)$. Then we have $K_v-L=(K_{v-6}-L^1)+K_{6,v-11}+[K_{5,6}+(K_6-C_3^1-C_3^2)-y_1y_l-y_3y_4+y_1y_3+y_4y_l-x_1x_t-x_3x_4+x_1x_3+x_4x_t]$ where $K_{v-6}-L^1$ is defined on $V(K_v\backslash\{y_1,y_2,y_3,x_1,x_2,x_3\})$, $K_{6,v-11}$ is defined on $\{y_1,y_2,y_3,x_1,x_2,x_3\}\cup V(K_v\setminus\{y_1,y_2,y_3,x_1,x_2,x_3,x_4,x_t,e,y_1,y_4\})$, $K_{5,6}$ is defined on $\{y_1,y_4,x_4,x_t,e\}\cup\{y_1,y_2,y_3,x_1,x_2,x_3\}$ and $K_6-C_3^1-C_3^2$ is defined on $\{y_1,y_2,y_3,x_1,x_2,x_3\}$ respectively.

When $|E(K_v-L)|\equiv 2 \pmod 6$, $|E(K_{v-6}-L^1)|\equiv 5 \pmod 6$, by induction, we can get 6-cycle collection T_1 and a C_5 as a leave.

When $|E(K_v-L)|\equiv 3 \pmod 6$, $|E(K_{v-6}-L^1)|\equiv 0 \pmod 6$, by induction, we can get 6-cycle collection T_1^*.

By Lemma 3.2, we can get a collection T_2 of hexagons from $K_{6,v-11}$. By Lemma 2.1, we can get a collection T_3 of hexagons and a C_3 from $K_{5,6}+(K_6-C_3^1-C_3^2)-y_1y_l-y_3y_4+y_1y_3+y_4y_l-x_1x_t-x_3x_4+x_1x_3+x_4x_t$.

Thus $K_v-L=T_1\cup T_2\cup T_3\cup C_3\cup C_5$ when $|E(K_v-L)|\equiv 2 \pmod 6$.

$K_v-L=T_1^*\cup T_2\cup T_3\cup C_3$ when $|E(K_v-L)|\equiv 3 \pmod 6$.

Case(3). $|E(K_v-L)|\equiv 4 \pmod 6$

(i) If L contains two 3-cycles, let $L=L^1+C_3^1+C_3^2$ where $C_3^1=(y_1,y_2,y_3)$ and $C_3^2=(x_1,x_2,x_3)$, then we have $K_v-L=(K_{v-6}-L^1)+K_{6,v-9}+K_{3,6}+(K_6-C_3^1-C_3^2)$ where $K_{v-6}-L^1$ is defined on $V(K_v\setminus\{y_1,y_2,y_3,x_1,x_2,x_3\})$, $K_{6,v-9}$ is defined on $\{y_1,y_2,y_3,x_1,x_2,x_3\}\cup V(K_v\setminus\{y_1,y_2,y_3,x_1,x_2,x_3,e_1,e_2,e_4\})$, $K_{3,6}$ is defined on $\{e_1,e_2,e_4\}\cup\{y_1,y_2,y_3,x_1,x_2,x_3\}$ and $K_6-C_3^1-C_3^2$ is defined on $\{y_1,y_2,y_3,x_1,x_2,x_3\}$ respectively. Since $|E(K_{v-6}-L^1)|\equiv 1 \pmod 6$, by induction, we can get a collection T_1 of hexagons and a C_7 as a leave. Without lose generality, let $C_7=(e_1,e_2,e_3,e_4,e_5,e_6,e_7)$. By Lemma 3.2, we can get a collection T_2

of hexagons from $K_{6,v-7}$. By Lemma 2.2, we can get a collection T_3 of hexagons and a C_4 from $C_7 + K_{3,6} + (K_6 - C_3^1 - C_3^2)$. Thus $K_v - L = T_1 \cup T_2 \cup T_3 \cup C_4$.

(ii) If L contains one 3-cycle and one t-cycle ($t \geq 4$), let $L = L^* + C_3^1 + C_t$ where $C_3^1 = (y_1, y_2, y_3)$ and $C_t = (x_1, x_2, x_3, ..., x_t)$. Furthermore, let $L = L^1 + C_3^1 + C_3^2 + x_1x_t + x_3x_4 - x_1x_3 - x_4x_t$ where $L^1 = (x_4, x_5, ..., x_t) + L^*$ and $C_3^2 = (x_1, x_2, x_3)$. Then we have $K_v - L = (K_{v-6} - L^1) + K_{6,v-9} + [K_{3,6} + (K_6 - C_3^1 - C_3^2) - x_1x_t - x_3x_4 + x_1x_3 + x_4x_t]$ where $K_{v-6} - L^1$ is defined on $V(K_v \setminus \{y_1, y_2, y_3, x_1, x_2, x_3\})$, $K_{6,v-9}$ is defined on $\{y_1, y_2, y_3, x_1, x_2, x_3\} \cup V(K_v \setminus \{y_1, y_2, y_3, x_1, x_2, x_3, x_4, x_t, e_2\})$, $K_{3,6}$ is defined on $\{x_4, x_t, e_2\} \cup \{1, 2, 3, x_1, x_2, x_3\}$ and $K_6 - C_3^1 - C_3^2$ is defined on $\{y_1, y_2, y_3, x_1, x_2, x_3\}$ respectively. Since $|E(K_{v-6} - L^1)| \equiv 1 \pmod 6$, by induction, we can get a collection T_1 of hexagons and a C_7 as a leave. Without lose generality, let $C_7 = (x_4, e_2, e_3, x_l, e_5, e_6, e_7)$. By Lemma 3.2, we can get a collection T_2 of hexagons from $K_{6,v-7}$. By Lemma 2.2, we can get a collection T_3 of hexagons and a C_4 from $C_7 + K_{3,6} + (K_6 - C_3^1 - C_3^2) - x_1x_t - x_3x_4 + x_1x_3 + x_4x_t$. Thus $K_v - L = T_1 \cup T_2 \cup T_3 \cup C_4$.

(iii) If L contains one l-cycle and one t-cycle ($l, t \geq 4$), let $L = L^* + C_l + C_t$ where $C_l = (y_1, y_2, y_3, ..., y_l)$ and $C_t = (x_1, x_2, x_3, ..., x_t)$. Furthermore, let $L = L^1 + C_3^1 + C_3^2 + y_1y_l + y_3y_4 - y_1y_3 - y_4y_l + x_1x_t + x_3x_4 - x_1x_3 - x_4x_t$ where $L^1 = (y_4, y_5, ..., y_l) + (x_4, x_5, ..., x_t) + L^*$, $C_3^1 = (y_1, y_2, y_3)$ and $C_3^2 = (x_1, x_2, x_3)$. Then we have $K_v - L = (K_{v-6} - L^1) + K_{6,v-11} + [K_{5,6} + (K_6 - C_3^1 - C_3^2) - y_1y_l - y_3y_4 + y_1y_3 + y_4y_l - x_1x_t - x_3x_4 + x_1x_3 + x_4x_t]$ where $K_{v-6} - L^1$ is defined on $V(K_v \setminus \{y_1, y_2, y_3, x_1, x_2, x_3\})$, $K_{6,v-11}$ is defined on $\{y_1, y_2, y_3, x_1, x_2, x_3\} \cup V(K_v \setminus \{y_1, y_2, y_3, x_1, x_2, x_3, x_4, x_t, e_1, y_l, y_4\})$, $K_{5,6}$ is defined on $\{y_l, y_4, x_4, x_t, e_1\} \cup \{y_1, y_2, y_3, x_1, x_2, x_3\}$ and $K_6 - C_3^1 - C_3^2$ is defined on $\{y_1, y_2, y_3, x_1, x_2, x_3\}$ respectively. Since $E(K_{v-6} - L^1) \equiv 1 \pmod 6$, by induction, we can get a collection T_1 of hexagons and a C_7 as a leave. Without lose generality, let $C_7 = (x_4, y_4, x_l, x_t, e_1, e_2, e_3)$. By Lemma 3.2, we can get a collection T_2 of hexagons from $K_{6,v-11}$. By Lemma 2.2, we can get a collection T_3 of hexagons and a C_4 from $C_7 + K_{5,6} + (K_6 - C_3^1 - C_3^2) - y_1y_l - y_3y_4 + y_1y_3 + y_4y_l - x_1x_t - x_3x_4 + x_1x_3 + x_4x_t$. Thus $K_v - L = T_1 \cup T_2 \cup T_3 \cup C_4$.

Case(4). $|E(K_v - L)| \equiv 5 \pmod 6$

(i) If L contains two 3-cycles, let $L = L^1 + C_3^1 + C_3^2$ where $C_3^1 =$

(y_1, y_2, y_3) and $C_3^2 = (x_1, x_2, x_3)$, then we have $K_v - L = (K_{v-6} - L^1) + K_{6,v-9} + K_{3,6} + (K_6 - C_3^1 - C_3^2)$ where $K_{v-6} - L^1$ is defined on $V(K_v \setminus \{y_1, y_2, y_3, x_1, x_2, x_3\})$, $K_{6,v-9}$ is defined on $\{y_1, y_2, y_3, x_1, x_2, x_3\} \cup V(K_v \setminus \{y_1, y_2, y_3, x_1, x_2, x_3, e_1, e_2, e_5\})$, $K_{3,6}$ is defined on $\{e_1, e_2, e_5\} \cup \{y_1, y_2, y_3, x_1, x_2, x_3\}$ and $K_6 - C_3^1 - C_3^2$ is defined on $\{y_1, y_2, y_3, x_1, x_2, x_3\}$ respectively. Since $E(K_{v-6} - L^1) \equiv 2 \pmod 6$, by induction, we can get a collection T_1 of hexagons and a C_8 as a leave. Without lose generality, let $C_8 = (e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8)$. By Lemma 3.2, we can get a collection T_2 of hexagons from $K_{6,v-9}$. By Lemma 2.3, we can get a collection T_3 of hexagons and a C_5 from $C_8 + K_{3,6} + (K_6 - C_3^1 - C_3^2)$. Thus $K_v - L = T_1 \cup T_2 \cup T_3 \cup C_5$.

(ii) If L contains one 3-cycle and one t-cycle ($t \geq 4$), let $L = L^* + C_3^1 + C_t$ where $C_3^1 = (y_1, y_2, y_3)$ and $C_t = (x_1, x_2, x_3, ..., x_t)$. Furthermore, let $L = L^1 + C_3^1 + C_3^2 + x_1x_t + x_3x_4 - x_1x_3 - x_4x_t$ where $L^1 = (x_4, x_5, ..., x_t) + L^*$ and $C_3^2 = (x_1, x_2, x_3)$. Then we have $K_v - L = (K_{v-6} - L^1) + K_{6,v-9} + [K_{3,6} + (K_6 - C_3^1 - C_3^2) - x_1x_t - x_3x_4 + x_1x_3 + x_4x_t]$ where $K_{v-6} - L^1$ is defined on $V(K_v \setminus \{y_1, y_2, y_3, x_1, x_2, x_3\})$, $K_{6,v-9}$ is defined on $\{y_1, y_2, y_3, x_1, x_2, x_3\} \cup V(K_v \setminus \{y_1, y_2, y_3, x_1, x_2, x_3, x_4, x_t, e_1\})$, $K_{3,6}$ is defined on $\{x_4, x_t, e_1\} \cup \{y_1, y_2, y_3, x_1, x_2, x_3\}$ and $K_6 - C_3^1 - C_3^2$ is defined on $\{y_1, y_2, y_3, x_1, x_2, x_3\}$ respectively. Since $E(K_{v-6} - L^1) \equiv 2 \pmod 6$, by induction, we can get 6-cycle collection T_1 and a C_8 as a leave. Without lose generality, let $C_8 = (e_1, x_4, e_3, e_4, x_t, e_6, e_7, e_8)$. By Lemma 3.2, we can get a collection T_2 of hexagons from $K_{6,v-7}$. By Lemma 2.3, we can get a collection T_3 of hexagons and a C_5 from $C_8 + K_{3,6} + (K_6 - C_3^1 - C_3^2) - x_1x_t - x_3x_4 + x_1x_3 + x_4x_t$. Thus $K_v - L = T_1 \cup T_2 \cup T_3 \cup C_5$.

(iii) If L contains one l-cycle and one t-cycle ($l, t \geq 4$), let $L = L^* + C_l + C_t$ where $C_l = (y_1, y_2, y_3, ..., y_l)$ and $C_t = (x_1, x_2, x_3, ..., x_t)$. Furthermore, let $L = L^1 + C_3^1 + C_3^2 + y_1y_l + y_3y_4 - y_1y_3 - y_4y_l + x_1x_t + x_3x_4 - x_1x_3 - x_4x_t$ where $L^1 = (y_4, y_5, ..., y_l) + (x_4, x_5, ..., x_t) + L^*$, $C_3^1 = (y_1, y_2, y_3)$ and $C_3^2 = (x_1, x_2, x_3)$. Then we have $K_v - L = (K_{v-6} - L^1) + K_{6,v-11} + [K_{5,6} + (K_6 - C_3^1 - C_3^2) - y_1y_l - y_3y_4 + y_1y_3 + y_4y_l - x_1x_t - x_3x_4 + x_1x_3 + x_4x_t]$ where $K_{v-6} - L^1$ is defined on $V(K_v \setminus \{y_1, y_2, y_3, , x_1, x_2, x_3\})$, $K_{6,v-11}$ is defined on $\{y_1, y_2, y_3, x_1, x_2, x_3\} \cup V(K_v \setminus \{y_1, y_2, y_3, x_1, x_2, x_3, e_1, y_4, y_l, x_t, x_4\})$, $K_{5,6}$ is defined on $\{e_1, y_4, y_l, x_t, x_4\} \cup \{y_1, y_2, y_3, , x_1, x_2, x_3\}$ and $K_6 - C_3^1 - C_3^2$ is defined on $\{y_1, y_2, y_3, x_1, x_2, x_3\}$ respectively. Since $|E(K_{v-6} - L^1)| \equiv 2 \pmod 6$, by induction, we can get a collection T_1 of hexagons and a C_8

as a leave. Without lose generality, let $C_8 = (e_1, x_4, e_3, e_4, x_t, y_4, y_l, e_8)$. By Lemma 3.2, we can get a collection T_2 of hexagons from $K_{6,v-11}$. By Lemma 2.3, we can get a collection T_3 of hexagons and a C_5 from $C_8 + K_{5,6} + (K_6 - C_3^1 - C_3^2) - y_1y_l - y_3y_4 + y_1y_3 + y_4y_l - x_1x_t - x_3x_4 + x_1x_3 + x_4x_t$. Thus $K_v - L = T_1 \cup T_2 \cup T_3 \cup C_5$. □

4. Open Problems

We consider the packing of $K_v - L$ where L is a 2-regular subgraph with hexagons. If there is a method to consider covering K_v with hexagons? The author once extended the result in [6] to directed versions [12]. If there is a method to extend [1] to directed versions ?

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