Decomposition of Bipartite Graphs Into Spiders All of Whose Legs Are Two in Length

Tay-Woei Shyu*†
Department of Mathematics and Science,
National Taiwan Normal University,
Linkou, New Taipei City 24449, Taiwan, R.O.C.

Ying-Ren Chen, Chiang Lin‡
Department of Mathematics
National Central University
Chung-Li 32001, Taiwan, R.O.C.

Ming-Hong Zhong
National Lo-Tung Senior High School
Luodong, Yilan County 26542, Taiwan, R.O.C.

Abstract. As usual, $K_{m,n}$ denotes the complete bipartite graph with parts of sizes m and n. For positive integers $k \le n$, the crown $C_{n,k}$ is the graph with vertex set $\{a_0, a_1, \ldots, a_{n-1}, b_0, b_1, \ldots, b_{n-1}\}$ and edge set $\{a_i b_j\colon\ 0 \le i \le n-1,\ j = i, i+1, \ldots, i+k-1 \pmod{n}\}$. A spider is a tree with at most one vertex of degree more than two, called the center of the spider. A leg of a spider is a path from the center to a vertex of degree one. Let $S_l(t)$ denote a spider of l legs, each of length t. An H-decomposition of a graph G is an edge-disjoint decomposition of G into copies of H. In this paper we investigate the problems of $S_l(2)$-decompositions of complete bipartite graphs and crowns, and prove that: (1) $K_{n,tl}$ has an $S_l(2)$-decomposition if and only if $nt \equiv 0 \pmod{2}$, $n \ge 2l$ if $t = 1$, and $n \ge l$ if $t \ge 2$, (2) for $t \ge 2$ and $n \ge tl$, $C_{n,tl}$ has an $S_l(2)$-decomposition

*Corresponding author. E-mail address: twhsu@ntnu.edu.tw (Tay-Woei Shyu)

†Research supported by National Science Council of ROC under grant NSC 101-2115-M-003-005.

‡Research supported by National Science Council of ROC under grant NSC 96-2115-M-008-005.

if and only if $nt \equiv 0 \pmod 2$, (3) for $n \geq 3t$, $C_{n,3t}$ has an $S_3(2)$-decomposition if and only if $nt \equiv 0 \pmod 2$ and $n \equiv 0 \pmod 4$ if $t = 1$.

Keywords: decomposition; complete bipartite graph; crown; spider
Mathematics Subject Classification (2010): 05C70

1 Introduction

All graphs considered here are finite and undirected, unless otherwise noted. For standard graph-theoretic terminology the reader is referred to [1].

As usual, $K_{m,n}$ denotes the complete bipartite graph with parts of sizes m and n. For positive integers $k \leq n$, the *crown* $C_{n,k}$ is the graph with vertex set $\{a_0, a_1, \ldots, a_{n-1}, b_0, b_1, \ldots, b_{n-1}\}$ and edge set $\{a_i b_j\colon 0 \leq i \leq n-1,\ j = i, i+1, \ldots, i+k-1 \pmod n\}$. A *spider* is a tree with at most one vertex of degree more than two, called the *center* of the spider (if no vertex of degree more than two, then any vertex can be center). A *leg* of a spider is a path from the center to a vertex of degree one. Thus, a star with l edges is a spider of l legs, each of length one, and a path is a spider of one or two legs. Let $S_l(t)$ denote a spider of l legs, each of length t. For $l \geq 3$ and $t \geq 2$, for convenience, vertices of degree one in $S_l(t)$ are called *end vertices* of $S_l(t)$, and vertices of degree two in $S_l(t)$ are called *internal vertices* of $S_l(t)$. Suppose that H is a subgraph of a graph G. An *H-decomposition* of G is an edge-disjoint decomposition of G into copies of H. Let C_k denote a cycle of length k, P_{k+1} denote a path of length k, and S_{k+1} denote a star with k edges, i.e., $S_{k+1} \cong K_{1,k}$.

There are several papers concerned with decompositions of bipartite graphs into subgraphs. Sotteau [8] gave necessary and sufficient conditions for the existence of a C_{2k}-decomposition of $K_{m,n}$. Fronček [2] and Vanden Eynden [10] gave some results on the decompositions of complete bipartite graphs into cubes. Parker [5], Lin and Shyu [6, 7] gave necessary and sufficient conditions for the existence of a P_{k+1}-decomposition of $K_{m,n}$, $\lambda K_{n,n}$, and $C_{n,k}$, respectively. Truszczyński [9] gave some necessary and/or sufficient conditions for the existence of a P_{k+1}-decomposition of $\lambda K_{m,n}$. About the S_{k+1}-decomposition of bipartite graphs, Yamamoto et al. [11], Lin and Shyu [4] established necessary and sufficient conditions for the existence of a S_{k+1}-decomposition of $K_{m,n}$ and $C_{n,k}$, respectively. Finally, there is a paper by Jacobson, Truszczyński and Tuza [3] which is concerned with the decompositions of regular bipartite graphs. Of the many results proved in that paper three are particularly nice and give the flavor of the subject. (1) Every r-regular bipartite graph decomposes into any double star of size r (a double star is a tree of diameter at most 3). (2) Every 4-regular bipartite graph decomposes into the path P_5 on 4 edges. (3) The

r-dimensional cube Q_r decomposes into any tree T of size r.

In this paper we investigate the problems of $S_l(2)$-decompositions of complete bipartite graphs and crowns. In Section 2, we give necessary and sufficient conditions for the existence of an $S_l(2)$-decomposition of $K_{n,tl}$ (see Theorem 2.1). In Section 3, we give necessary and sufficient conditions for the existence of an $S_l(2)$-decomposition of $C_{n,tl}$ when $n \geq tl$ and l is even (see Theorem 3.4). Besides, we give necessary and sufficient conditions for the existence of an $S_l(2)$-decomposition of $C_{n,tl}$ when $t \geq 2$ and $n \geq tl$ (see Theorem 3.6). Finally, we give necessary and sufficient conditions for the existence of an $S_3(2)$-decomposition of $C_{n,3t}$ (see Theorem 3.7).

2 Decomposition of $K_{m,n}$ into copies of $S_l(2)$

In this section we investigate the problem of $S_l(2)$-decompositions of $K_{m,n}$. Before going into more detail, we need the following notations for our discussion. Let $S(x; x_1, x_2, \ldots, x_k)$ denote the star S_{k+1} centered at vertex x and having $x_1, x_2, \ldots, x_k$ as its other vertices. Suppose that (A, B) is the bipartition of a spanning subgraph G of $K_{n,n}$, where $A = \{a_1, a_2, \ldots, a_n\}$ and $B = \{b_1, b_2, \ldots, b_n\}$. The *label* of an edge a_ib_j in G is the number $j - i$ if $j - i$ is a nonnegative integer or $n + (j - i)$ if $j - i$ is a negative integer.

In the following theorem we will give necessary and sufficient conditions for the existence of a decomposition of $K_{n,m}$ into copies of $S_l(2)$ when $m \equiv 0 \pmod{l}$.

Theorem 2.1 *Let n, t and l be positive integers. $K_{n,tl}$ has an $S_l(2)$-decomposition if and only if the following conditions are fulfilled:*
(1) $nt \equiv 0 \pmod 2$;
(2) $n \geq 2l$ *if* $t = 1$;
(3) $n \geq l$ *if* $t \geq 2$.

Proof. (Necessity) Since $ntl \equiv 0 \pmod{2l}$, we have $nt \equiv 0 \pmod 2$. Suppose that (A, B) is the bipartition of $K_{n,tl}$, where $|A| = n$ and $|B| = tl$. Since for each $S_l(2)$ in the decomposition, either its internal vertices or its end vertices are contained in A, it implies $|A| \geq l$.

Assume $t = 1$. There are $\frac{nl}{2l} = \frac{n}{2}$ copies of $S_l(2)$ in the decomposition. Since $|B| = l$, each $S_l(2)$ in the decomposition must be centered at a different vertex in A. Let A_1 denote the subset consisting of all centers of $S_l(2)$'s in the decomposition. It follows that the end vertices of each $S_l(2)$ in the decomposition must be contained in $A \setminus A_1$, and so $|A \setminus A_1| \geq l$, i.e., $|A| - |A_1| \geq l$. On the other hand, since $|A| = n$ and $|A_1| = \frac{n}{2}$, we have that $n - \frac{n}{2} \geq l$, i.e., $n \geq 2l$.

(Sufficiency) The proof is by construction. We consider four cases as follows.

Case 1. $t = 1$.

By assumption, $\frac{n}{2}$ is an integer. Suppose that (A, B) is the bipartition of $K_{n,l}$, where $A = \{a_1, a_2, \ldots, a_{\frac{n}{2}}, c_1, c_2, \ldots, c_{\frac{n}{2}}\}$ and $B = \{b_1, b_2, \ldots, b_l\}$. For $i \in \{1, 2, \ldots, \frac{n}{2}\}$, let $S(i)$ be the star $S(a_i; b_1, b_2, \ldots, b_l)$ and $M(i)$ be the graph with edge set $\{b_1c_i, b_2c_{i+1}, \ldots, b_jc_{i+j-1}, \ldots, b_lc_{i+(l-1)}\}$, where the subscripts of c_i's are taken modulo $\frac{n}{2}$. Since $n \geq 2l$, $M(i)$ is a matching with l edges. We use $H(i)$ to denote the subgraph induced by the edge set $E[S(i)] \cup E[M(i)]$. It is easily seen that $H(i)$ is an $S_l(2)$; $H(i)$ and $H(j)$ are edge disjoint for $1 \leq i < j \leq \frac{n}{2}$; and so $\{H(i) |\ i = 1, 2, \ldots, \frac{n}{2}\}$ form an $S_l(2)$-decomposition of $K_{n,l}$.

Case 2. $t = 2$.

In this case we prove that $K_{2l,n}$ can be decomposed into n copies of $S_l(2)$. First assume $l \leq n \leq 2l$. Suppose that (A, B) is the bipartition of $K_{2l,n}$, where $A = \{a_1, a_2, \ldots, a_n, c_1, c_2, \ldots, c_{2l-n}\}$ and $B = \{b_1, b_2, \ldots, b_n\}$. For $i \in \{1, 2, \ldots, n\}$, let $S(i)$ be the star $S(a_i; b_i, b_{i+1}, \ldots, b_{i+l-1})$; $M_1(i)$ be the graph with edge set $\{b_ia_{i+1}, b_{i+1}a_{i+3}, \ldots, b_{i+j}a_{i+2j+1}, \ldots, b_{i+(n-l-1)} a_{i+2(n-l-1)+1}\}$ (except in the case $n = l$, in which case we get the null graph); and $M_2(i)$ be the graph with edge set $\{b_{i+(n-l)}c_1, b_{i+(n-l)+1}c_2, \ldots, b_{i+(n-l)+j}c_{j+1}, \ldots, b_{i+(n-l)+2l-n-1}c_{2l-n} (= b_{i+l-1}c_{2l-n})\}$ (except in the case $n = 2l$, in which case we get the null graph), where the subscripts of a_i's and b_i's are taken modulo n. It is not difficult to see that $M_1(i)$ is a matching with $n - l$ edges and $M_2(i)$ is a matching with $2l - n$ edges. We use $H(i)$ to denote the subgraph induced by the edge set $E(S(i)) \cup E(M_1(i)) \cup E(M_2(i))$. Since $l \leq n \leq 2l$, $l = (n - l) + (2l - n)$ and $2(n-l-1)+1 < n$, we have that $H(i)$ is an $S_l(2)$. On the other hand, the set $\{a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n\}$ induces a subgraph G_1 isomorphic to $K_{n,n}$, the set $\{c_1, c_2, \ldots, c_{2l-n}, b_1, b_2, \ldots, b_n\}$ induces a subgraph G_2 isomorphic to $K_{2l-n,n}$, and so $K_{2l,n}$ can be viewed as an edge-disjoint union of G_1 and G_2. It is easily seen that $\{M_2(i) |\ i = 1, 2, \ldots, n\}$ form a decomposition of G_2.

By the definition of labels of edges in G_1, it is not difficult to see that $S(i)$ consists of edges with the following labels in order of $0, 1, 2, \ldots, l-1$, and $M_1(i)$, in order of $-1, -2, \ldots, -(n-l)$, i.e., $n-1, n-2, \ldots, l$. Therefore, $\{S(i) \cup M_2(i) |\ i = 1, 2, \ldots, n\}$ form a decomposition of G_1, and so $\{H(i) |\ i = 1, 2, \ldots, n\}$ form an $S_l(2)$-decomposition of $K_{2l,n}$.

Now assume $n \geq 2l+1$. Let $n = tl + r$ for positive integers t and r with $t \geq 1$ and $l + 1 \leq r \leq 2l$. Since $K_{2l,n}$ can be viewed as an edge-disjoint union of t copies of $K_{2l,l}$ and one copy of $K_{2l,r}$, by Case 1 and the case mentioned above, we are done.

Case 3. $t = 3$.

In this case we prove that $K_{3l,n}$ can be decomposed into $\frac{3n}{2}$ copies of $S_l(2)$. Firstly, assume $l \leq n \leq 2l$. Suppose that (A, B) is the bipartition of $K_{3l,n}$, where $A = \{a_1, a_2, \ldots, a_n, c_1, c_2, \ldots, c_{2l-n}, d_1, d_2, \ldots, d_l\}$ and $B = \{b_1, b_2, \ldots, b_n\}$. The set $\{a_1, a_2, \ldots, a_n, c_1, c_2, \ldots, c_{2l-n}, b_1, b_2, \ldots, b_n\}$ induces a subgraph G isomorphic to $K_{2l,n}$. We define the graphs $H(i)$ the same as in Case 2, and so $\{H(i) |\ i = 1, 2, \ldots, n\}$ form an $S_l(2)$-decomposition of G. For $i \in \{1, 2, \ldots, l\}$, let $M(i)$ be the graph with edge set $\{b_{n-l+1}d_i, b_{n-l+2}d_{i+1}, \ldots, b_{n-l+j}d_{i+j-1}, \ldots, b_n d_{i+(l-1)}\}$, where the subscripts of d_i's are taken modulo l. It is not difficult to see that $M(i)$ is a matching with l edges, and $M(i)$ and $M(j)$ are edge-disjoint for $1 \leq i < j \leq l$. Now we obtain the graph $H'(k)$ from the graph $H(k)$ as follows. If $b_{n-l+i}c_j \in E(H(k))$ with $1 \leq i \leq l$ and $1 \leq j \leq l - \frac{n}{2}$, then we change the edge $b_{n-l+i}c_j$ for the edge $b_{n-l+i}d_{i+j-1}$ instead. Assume $b_{n-l+i}c_j$, $b_{n-l+i'}c_{j'} \in E(H(k))$ with $i' > i$, $(n-l)+1 \leq (n-l)+i$, $(n-l)+i' \pmod{n} \leq n$ and $1 \leq j$, $j' \leq l - \frac{n}{2}$. By the definition of $H(k)$, we have $j' > j$, and by the definition of $H'(i)$, we change the edges $b_{n-l+i}c_j$, $b_{n-l+i'}c_{j'}$ for the edges $b_{n-l+i}d_{i+j-1}$, $b_{n-l+i'}d_{i'+j'-1}$ instead. On the other hand, by the definition of $H(k)$ and $H'(k)$, we have $1 \leq i'-i$, $j'-j \leq l - \frac{n}{2}$. It implies that $0 < |(i+j-1) - (i'+j'-1)| \leq |i-i'| + |j-j'| < 2(l - \frac{n}{2}) = 2l - n \leq l$, i.e., $d_{i+j-1} \neq d_{i'+j'-1}$, and so $H'(k)$ is also an $S_l(2)$ for $k \in \{1, 2, \ldots, n\}$. In fact, $\{b_{n-l+i}c_j |\ 1 \leq i \leq l \text{ and } 1 \leq j \leq l - \frac{n}{2}\} \cong E(\bigcup_{i=1}^{l-\frac{n}{2}} S(c_i; b_{n-l+1}, b_{n-l+2}, \ldots, b_n))$ and $\{b_{n-l+i}d_{i+j-1} |\ 1 \leq i \leq l \text{ and } 1 \leq j \leq l - \frac{n}{2}\} \cong E\left(\bigcup_{i=1}^{l-\frac{n}{2}} M(i)\right)$. Suppose that $G' = G - E(\bigcup_{i=1}^{l-\frac{n}{2}} S(c_i; b_{n-l+1}, b_{n-l+2}, \ldots, b_n)) + E\left(\bigcup_{i=1}^{l-\frac{n}{2}} M(i)\right)$. We have that $\{H'(i) |\ i = 1, 2, \ldots, n\}$ form an $S_l(2)$-decomposition of G'.

Now we show that $K_{3l,n} - E(G')$ can be decomposed into $\frac{n}{2}$ copies of $S_l(2)$. Let $S(n+i)$ be the star $S(b_i; d_1, d_2, \ldots, d_l)$ for $i \in \{1, 2, \ldots, n-l\}$ (except in the case $n = l$, in which case we get the null graph), and let $S(n+(n-l)+i)$ be the star $S(c_i; b_{n-l+1}, b_{n-l+2}, \ldots, b_n)$ for $i \in \{1, 2, \ldots, l - \frac{n}{2}\}$ (except in the case $n = 2l$, in which case we get the null graph). For $i \in \{n+1, n+2, \ldots, n+\frac{n}{2}\}$, we use $H'(i)$ to denote the subgraph induced by the edge set $E(S(i)) \cup E(M(i - (\frac{3n}{2} - l)))$. It is easily seen that $H'(i)$ is an $S_l(2)$, and $\{H'(i) |\ i = n+1, n+2, \ldots, \frac{3n}{2}\}$ form an $S_l(2)$-decomposition of $K_{3l,n} - E(G')$.

Now we assume $n \geq 2l+1$. Since $K_{3l,n}$ can be viewed as an edge-disjoint union of $K_{2l,n}$ and $K_{l,n}$, we are done, by Case 1 and Case 2.

Case 4. $t \geq 4$.

Assume t is odd. Write $t = 2r + 3$ for positive integer r. Since $K_{tl,n}$ can be viewed as an edge-disjoint union of r copies of $K_{2l,n}$ and one copy of $K_{3l,n}$, we are done, by Case 2 and Case 3. Now assume t is even. Write $t = 2r$

for positive integer r. Since $K_{tl,n}$ can be viewed as an edge-disjoint union of r copies of $K_{2l,n}$, we are done, by Case 2. □

3 Decomposition of $C_{n,k}$ into copies of $S_l(2)$

In this section we investigate the problem of decomposing crowns $C_{n,k}$ into copies $S_l(2)$. In the following theorem we will give necessary conditions for the existence of an $S_l(2)$-decomposition of $C_{n,tl}$. It is easily seen that $S_l(2)$ is a path for $l \in \{1, 2\}$. Therefore, assume $l \geq 3$.

Theorem 3.1 *Let n, t and l be positive integers such that $l \geq 3$ and $n \geq tl$. If $C_{n,tl}$ has an $S_l(2)$-decomposition, then the following conditions hold:*
(1) $nt \equiv 0 \pmod 2$;
(2) $n \geq 2l$ *if* $t = 1$ *and* l *is even;*
(3) $n \geq 4l - 8$ *if* $t = 1$ *and* l *is odd.*

Proof. Since $ntl \equiv 0 \pmod{2l}$, we have $nt \equiv 0 \pmod 2$. Assume $t = 1$. Suppose that (A, B) is the bipartition of $C_{n,l}$, where $A = \{a_0, a_1, \ldots, a_{n-1}\}$ and $B = \{b_0, b_1, \ldots, b_{n-1}\}$. Let D be an arbitrary decomposition of $C_{n,l}$ into $\frac{n}{2}$ copies of $S_l(2)$. Firstly, assume that either A or B contains all centers of $S_l(2)$'s in D. By the same argument in the proof of Theorem 2.1, we have $n \geq 2l$.

Now assume that both A and B contain centers of $S_l(2)$'s in D. Let $S_l^*(2)$ and $S_l^{**}(2)$ denote two copies of $S_l(2)$ in D which are centered at a_i and b_j, respectively. By the definition of $C_{n,l}$, the internal vertices of $S_l^*(2)$ and $S_l^{**}(2)$ are $b_i, b_{i+1}, \ldots, b_{i+(l-1)}$ and $a_{j-(l-1)}, a_{j-(l-1)+1}, \ldots, a_j$, respectively, where the subscripts of a_i's and b_i's are taken modulo n. Since the degree of each vertex in $C_{n,l}$ is l, we have that a_i can not be an internal vertex of $S_l^{**}(2)$ (b_j can not be an internal vertex of $S_l^*(2)$), i.e., $i \notin \{j-(l-1), j-(l-1)+1, \ldots, j\}$, and $j \notin \{i, i+1, \ldots i+(l-1)\}$. Without loss of generality we can assume that there does not exist an i' in $\{i+1, i+2, \ldots, j-(l-1)-1\}$ such that $a_{i'}$ is a center of $S_l(2)$ in D, and there does not exist an j' in $\{i+(l-1)+1, i+(l-1)+2, \ldots, j-1\}$ such that $b_{j'}$ is a center of $S_l(2)$ in D. It follows that vertex $a_{j-(l-1)}$ is not an internal vertex of any $S_l(2)$ in D except $S_l^{**}(2)$; vertex $b_{i+(l-1)}$ is not an internal vertex of any $S_l(2)$ in D except $S_l^*(2)$. Since the degree of each vertex in $C_{n,l}$ is l, we have that vertex $a_{j-(l-1)}$ is an end vertex of $l-2$ copies of $S_l(2)$ centered at A in D, and vertex $b_{i+(l-1)}$ is an end vertex of $l-2$ copies of $S_l(2)$ centered at B in D. Therefore, there are at least $2(l-2)$ copies $S_l(2)$ in D. Since the cardinality of D is $\frac{n}{2}$, we have $\frac{n}{2} \geq 2(l-2)$, i.e., $n \geq 4l-8$.

If l is odd, then the degree of each vertex in $C_{n,l}$ is odd. Since the degree of each internal vertex of $S_l(2)$ is two, we have that each vertex in

$C_{n,l}$ can not only be an internal vertices of $S_l(2)$'s in D. It follows that all centers of $S_l(2)$'s in D can not be contained in the same part set. By the argument mentioned above, we have $n \geq 4l - 8$. □

In the following lemma we will prove that $C_{n,tl}$ can be decomposed into copies of $S_l(2)$ when t is even.

Lemma 3.2 *Let n, t and l be positive integers such that $n \geq tl$. If t is even, then $C_{n,tl}$ has an $S_l(2)$-decomposition.*

Proof. Since $C_{n,tl} - E(C_{n,2l}) \cong C_{n,(t-2)l}$, we have that $C_{n,tl}$ can be view as an edge-disjoint union of $\frac{t}{2}$ copies of $C_{n,2l}$, and so it is sufficient to show that $C_{n,2l}$ has an $S_l(2)$-decomposition.

Suppose that (A, B) is the bipartition of $C_{n,2l}$, where $A = \{a_0, a_1, \ldots, a_{n-1}\}$ and $B = \{b_0, b_1, \ldots, b_{n-1}\}$. For $i \in \{0, 1, \ldots, n-1\}$, let $S(i)$ be the star $S(a_i; b_i, b_{i+1}, \ldots, b_{i+l-1})$ and $M(i)$ be the matching with edge set $\{b_i a_{n-(2l-1)+i}, b_{i+1} a_{n-(2l-1)+i+2}, \ldots, b_{i+j} a_{n-(2l-1)+i+2j}, \ldots, b_{i+l-1} a_{n-(2l-1)+i+2(l-1)}\}$. It is not difficult to see that $S(i)$ consists of edges with the following labels in order of $0, 1, 2, \ldots, l-1$, and $M(i)$, in order of $2l-1, 2l-2, \ldots, l$. Since the largest subscript of a_i's in $M(i)$ is $n - (2l-1) + i + 2(l-1) = n + i - 1 \equiv i - 1 \pmod{n}$, we have that the edge set $E[S(i)] \cup E[M(i)]$ induces a subgraph $H(i)$ isomorphic to $S_l(2)$. It is easy to see that $\{H(i) | \; i = 0, 1, \ldots, n-1\}$ form an $S_l(2)$-decomposition of $C_{n,2l}$. □

In the following lemma we will prove that $C_{n,l}$ can be decomposed into copies of $S_l(2)$ when n and l are even.

Lemma 3.3 *Let n and l be positive integers such $n \geq 2l$. If n and l are even, then $C_{n,l}$ has an $S_l(2)$-decomposition.*

Proof. Suppose that (A, B) is the bipartition of $C_{n,l}$, where $A = \{a_0, a_1, \ldots, a_{n-1}\}$ and $B = \{b_0, b_1, \ldots, b_{n-1}\}$. For $i \in \{1, 3, \ldots, n-1\}$, let $S(i)$ be the star $S(a_i; b_i, b_{i+1}, \ldots, b_{i+l-1})$ and $M(i)$ be the graph with edge set $\{b_i a_{n-(l-1)+i}, b_{i+1} a_{n-(l-2)+i+1}, \ldots, b_{i+j} a_{n-(l-j-1)+i+j}, \ldots, b_{i+l-1} a_{n+i+(l-1)}\}$. Since $n \geq 2l$, $M(i)$ is a matching. Besides, since i is odd and both n and l are even, we have that all of the subscripts of a_j's in $M(i)$ are even. It implies that the edge set $E[S(i)] \cup E[M(i)]$ induces a subgraph $H(i)$ isomorphic to $S_l(2)$. It is not difficult to see that $S(i)$ consists of edges with the following labels in order of $0, 1, 2, \ldots, l-1$, and $M(i)$, in order of $l-1, l-2, \ldots, 0$. It is easy to see that $\{H(i) | \; i = 1, 3, \ldots, n-1\}$ form an $S_l(2)$-decomposition of $C_{n,l}$. □

In the following theorem we will give necessary and sufficient conditions for the existence of an $S_l(2)$-decomposition of $C_{n,tl}$ when l is even.

Theorem 3.4 *Let n, t and l be positive integers such that $n \geq tl$ and l is even. $C_{n,tl}$ has an $S_l(2)$-decomposition if and only if $nt \equiv 0 \pmod 2$ and $n \geq 2l$ if $t = 1$.*

Proof. (Necessity) By Theorem 3.1, we have that $nt \equiv 0 \pmod 2$ and $n \geq 2l$ if $t = 1$.

(Sufficiency) If t is even, then we are done by Lemma 3.2. Now assume t is odd. Write $t = 2u+1$ for nonnegative integer u. Since $C_{n,tl} - E(C_{n,2l}) \cong C_{n,(t-2)l}$, $C_{n,tl}$ can be view as an edge-disjoint union of u copies of $C_{n,2l}$ and one copy of $C_{n,l}$. By Lemma 3.2-3.3, we are done. □

In the following lemma we will prove that $C_{n,3l}$ can be decomposed into copies of $S_l(2)$ when l is odd.

Lemma 3.5 *Let n and l be positive integers such that $n \geq 3l$ and l is odd. If n is even, then $C_{n,3l}$ has an $S_l(2)$-decomposition.*

Proof. We prove that $C_{n,3l}$ can be decomposed into $\frac{3n}{2}$ copies of $S_l(2)$ as follows. Suppose that (A, B) is the bipartition of $C_{n,3l}$, where $A = \{a_0, a_1, \ldots, a_{n-1}\}$ and $B = \{b_0, b_1, \ldots, b_{n-1}\}$. Assume that for $i \in \{0, 1, \ldots, \frac{n}{2}-1\}$, $S(i)$ denotes the star $S(a_{2i}; b_{2i}, b_{2i+1}, \ldots, b_{2i+l-1})$ and $M(i)$ denotes the graph with edge set $\{b_{2i}a_{2i-l}, b_{2i+1}a_{2i+1-(l-1)}, \ldots, b_{2i+j}a_{2i+j-(l-j)}, \ldots, b_{2i+(l-1)}a_{2i+(l-1)-1}\}$. Since l is odd, we have that all of the subscripts of a_j's in $M(i)$ are odd. On the other hand, since $n \geq 3l$, it follows that $M(i)$ is a matching and the edge set $E[S(i)] \cup E[M(i)]$ induces a subgraph $H(i)$ isomorphic to $S_l(2)$. Besides, it is not difficult to see that $S(i)$ consists of edges with the following labels in order of $0, 1, 2, \ldots, l-1$, and $M(i)$, in order of $l, l-1, \ldots, 1$. It is easily seen that $H(i)$ and $H(j)$ are edge disjoint for $0 \leq i < j \leq \frac{n}{2} - 1$.

Now assume that for $i \in \{0, 1, \ldots, n-1\}$, $S^*(i)$ denotes the star $S(b_i; a_{i-(l+1)}, a_{i-(1+2)}, \ldots, a_{i-(2l)})$; for $i \in \{0, 2, \ldots, n-2\}$ (i is even), $M^*(i)$ denotes the graph with edge set $\{a_{i-(2l)}b_{i+1}, a_{i-(2l-1)}b_{i+3}, \ldots, a_{i-(2l-j)}b_{i+2j+1}, \ldots, a_{i-(l+2)}b_{i+2l-3}, a_{i-(l+1)}b_{i-1}\}$; for $i \in \{1, 3, \ldots, n-1\}$ (i is odd), $M^*(i)$ denotes the graph with edge set $\{a_{i-(2l)}b_{i+1}, a_{i-(2l-1)}b_{i+3}, \ldots, a_{i-(2l-j)}b_{i+2j+1}, \ldots, a_{i-(l+2)}b_{i+2l-3}, a_{i-(l+1)}b_{i-(l+1)}\}$. It is clearly that $M^*(i)$ is a matching. Since $n \geq 3l$, we have that $b_i \notin V(M^*(i))$, and so the edge set $E[S^*(i)] \cup E[M^*(i)]$ induces a subgraph $H^*(i)$ isomorphic to $S_l(2)$. On the other hand, it is not difficult to see that $S^*(i)$ consists of edges with the following labels in order of $l+1, l+2, \ldots, 2l$, if i is even, then $M^*(i)$, in order of $2l+1, 2l+2, \ldots, 3l-1, l$, and if i is odd, then $M^*(i)$, in order of $2l+1, 2l+2, \ldots, 3l-1, 0$. It is easily seen that for $0 \leq i < j \leq n-1$, $H^*(i)$ and $H^*(j)$ are edge disjoint.

For $i \in \{0, 1, \ldots, \frac{n}{2}-1\}$, the edge with label l in $H(i)$ is $b_{2i}a_{2i-l}$. For $j \in \{0, 2, \ldots, n-2\}$ (j is even), the edge with label l in $H^*(j)$ is $a_{j-(l+1)}b_{j-1}$.

Since j is even and l is odd, $\{b_{2i}a_{2i-l} \mid i = 0, 1, \dots, \frac{n}{2}-1\} \cap \{a_{j-(l+1)}b_{j-l} \mid j = 0, 2, \dots, n-2\}$ is a null set. On the other hand, for $i \in \{0, 1, \dots, \frac{n}{2}-1\}$, the edge with label 0 in $H(i)$ is $b_{2i}a_{2i}$. For $j \in \{1, 3, \dots, n-1\}$ (j is odd), the edge with label 0 in $H^*(j)$ is $a_{j-(l+1)}b_{j-(l+1)}$. Since j is odd and l is odd, $\{b_{2i}a_{2i} \mid i = 0, 1, \dots, \frac{n}{2}-1\} \cap \{a_{j-(l+1)}b_{j-(l+1)} \mid j = 1, 3, \dots, n-1\}$ is also a null set. It implies that for $0 \le i \le \frac{n}{2}-1$ and $0 \le j \le n-1$, $H(i)$ and $H^*(j)$ are edge disjoint. Therefore, $C_{n,3l}$ can be decomposed into $\frac{3n}{2}$ copies of $S_l(2)$ as follows: $H(0), H(1), \dots, H(\frac{n}{2}-1), H^*(0), H^*(1), \dots, H^*(n-1)$. □

The following theorem follows from Lemma 3.2 and Lemma 3.5.

Theorem 3.6 *Let n, t and l be positive integers such that $t \ge 2$ and $n \ge tl$. $C_{n,tl}$ has an $S_l(2)$-decomposition if and only if $nt \equiv 0 \pmod 2$.*

Proof. (Necessity) Condition $nt \equiv 0 \pmod 2$ is trivial.

(Sufficiency) By Lemma 3.2, it is sufficient to deal with the case where t is odd. Assume that $t = 2s + 3$ where s is a nonnegative integer. It is easy to see that $C_{n,tl}$ can be decomposed into $C_{n,2sl}$ and $C_{n,3l}$. By Lemma 3.2 and 3.5, we are done. □

In the following theorem we will give necessary and sufficient conditions for the existence of an $S_3(2)$-decomposition of $C_{n,3t}$.

Theorem 3.7 *Let n and t be positive integers such that $n \ge 3t$. $C_{n,3t}$ has an $S_3(2)$-decomposition if and only if $nt \equiv 0 \pmod 2$ and $n \equiv 0 \pmod 4$ if $t = 1$.*

Proof. Suppose that (A, B) is the bipartition of $C_{n,3t}$, where $A = \{a_0, a_1, \dots, a_{n-1}\}$ and $B = \{b_0, b_1, \dots, b_{n-1}\}$.

(Necessity) By Theorem 3.1, we have that $nt \equiv 0 \pmod 2$, and if $t = 1$, then $n \ge 4$ and n is even. Assume $t = 1$. Let D be an arbitrary decomposition of $C_{n,3}$ into $\frac{n}{2}$ copies of $S_3(2)$. Suppose that there are k copies of $S_3(2)$ in D, each centered at a different vertex in A. Since the degree of each vertex in $C_{n,3}$ is 3, and the degree of each internal vertex in $S_3(2)$ is 2, it follows that there are $3k$ vertices in B which are internal vertices of the k copies of $S_3(2)$ in D, each centered at a different vertex in A, and those $3k$ internal vertices in B must be also end vertices of k copies of $S_3(2)$ in D, each centered at a different vertex in B. It implies that $|D| = 2k$. Therefore, $\frac{n}{2}(= |D|)$ is even, i.e., $n \equiv 0 \pmod 4$.

(Sufficiency) If $t \ge 2$, then we are done by Theorem 3.6. Now assume $t = 1$. In this case we will show that $C_{n,3}$ can be decomposed into $\frac{n}{2}$ copies of $S_3(2)$. Let $S(i)$ denote the star $S(a_{4i}; b_{4i}, b_{4i+1}, b_{4i+2})$ and $M(i)$ denote the graph with edge set $\{b_{4i}a_{4i-1}, b_{4i+1}a_{4i+1}, b_{4i+2}a_{4i+2}\}$ for $i \in \{0, 1, \dots, \frac{n}{4}-1\}$. It is clearly that the edge set $E[S(i)] \cup E[M(i)]$ induces a subgraph $H(i)$ isomorphic to $S_3(2)$. On the other hand, Let $S^*(i)$ denote the star $S(b_{4i+3}; a_{4i+3}, a_{4i+2}, a_{4i+1})$ and $M^*(i)$ denote the graph with edge set $\{a_{4i+3}b_{4i+5}, a_{4i+2}b_{4i+4}, a_{4i+1}b_{4i+2}\}$. It is clearly that the edge set $E[S^*(i)] \cup E[M^*(i)]$ induces a subgraph $H^*(i)$ isomorphic to $S_3(2)$. It is not difficult to see that $H(0), H(1), \ldots, H(\frac{n}{4}-1), H^*(0), H^*(1), \ldots, H^*(\frac{n}{4}-1)$ are pairwise edge disjoint. Therefore, $\{H(0), H(1), \ldots, H(\frac{n}{4}-1), H^*(0), H^*(1), \ldots, H^*(\frac{n}{4}-1)\}$ form an $S_3(2)$-decomposition of $C_{n,3}$. □

References

[1] J. A. Bondy and U. S. R. Murty, *Graph theory with applications*, Macmillan Press, London, 1976.

[2] D. Fronček, Note on cyclic decompositions of complete bipartite graphs into cubes, The Seventh Workshop "3 in 1" Graphs'98 (Krynica), Discuss. Math. Graph Theory 19 (1999), no. 2, 219-227.

[3] M. Jacobson, M. Truszczyński and Z. Tuza, Decompositions of regular bipartite graphs, Discrete Math. 89 (1991), 17-27.

[4] C. Lin , J. J. Lin and T.-W. Shyu, Isomorphic star decompositions of multicrowns and the power of cycles, Ars Combin. 53 (1999), 249-256.

[5] C. A. Parker, *Complete bipartite graph path decompositions*, Ph. D. Dissertation, Auburn University, Auburn, Alabama, 1998.

[6] T.-W. Shyu and C. Lin, Isomorphic path decompositions of crowns, Ars Combin. 67 (2003), 97-103.

[7] T.-W. Shyu, Path decompositions of $\lambda K_{n,n}$, Ars Combin. 85 (2007), 211-219.

[8] D. Sotteau, Decomposition of $K_{m,n}(K^*_{m,n})$ into cycles (circuits) of length $2k$, J. Comb. Theory, Ser. B 30 (1981), 75-81.

[9] M. Truszczyński, Note on the decomposition of $\lambda K_{m,n}(\lambda K^*_{m,n})$ into paths, Discrete Math. 55 (1985), 89-96.

[10] C. Vanden Eynden, Decomposition of complete bipartite graphs. Ars Combin. 46 (1997), 287-296.

[11] S. Yamamoto, H. Ikeda, S. Shige-ede, K. Ushio, N. Hamada, On claw decomposition of complete graphs and complete bipartite graphs, Hiroshima Math. J. 5 (1975), 33-42.