Upper bounds for the vertex Folkman number $F_v(3,3,3;4)$ and $F_v(3,3,3;5)$

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Abstract. For positive integers t and k, the vertex(resp. edge) Folkman number $F_v(t,t,t;k)$ (resp. $F_e(t,t,t;k)$) is the smallest integer n such that there is a K_k-free graph of order n for which any three coloring of its vertices(resp. edges) yields a monochromatic copy of K_t. In this note, an algorithm for testing $(t,t,\cdots,t)^v$ in cyclic graphs is presented and it is applied to find new upper bounds for some vertex or edge Folkman numbers. By using this method, we obtain $F_v(3,3,3;4) \le 66$ and $F_v(3,3,3;5) \le 24$, which leads to $F_v(6,6,6;7) \le 726$ and $F_e(3,3,3;8) \le 727$.

1 Introduction

In this note, we shall only consider graphs without multiple edges or loops. If G is a graph, then the set of vertices of G is denoted by $V(G)$, the set of edges of G by $E(G)$, the cardinality of $V(G)$ by $|V(G)|$. The subgraph of G induced by $S \subseteq V(G)$ is denoted by $G[S]$. A complete graph of order n is denoted by K_n. $S \subseteq V(G)$ is called a *K_3-free set* if $G[S]$ contains no K_3.

Given a positive integer n, $Z_n = \{0, 1, 2, \cdots, n-1\}$, and $S \subseteq \{1, 2, \cdots, \lfloor n/2 \rfloor\}$, let G be a graph with the vertex set $V(G) = Z_n$ and the edge set $E(G) = \{(x, y) : \min\{|x-y|, n-|x-y|\} \in S\}$, then G is called a cyclic graph of order n, denoted by $G_n(S)$. G is an (s,t)-graph if G contains neither clique of order s nor independent set of order t. An (s,t)-graph of order n is denoted by $(s,t;n)$-graph. For integers $s, t \geq 1$, the classical Ramsey number $R(s,t)$ is defined to be the least positive integer n such that every graph on n vertices contains either a clique of order s or an independent set of order t.

For a graph G and positive integers $a_1, a_2, \cdots, a_r$, we write $G \to (a_1, a_2, \cdots, v_r)^v$ if every r-coloring of the vertices of G must result in a monochromatic a_i-clique of color i for some $i \in \{1, 2, \cdots, r\}$; we write $G \to (a_1, a_2, \cdots, v_r)^e$ if every r-coloring of the edges of G must contain a monochromatic a_i-clique of color i for some $i \in \{1, 2, \cdots, r\}$. Let

$$\mathcal{F}_v(a_1, a_2, \cdots, a_r; k) = \{G : G \to (a_1, a_2, \cdots, a_r)^v \text{ and } K_k \not\subseteq G\}.$$

The vertex Folkman number is defined as

$$F_v(a_1, a_2, \cdots, a_r; k) = \min\{|V(G)| : G \in \mathcal{F}_v(a_1, a_2, \cdots, a_r; k)\}.$$

Let

$$\mathcal{F}_e(a_1, a_2, \cdots, a_r; k) = \{G : G \to (a_1, a_2, \cdots, a_r)^e \text{ and } K_k \not\subseteq G\}.$$

The edge Folkman number is defined as

$$F_e(a_1, a_2, \cdots, a_r; k) = \min\{|V(G)| : G \in \mathcal{F}_e(a_1, a_2, \cdots, a_r; k)\}.$$

In 1970, Folkman [3] proved that for positive integers k and $a_1, a_2, \cdots, a_r$, $F_v(a_1, a_2, \cdots, a_r; k)$ exists if and only if $k > \max\{a_1, \cdots a_r\}$. Recently Dudek and Rödl gave a new proof with a relatively small (cubic) upper bound (see [2]). Until now, even with the help of computer for two color vertex Folkman numbers, very little is known about the exact values of them (see [6,8,11,13,14,16,18]).

By the definition of Folkman numbers, we can see that it is more difficult to determine the values of three color Folkman numbers. For positive integers k and $a_1, a_2, \cdots, a_r$, most known exact multicolor Folkman numbers

$F_v(a_1, a_2, \cdots, a_r; k)$ contains some a_i with $a_i = 2$ or k is relatively bigger. For example, $F_v(2,2,2,4;6)$ and $F_v(2,3,4;5)$ (see [15]) were determined in 2002. In [13], it was prove that $F_e(3,3,3;14) = 25$, $F_e(3,3,3;15) = 23$, $F_e(3,3,3;16) = 21$. The gap between lower and upper bounds for some other multicolor Folkman numbers remains not small. In [19], it was shown that $F_v(2,3,3;4) \geq 19$ and $F_v(3;3;3;4) \geq 24$. In [20], it was proved that $F_v(2,3,3;4) \leq 30$. In [10], it was shown that $F_v(3,3,3;q) = 7$ if $q \geq 8$ and it was also proved that $F_v(3,3,3;7) = 10$. The number $F_v(3,3,3;6) = 13$ was determined in [21]. So only in the case when $q = 4$ and 5 we have no upper bound.

In the set of the Folkman numbers of type $F_v(k,k,k;q)$, we can observe that it is more difficult to determine for smaller parameter q. For the case $k = 3$, $F_v(3,3,3;4)$ is the most difficult one to determine, even its bounds, among $F_v(3,3,3;q)$ for $q \geq 4$.

For a given graph G, we need to test if $G \to (3,3,3)^v$. It can be solved by coloring each vertex of G using a direct backtrack procedure. Even with a pruning strategy, this method is still time-consuming. The SAT method is a better way to test if $G \to (3,3,3)^v$ [9]. By transforming into an SAT instance, we can use SAT program which is available at `http://www.princeton.edu/~chaff/zchaff.html`. Since the cyclic graphs have special structure, we can direct color the vertices by avoiding equivalent colorings instead of SAT method. Since they have a large mount of equivalent colorings, the computational steps are reduces significantly.

2 An algorithm for testing $G \to (t, t, \cdots, t)^v$ in cyclic graphs

Let G be a cyclic graph with n vertices, and a r-coloring χ of $V(G)$ be a function $\chi : V(G) \to \{1, 2, \cdots, r\}$. We also say a r-coloring to be a r-coloring pattern, or simply pattern, and we give some definitions:

Definition 1 *For a r-coloring pattern $P = a_1 a_2 \cdots a_q$, we say the subpattern $a_i a_{i+1} \cdots a_{i+L-1}$ to be a (P, i, L)-pattern, and the subpattern $a_i a_{i-1} \cdots a_{i-L+1}$ to be a $(P, i, -L)$-pattern.*

Example: Let $r = 3$ and $P = 12132133321$, then the $(P, 6, -4)$-pattern is 1231 and the $(P, 6, 4)$-pattern is 1333.

Definition 2 *The reverse pattern of $P = a_1 a_2 \cdots a_q$ is defined to be the same pattern of P but in reverse and we write $reverse(P) = a_q a_{q-1} \cdots a_1$.*

Two patterns are said to be equivalent if one can be transformed into the other by applying the function *reverse* and a permutation of $\{1, 2, \cdots, 3\}$.

Example: $P = 123132$ is equivalent to 231321 (by reverse), 132312 (by reverse and $1 \leftrightarrow 2$).

Definition 3 *Given two patterns* $\chi_1 = c_1c_2\cdots c_k, \chi_2 = d_1d_2\cdots d_k$ *of* $S \subseteq V(G)$, *we define* $\chi_1 < \chi_2$ *if there exists* $m > 0$ *such that for* $i < m$, $c_i = d_i$ *and* $c_m < d_m$, *and we say* χ_1 *is* lexicographically smaller *than* χ_2.

Definition 4 *For a pattern p, we define minimal(p) to be the equivalent pattern having the smallest value.*

Example: $p = 223$ is equivalent to 112, 113, 221, 331, 332, 122, 133, 211, 233, 311, 322. Since 112 is the smallest, so we have $minimal(p) = 112$.

In order to test $G \rightarrow (t, t, \cdots, t)^v$ with r colorings, we will try to find a r-coloring of $V(G)$ such that there exists no monochromatic induced subgraph containing K_t. If such a coloring does not exist, we have $G \rightarrow (t, t, \cdots, t)^v$.

For a pattern $P = a_1a_2\cdots a_q$, we consider it as a r-coloring χ of $V(G)$ with the property that $\chi(i-1) = a_i$ for $1 \leq i \leq q$ and $\chi(i) = 0$ for $q \leq i \leq |V(G)|$. We denote the colored graph by $\chi(G)$, and by $Sub(P, c, G)$ the induced subgraph of $\chi(G)$ by the vertices with color c caused by the pattern P.

Let us take $r = 3$ for example to show how to extend the pattern. We starting from a single 1, and then we have a choice to expand this pattern on the right leading to the following patterns in minimal form:

(a) $1 \rightarrow 11, 12$;

(b) $11 \rightarrow 111, 112, 113 \rightarrow 111, 112$;

(c) $12 \rightarrow 121, 122, 123 \rightarrow 121, 112, 123$;

The case (a) shows that 1 can be extended to 11,12. The case (b) shows that 11 can be extended to 111,112, 113, since 112 is equivalent to 113, we have 11 is sufficient to be extended to 111 and 112. The case (c) shows that 11 can be extended to 121,122, 123, since 122 is equivalent to 112, we have 12 is extended to 111, 112 and 123. By the above minimal testing, some computation steps can be reduced.

If the current pattern is $P = a_1a_2\cdots a_q$, the pattern P can not be extended to Px with $x \in \{1, 2, \cdots, r\}$ as follows.

R1) If $Sub(Px, x, G)$ contains a K_t with color x, then P can not be extended to Px.

R2) If for some L, minimal($(P, |P|, -L)$-pattern) $<$ $(P, 1, L)$-pattern, then P can not be extended to Px.

Example 1: Let the current pattern be $P = 1121233$ and $L = 3$. Observing that $P3$ is the pattern 11212333, then the $(P, |P|, -L)$-pattern is 33321211. Therefore, minimal($(P, |P|, -L)$-pattern)=111 and $(P, 1, L)$-pattern is 112. Since $111 < 112$, we have P can not be extended to $P3$. Fig.1 shows the search tree of patterns for three numbers.

Example 2: By applying rules R1 and R2, we have

1) 1 is extended to 11 and 12 since 13 is equivalent to 12 and $13 > 12$;

2) 11 is extended to 111 and 112 since 113 is equivalent to 112 and $113 > 112$;

3) 12 is extended to 121 and 123 since 122 is equivalent to 112 and $122 > 112$.

Fig.1 shows the search tree of patterns for three numbers, where the branches 13, 113 and 122 are eliminated.

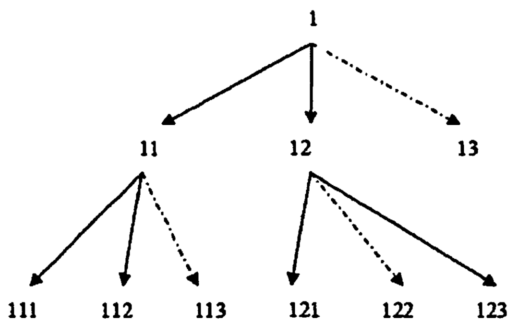

Figure 1: Search tree of patterns for three numbers

Algorithm 1 ExtendColoring(σ, i, c, G, r, t)

```
if i > |V(G)| then
  return TRUE;
end if
if (Sub(σc, c, G) contains a K_t of color c) or (σc is not minimal ) then
  return FALSE;
end if
if i > max then
  max ← i;
end if
for j ← 1; j ≤ r; j ← j + 1 do
  if ExtendColoring(σj, i + 1, j, G, r, t)=TRUE then
    return TRUE;
  end if
end for
```

The coloring extending can be implemented by a backtracking algorithm shown in Algorithm 1. If we want to test if $G \rightarrow (t, t, \cdots, t)^v$ with r colors, we call the function $ExtendColoring(\sigma, 1, 1, G, r, t)$ with $\sigma = 1$. In line 4-6 of Algorithm 1, the rules R1 and R2 are applied. If it returns TRUE, we have $G \nrightarrow (t, t, \cdots, t)^v$. Otherwise, $G \rightarrow (t, t, \cdots, t)^v$.

3 Upper bounds for some vertex and edge Folkman numbers

3.1 Upper bound for $F_v(3, 3, 3; 4)$ and $F_v(3, 3, 3; 5)$

In order to obtain an upper bound for $F_v(3, 3, 3; 4)$, we take the cyclic graph $G_{91}(S)$, where $S = \{1, 2, 4, 7, 8, 14, 16, 17, 23, 27, 28, 32, 34, 37, 45\}$, to test if $G_{91}(S) \rightarrow (3, 3, 3)^v$. This graph witnesses that $R(4, 10) \geq 92$ (see [1]), and we assume $G = G_{91}(S)$. By applying Algorithm 1, we obtain an upper bound: After we call the procedure $ExtendColoring(\sigma, 1, 1, G, 3, 3)$ with $\sigma = 1$, it returns FALSE. Therefore, we have $G \rightarrow (3, 3, 3)^v$. We further observe that the subgraph H of G induced by the vertex set $\{1, 2, 3, 4, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 22, 23, 24, 25, 27, 28, 29, 30, 31, 32, 33, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 49, 50, 51, 54, 55, 56, 57, 59, 60, 61, 62, 63, 64, 65, 68, 69, 70, 72, 75, 76, 77, 78, 85, 86, 87, 88, 89, 90, 91\}$ arrows $(3, 3, 3)^v$, which have 66 vertices, thus we have

Theorem 1 $F_v(3, 3, 3; 4) \leq 66$.

We take the cyclic graph $G_{24}(S)$, where $S = \{1, 2, 3, 5, 6, 11, 12\}$. This graph witnesses that $R(5, 4) \geq 25$. The result obtained by the above procedure shows that it arrows $(3, 3, 3)^v$. Therefore,

Theorem 2 $F_v(3, 3, 3; 5) \leq 24$.

3.2 Upper bounds for other vertex and edge Folkman numbers

Theorem 3 *$F_v(3, 3, \cdots, 3; 2r - 2) \leq 60 + 2r$ and $F_v(3, 3, \cdots, 3; 2r - 1) \leq 18 + 2r$, where r is the number of the colors, for $r \geq 3$.*

Proof. Let H be the subgraph of $G_{91}(S)$ (described above in Section 3.1) which arrows $(3, 3, 3)^v$ and A be an independent set in the graph H, then $H - A \rightarrow (2, 3, 3)^v$. According to Lemma 3 from [21] $H + K_{2r-6} \rightarrow (3, \cdots, 3)^v$ and $G_{91}(S) + K_{2r-6} \rightarrow (3, \cdots, 3)^v$. Therefore, $F_v(3, 3, \cdots, 3; 2r - 2) \leq 60 + 2r$. Similarly, $F_v(3, 3, \cdots, 3; 2r - 1) \leq 18 + 2r$. □

In [6], it was proved that

Theorem 4 *Let* $a_i, b_i, c_i, i \in \{1, 2, \cdots, r\}, s, t$ *be positive integers and* $c_i = a_i b_i$, $1 \le a_i \le s$, $1 \le b_i \le t$. *Then*
$F_v(c_1, c_2, \cdots, c_r; st+1) \le F_v(a_1, a_2, \cdots, a_r; s+1) F_v(b_1, b_2, \cdots, b_r; t+1)$.

By Theorem 4 and $F_v(3,3,3;4) \le 68$, we have

Theorem 5 $F_v(6,6,6;7) \le 726$, $F_e(3,3,3;8) \le 727$.

Proof. By Theorem 4, we have $F_v(6,6,6;7) \le F_v(3,3,3;4) \times F_v(2,2,2;3) \le 66 \times 11 = 726$. Thus $F_e(3,3,3;8) \le F_v(6,6,6;7) + 1 \le 727$. □

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