Minimum embedding of a K_3-design into a balanced incomplete block design of index $\lambda \geq 2$

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Abstract

Let H be a subgraph of G. An H-design $(V, \mathcal{C})$ of order v and index μ is *embedded* into a G-design $(X, \mathcal{B})$ of order $v + w$, $w \geq 0$, and index λ, if $\mu \leq \lambda$, $V \subseteq X$ and there is an injective mapping $f : \mathcal{C} \to \mathcal{B}$ such that B is subgraph of $f(B)$ for every $B \in \mathcal{C}$.

For every pair of positive integers v, λ, we determine the minimum value of w such that there exists a balanced incomplete block design of order $v+w$, index $\lambda \geq 2$ and block-size 4 which embeds a K_3-design of order v and index $\mu = 1$.

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1 Introduction and Definitions

Let G be a finite and simple graph. A G-design of order v and index λ is a pair $(V, \mathcal{C})$ where V is the vertex set of K_v (the complete graph on v vertices) and $\mathcal{C}$ is a collection of isomorphic copies of the graph G, called *blocks*, which partition the edges of λK_v (the complete multigraph on v vertices).

A K_4-design of order v and index λ is well-known as a balanced incomplete block design of order v, index λ and block-size 4. We denote such a design as $S_\lambda(2,4,v)$. Hanani [7] proved that an $S_\lambda(2,4,v)$ exists if and only if

- $v \equiv 1,4 \pmod{12}$ if $\lambda \equiv 1,5 \pmod 6$;
- $v \equiv 1 \pmod 3$ if $\lambda \equiv 2,4 \pmod 6$;
- $v \equiv 0,1 \pmod 4$ if $\lambda \equiv 3 \pmod 6$;
- any $v \geq 4$ if $\lambda \equiv 0 \pmod 6$.

A *Steiner triple system* of order v and index $\lambda = 1$, or $S(2,3,v)$, is a K_3-design of order v and index $\lambda = 1$. An $S(2,3,v)$ exists if and only if $v \equiv 1,3 \pmod 6$.

Definition 1.1 *Let H be a subgraph of G, and let $V \subseteq X$. We say that an H-design $(V, \mathcal{C})$ of order v and index μ is embedded into a G-design $(X, \mathcal{B})$ of order $v + w$ and index λ, $\mu \leq \lambda$, if there is an injective mapping*

$$f : \mathcal{C} \to \mathcal{B}$$

such that B is a subgraph of $f(B)$ for every $B \in \mathcal{C}$.

The mapping f is called the embedding of $(V, \mathcal{C})$ into $(X, \mathcal{B})$. When w attains the minimum possible value we say that f is a minimum embedding.

If $H = G$ and $\mu = \lambda$ then we obtain the usual embedding definition for G-designs.

When $\mu = \lambda = 1$, the (minimum) embedding of an H-design into a G-design has been studied for many pairs of graphs H and G with H a subgraph of G [2, 3, 4, 6, 8, 9, 11, 12, 14, 16, 17]. When $\mu = 1$ and $\lambda > 1$ the minimum embedding has been studied by Milici [13] for $H = P_3$ and $G = K_3$ and by Danziger, Milici, Quattrocchi [5] for $H = P_4$ and $G = K_4$. Milici, Quattrocchi and Shen have studied embeddings of simple maximum packing of triples with index λ even [15]. The case $\lambda = \mu = 1$, $H = K_3$ and $G = K_4$ is very difficult to solve. M.Meszka and A. Rosa [12] solved this problem for $v = 7, 9$. Of course there are well known geometrical examples obtained from embedding affine planes into projective planes. In this paper we wish to consider the minimum embedding of an $S(2,3,v)$ into an $S_\lambda(2,4,v+w)$, $\lambda \geq 2$. In particular, we will prove the following results.

Main Theorem. *Let $v \equiv 1,3 \pmod 6$ and $\lambda \geq 2$. Then there exists a minimum embedding of an $S(2,3,v)$ into an $S_\lambda(2,4,v+w)$ if and only if the conditions in Table 1 except possibly when $\lambda = 5$ and $v = 19$ are satisfied.*

Table 1

$v \pmod{12}$	$v \geq$	$\lambda \pmod 6 \geq 2$	w
1,9	4	3	0
3,7	3	3	1
7	19	$\lambda \geq 9$	6
3,9	3	2,4	1
1,7	7	2,4	0
1	13	1,5	0
3	3	1,5	1
7	7	1,5	6
9	9	1,5	4
1,3,7,9	3	6	0

2 Preliminaries

In this section we recall some useful definitions and results. With regards to terms not defined in this paper or results not explicitly cited the reader is referred to *CRC Handbook of Combinatorial Designs* [1] and its online updates.

A *partial balanced K_4-design* of order v and index λ, with a *hole* of order w and index μ, $w \leq v$ and $\mu \leq \lambda$, is a v-set V with a w-subset $W \subseteq V$ (the *hole*) and a set $\mathcal{B}$ of blocks such that every pairs x and y of elements from V appears in $\lambda - \mu$ blocks if $x, y \in W$ and in λ blocks otherwise.

A 4-GDD is a triple $(V, \mathcal{G}, \mathcal{B})$, where V is a finite set, $\mathcal{G} = \{G_1, G_2, \ldots, G_n\}$ is a partition of V into subsets, the elements of $\mathcal{G}$ are called *groups*, and $\mathcal{B}$ is a collection of isomorphic copies of K_4, called *blocks*, which partition the edges of $K_{g_1,g_2,\ldots,g_n}$ ($|G_i| = g_i$) on the vertex set V. If for $i = 1, 2, \ldots, t$, there are u_i groups of size g_i, we say that the 4-GDD is of type $g_1^{u_1} g_2^{u_2} \ldots g_t^{u_t}$.

Let $\mathcal{B}$ be the block set of a design or a GDD. A *parallel class* or *resolution class* is a collection of blocks which partition the point-set of the design or the GDD. A design or a GDD is *resolvable* if $\mathcal{B}$ can be partitioned into parallel classes.

We recall the existence of some 4-GDD and 4-RGDD we need in the following.

Lemma 2.1 [1] *There exists a 4-GDD of type*

- 4^t *for each* $t \equiv 1 \pmod 3$, $t \geq 4$;
- $7^1 4^t$ *for each* $t \equiv 0 \pmod 3$, $t \geq 5$;
- $10^1 4^t$ *for each* $t \equiv 0 \pmod 3$, $t \geq 6$;
- $13^1 4^t$ *for each* $t \equiv 0 \pmod 3$,$t \geq 8$;
- $16^1 1^t$ *for each* $t \equiv 0, 9 \pmod{12}$, $t \geq 33$;
- $19^1 1^t$ *for each* $t \equiv 0, 3 \pmod{12}$, $t \geq 39$;
- $10^1 1^t$ *for each* $t \equiv 0, 9 \pmod{12}$, $t \geq 21$;
- $m^1 4^t$ *for* $t = 3$ *and* $m = 4$ *or* $t \geq 6$, $t \equiv 0 \pmod 3$ *and* $m \equiv 1 \pmod 3$ *with* $1 \leq m \leq 2(t-1)$.

There exists a resolvable 3-GDD of type 6^t *for each* $t \geq 4$ *and of type* 12^t *for each* $t \geq 3$.

The following Lemma will be used in this paper.

Lemma 2.2 *Let* $(X, \mathcal{G}, \mathcal{B})$ *be a 4-GDD of type* $m^1 4^t$. *Suppose there exists, for* $\lambda = 3, 6$, *an* $S_\lambda(2,4,9)$ *which embeds an* $S(2,3,9)$ *and an* $S_\lambda(2,4,2m+1)$ *which embeds an* $S(2,3,2m+1)$. *Then there exists an* $S_\lambda(2,4,8t+2m+1)$ *which embeds an* $S(2,3,8t+2m+1)$.

Proof. Let $(X, \mathcal{G}, \mathcal{B})$ be a 4-GDD of type $m^1 4^t$ having groups G and G_i, $i = 1, 2, ..., t$, $\mid G \mid = m$, $\mid G_i \mid = 4$. Let $V = \{X \times \mathbb{Z}_2\} \cup \{\infty\}$, $\mid V \mid = v$. For each block $\{a, b, c, d\} \in \mathcal{G}$ construct the set $U = \{a, b, c, d\} \times \mathbb{Z}_2$ and place on U a K_4-decomposition of $\lambda K_{2,2,2,2}$ which embeds a K_3-decomposition of $K_{2,2,2,2}$ (see step 3 in Appendix). On $\{G \times \mathbb{Z}_2\} \cup \{\infty\}$, place an $S_\lambda(2,4,2m+1)$ which embeds an $S(2,3,2m+1)$. On $\{G_i \times \mathbb{Z}_2)\} \cup \{\infty\}$, $i = 1, 2, ..., t$, place an $S_\lambda(2,4,9)$ which embeds an $S(2,3,9)$ (see step 2 in Appendix). The result is an $S_\lambda(2,4,v)$ on V which embeds an $S(2,3,v)$. ∎

3 Proof of Main Theorem

The necessary part of the Main Theorem is straightforward. It is easy to see that the sufficiency of Main Theorem for $\lambda = 2,3,4,5,6$ implies its sufficiency for every λ, with $\lambda = a+6k, a = 1,2,..,6$. If $a = 1$ then minimum embedding of an $S(2,3,v)$ into an $S_{1+6k}(2,4,v+w)$ can be obtained pasting the blocks of an $S_5(2,4,v+w)$ which embeds an $S(2,3,v)$ to the blocks of an $S_{6k-4}(2,4,v+w)$. If $a \geq 2$ paste the blocks of an $S_a(2,4,v+w)$ which embeds an $S(2,3,v)$ to the blocks of an $S_{6k}(2,4,v+w)$.

3.1 $\lambda = 2,4$

Theorem 3.1 *Let* $\lambda = 2,4$. *For* $v \equiv 1 \pmod 6$ *there is an* $S_\lambda(2,4,v)$ *which embeds an* $S(2,3,v)$. *For* $v \equiv 3 \pmod 6$, $v \geq 3$, *there is an* $S_\lambda(2,4,v+1)$ *which embeds an* $S(2,3,v)$

Proof.
Let $\lambda = 2$. For $v \equiv 1 \pmod 6$ we obtain the required design by nesting an $S(2,3,v)$ [18]. For $v \equiv 3 \pmod 6$ let $(V,\mathcal{B})$ be an $S(2,3,v)$ and $\mathcal{T}$ be a parallel class of $\mathcal{B}$. Construct a nested partial triple system $(V,\mathcal{B}-\mathcal{T})$ [10] and take the blocks set $\{\infty,x,y,z\}, [x,y,z] \in \mathcal{T}$ each two-times repeated. The result is an $S_2(2,4,v+1)$ on $V \cup \{\infty\}$ which embeds the $S(2,3,v)$ $(V,\mathcal{B})$. Doubling the solution for $\lambda = 2$ we obtain the required result for $\lambda = 4$. ∎

3.2 $\lambda = 3$

Theorem 3.2 *Let* $v \equiv 1 \pmod{12}$. *Then there is an* $S_3(2,4,v)$ *which embeds an* $S(2,3,v)$.

Proof. Paste an $S(2,4,v)$ to an $S_2(2,4,v)$ which embeds an $S(2,3,v)$. ∎

Theorem 3.3 *Let* $v \equiv 3 \pmod{12}$. *Then there is an* $S_3(2,4,v+1)$ *which embeds an* $S(2,3,v)$.

Proof. Paste an $S(2,4,v+1)$ to an $S_2(2,4,v+1)$ which embeds an $S(2,3,v)$. ∎

Theorem 3.4 *Let* $v \equiv 7 \pmod{12}$. *Then there is an* $S_3(2,4,v+1)$ *which embeds an* $S(2,3,v)$.

Proof. For $v = 7, 19$ see steps 1 and 4 in Appendix. Let $v \geq 31$. Put $V = \mathbb{Z}_v$ and $W = \{a_0\}$. Embed an $S(2,3,v)$ on V into an $S_2(2,4,v)$. Take a 4-GDD of type $10^1 1^{12t}$, $t \geq 2$, on $V \cup \{\infty_0, \infty_1, \infty_2\}$. Let $G = \mathbb{Z}_7 \cup \{\infty_0, \infty_1, \infty_2\}$ be the group of size 10. Replace each infinite point with a_0 and take the blocks so obtained. For each $i \in \mathbb{Z}_7$, construct $\{a_0, i, 1+i, 3+i\}$. The result is an $S_3(2,4,v+1)$ which embeds an $S(2,3,v)$. ∎

Theorem 3.5 *Let $v \equiv 9 \pmod{12}$. Then there is an $S_3(2,4,v)$ which embeds an $S(2,3,v)$.*

Proof. For $v = 9, 21, 33$ see steps 2, 5 and 6 in Appendix. For $v = 9+24t \geq 33$ apply Lemma 2.2 with $m = 4$ to a 4-GDD $(X, \mathcal{G}, \mathcal{B})$ of type 4^{1+3t}, $t \geq 2$, and an $S_3(2,4,9)$ which embeds an $S(2,3,9)$ (step 2 in Appendix). For $v = 21 + 24t \geq 69$, apply Lemma 2.2 with $m = 10$ to a 4-GDD $(X, \mathcal{G}, \mathcal{B})$ of type $10^1 4^{3t}$, $t \geq 2$, and an $S_3(2,4,21)$ which embeds an $S(2,3,21)$ (step 5 in Appendix). ∎

3.3 $\lambda = 5$

For $v \equiv 1 \pmod{12}$ paste an $S_3(2,4,v)$ to an $S_2(2,4,v)$ which embeds an $S(2,3,v)$. For $v \equiv 3 \pmod{12}$ paste an $S(2,4,v+1)$ to an $S_4(2,4,v+1)$ which embeds a $S(2,3,v)$.

Theorem 3.6 *Let $v \equiv 7 \pmod{12}$, $v \neq 19$. Then there is an $S_5(2,4,v+6)$ which embeds an $S(2,3,v)$.*

Proof. For $v = 7, 31, 43$ see steps 8, 11 and 12 in Appendix. Let V be a v-set, G be a subset of size 7 and $v = 7 + 12t \geq 55$. Embed an $S(2,3,v)$ into an $S_2(2,4,v)$ $(V, \mathcal{B})$. Now take a 4-GDD of type $19^1 1^{12t}$, $t \geq 4$, on $V \cup \{\infty_{ij} \mid (i,j) \in \mathbb{Z}_6 \times \mathbb{Z}_2\}$ having $G \cup \{\infty_{ij} \mid (i,j) \in \mathbb{Z}_6 \times \mathbb{Z}_2\}$ as group of size 19. For each $i \in \mathbb{Z}_6$, replace ∞_{ij} with a_i and repeat two-times the blocks so obtained. On $G \cup \{a_0, a_1, \ldots, a_5\}$, place an incomplete $S_4(2,4,13)$ with a hole of order 7 and index 2 which embeds an $S(2,3,7)$ having G as vertex set (see step 7 in the Appendix). The result is an $S_4(2,4,v+6)$ which embeds an $S(2,3,v)$. Paste an $S(2,4,v+6)$ on $V \cup \{a_0, a_1, \ldots, a_5\}$. ∎

Theorem 3.7 *Let $v \equiv 9 \pmod{12}$. Then there is an $S_5(2,4,v+4)$ which embeds an $S(2,3,v)$.*

Proof. For $v = 9, 21$ see steps 9 and 10 in Appendix. Let $v = 9+12t \geq 33$. Embed an $S(2,3,v)$ on $V = \mathbb{Z}_v$ into an $S_2(2,4,v+1)$ on $V = \mathbb{Z}_v \cup \{\infty\}$.

Take a 4-GDD $(X, \mathcal{D})$ of type $10^1 1^v$, $v \geq 33$, with $X = \mathbb{Z}_v \cup \{\infty\} \cup \{\infty_{ij} \mid (i,j) \in \mathbb{Z}_3 \times \mathbb{Z}_3\}$ and such that $H = \{\infty_{ij} \mid (i,j) \in \mathbb{Z}_3 \times \mathbb{Z}_3\} \cup \{\infty\}$ is the group of size 10. For each $(i,j) \in \mathbb{Z}_3 \times \mathbb{Z}_3$, replace ∞_{ij} with a_i and take the blocks so obtained. At last paste an $S_3(2,4,4)$ on $\{a_0, a_1, a_2, \infty\}$ and an $S_2(2,4,v+4)$ on $\mathbb{Z}_v \cup \{a_0, a_1, a_2, \infty\}$. ∎

3.4 $\lambda = 6$

For $v \equiv 1, 7 \pmod{12}$ or $v \equiv 9 \pmod{12}$ we get the proof by tripling the solution for $\lambda = 2$ or by doubling the solution for $\lambda = 3$ respectively. So we suppose $v \equiv 3 \pmod{12}$.

Theorem 3.8 *Let $v \equiv 3 \pmod{12}$, $v \neq 3$. Then there is an $S_6(2,4,v)$ which embeds an $S(2,3,v)$.*

Proof. For $v = 15, 27, 39, 51, 75$ see steps 15, 16, 17, 18 and 20 in the Appendix. For $v = 3 + 24t \geq 99$, $v \neq 3$, there exists a 4-GDD $(X, \mathcal{G}, \mathcal{B})$ of type $13^1 4^{3t-3}$, $t \geq 4$, and an $S_6(2,4,27)$ which embeds an $S(2,3,27)$ (see step 15 in Appendix). Applying Lemma 2.2 with $m = 13$ we obtain the desired result. For $v = 15 + 24t \geq 63$ there exists a 4-GDD $(X, \mathcal{G}, \mathcal{B})$ of type $7^1 4^{3t}$, $t \geq 2$, and an $S_6(2,4,15)$ wich embeds an $S(2,3,15)$ (see step 15 in Appendix). Applying Lemma 2.2 with $m = 7$ we obtain the desired result.

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4 Appendix

In this appendix we list some minimum embeddings of an $S(2,3,v)$ $(V,\mathcal{C})$ into an $S_\lambda(2,4,v)$ $(V \cup W, \mathcal{B})$ for small values of v. We use the following notation: when V or W are not specified we suppose $V = \mathbb{Z}_v$ or $V = \mathbb{Z}_{v-1} \cup \{\infty\}$ and $W = \{a_0, a_1, \ldots, a_{w-1}\}$ if $w \geq 1$ or $W = \emptyset$ if $w = 0$. We list *only* the blocks of $\mathcal{B}$, using square brackets (braces) if the block is (is not) in $\mathcal{C}$. For example, $\{[x,y,z],t\}$ means that the K_4 on vertices x,y,z,t is a block of $\mathcal{B}$ and that the K_3 having the vertices x,y,z and edges $\{x,y\}$, $\{y,z\}$ and $\{x,z\}$ is a block of $\mathcal{C}$. Whereas $\{x,y,z,t\}$ denotes a block of $\mathcal{B}$ not inducing a triple in $\mathcal{C}$. When we list base blocks for $\mathcal{B}$ we intend them to be developed $\pmod v$ ($\pmod{v-1}$) where the vertex set is $\mathbb{Z}_v$ ($\mathbb{Z}_{v-1} \cup \{\infty\}$).

1. $\lambda = 3$, $v = 7$, $w = 1$. Base blocks: $\{[0,1,3],6\}$, $\{a_0,0,1,3\}$.

2. $\lambda = 3$, $v = 9$, $w = 0$. Blocks: $\{3,1,2,4\}$, $\{[0,1,8],3\}$, $\{5,1,4,7\}$, $\{6,1,7,0\}$, $\{7,6,2,3\}$, $\{[5,0,6],2\}$ $\{[2,8,6],4\}$ $\{[2,7,5],8\}$ $\{[4,3,6],0\}$ $\{[8,5,3],6\}$ $\{[0,3,7],5\}$, $\{8,0,5,4\}$, $\{0,1,2,8\}$, $\{[1,4,5],6\}$, $\{[1,2,3],5\}$, $\{[1,6,7],8\}$, $\{[0,2,4],7\}$, $\{[4,7,8],3\}$.

3. A K_4-decomposition of $3K_{2,2,2,2}$ having $V(K_{2,2,2,2})=\{a,b\} \cup \{1,2\} \cup \{x,y\} \cup \{r,s\}$ which embeds a K_3-decomposition of $K_{2,2,2,2}$.
   Blocks: $\{[1,a,s],x\}, \{[1,b,y],r\}, \{[1,x,r],a\}, \{[2,y,s],a\}, \{[2,a,x],s\}, \{[2,b,r],x\}, \{[a,y,r],1\}, \{[b,x,s],1\}, \{1,b,y,s\}, \{2,b,y,s\}, \{2,b,x,r\}, \{2,a,y,r\}\}$.

4. $\lambda = 3, v = 19, w = 1$. Take an $S_2(2,4,19)$ which embeds an $S(2,3,19)$ and add the blocks $\{a_0, i, 4+i, 10+i\}$, $\{i, 18+i, 11+i, 16+i\}$, $i \in \mathbb{Z}_{19}$.

5. $\lambda = 3$, $v = 21$, $w = 0$. Develop $\pmod{21}$ the following base blocks: $\{[7,3,1],13\}$, $\{[9,1,6],14\}$, $\{[1,10,11],13\}$, $\{1,2,3,7\}$. Note that the difference 7 is missing. Now construct the following blocks:
   (a) $\{[i,7+i,14+i],4+i\}, i = 0,1,\ldots,6.$
   (b) $\{i,7+i,14+i,4+i\}$, $i \in \mathbb{Z}_{21} \setminus Z_7$.

6. $\lambda = 3$, $v = 33$, $w = 0$. Let $V = \{\infty\} \cup \{Z_{16} \times \mathbb{Z}_2\}$. Take a 4-GDD $(Z_{16}, \mathcal{G}, \mathcal{B})$ of type 4^4, having groups G_i, $i = 1,2,3,4$, $|G_i| = 4$. For each block $\{a,b,c,d\} \in \mathcal{G}$ construct the set $U = \{a,b,c,d\} \times \mathbb{Z}_2$ and place on U a K_4-decomposition of $3K_{2,2,2,2}$ which embeds a K_3-decomposition of $K_{2,2,2,2}$ (see step 3 in Appendix). For each $i = 1,2,3,4$, on $(G_i \times \mathbb{Z}_2) \cup \{\infty\}$ place an $S_3(2,4,9)$ which embeds an $S(2,3,9)$ (see step 2 in Appendix). The result is an an $S_3(2,4,33)$ embedding an $S(2,3,33)$.

7. A *partial balanced K_4-design* of order 13 and index 4, with a *hole* of order 7 and index 2. Blocks $\{a_0, a_1, 1, 3\}$, $\{a_0, a_1, 4, 3\}$, $\{a_0, a_1, 2, 3\}$, $\{a_0, a_2, 5, 4\}$, $\{a_0, a_2, 6, 0\}$, $\{a_0, a_2, 4, 6\}$, $\{a_0, a_3, 2, 0\}$, $\{a_0, a_4, 0, 1\}$, $\{a_0, a_4, 2, 6\}$, $\{a_0, a_4, 3, 5\}$, $\{a_0, a_5, 1, 2\}$, $\{a_0, a_5, 0, 4\}$, $\{a_0, a_5, 5, 6\}$, $\{a_0, a_3, 1, 5\}$, $\{a_1, a_2, 5, 2\}$, $\{a_1, a_2, 0, 5\}$, $\{a_1, a_2, 1, 6\}$, $\{a_1, a_4, 0, 6\}$, $\{a_1, a_4, 1, 4\}$, $\{a_1, a_3, 2, 4\}$, $\{a_1, a_3, 0, 5\}$, $\{a_1, a_3, 3, 6\}$, $\{a_1, a_5, 1, 6\}$, $\{a_1, a_5, 2, 5\}$, $\{a_1, a_5, 0, 4\}$, $\{a_2, a_3, 2, 6\}$, $\{a_2, a_3, 0, 3\}$, $\{a_2, a_3, 1, 4\}$, $\{a_2, a_4, 0, 3\}$, $\{a_2, a_4, 1, 5\}$, $\{a_2, a_4, 2, 4\}$, $\{a_3, a_4, 1, 2\}$, $\{a_3, a_4, 3, 4\}$, $\{a_3, a_4, 5, 6\}$, $\{a_3, a_5, 0, 1\}$, $\{a_3, a_5, 3, 5\}$, $\{a_3, a_5, 4, 6\}$, $\{a_4, a_5, 0, 2\}$, $\{a_4, a_5, 3, 6\}$, $\{a_4, a_5, 4, 5\}$, $\{a_2, a_5, 1, 3\}$, $\{a_2, a_5, 2, 3\}$, $\{a_0, a_1, a_3, a_4\}$, $\{a_0, a_2, a_3, a_5\}$, $\{a_1, a_2, a_4, a_5\}$.

8. $\lambda = 5$, $v = 7$, $w = 6$. Let $V = \mathbb{Z}_7 \cup \{a_0, a_1, \ldots, a_5\}$. Develop (mod 7) the following base block: $\{[1, 2, 4], 7\}$. Add the blocks of step 2 in Appendix. The result is an $S_4(2, 4, 13)$ which embeds an $S(2, 3, 7)$. Paste an $S(2, 4, 13)$.

9. $\lambda = 5$, $v = 9$, $w = 4$. Embed an $S(2, 3, 9)$ into an $S(2, 4, 13)$ (see [12]). Paste an $S_4(2, 4, 13)$.

10. $\lambda = 5$, $v = 21$, $w = 4$. Embed an $S(2, 3, 21)$ into an $S_3(2, 4, 21)$ on $\mathbb{Z}_{21}$ (see step 5). Paste an $S_5(2, 4, 4)$ on $\{a_0, a_1, a_2, a_3\}$. Take a resolvable $S_2(2, 3, 21)$ on $\mathbb{Z}_{21}$ having the resolution classes $\mathcal{R}_j$, $j = 0, 1, \ldots, 19$. For each $i = 0, 1, 2, 3$, place $\{a_i, x, y, t\}$, for every $\{x, y, t\} \in \cup_{j=0}^{4} \mathcal{R}_{5i+j}$.

11. $\lambda = 5$, $v = 31$, $w = 6$. Let $V = \mathbb{Z}_{30} \cup \{\infty\}$ and $W = \{a_0, a_1, \ldots, a_5\}$. Embed an $S(2, 3, 31)$ on V into an $S_2(2, 4, 31)$. Paste an $S_4(2, 4, 7)$ on $W \cup \{\infty\}$. On $\mathbb{Z}_{30}$ take a resolvable 3-GDD of type 6^5 having groups $G_1, G_2, \ldots, G_5$ and parallel classes $\mathcal{R}_j$, $j = 0, 1, \ldots, 11$. For $i = 0, 1, \ldots, 5$ construct the following blocks $\{a_i, x, y, t\}$, for every $\{x, y, t\} \in \cup_{j=0}^{1} \mathcal{R}_{2i+j}$, each two-times repeated. The result is an $S_4(2, 4, 37)$ which embeds an $S(2, 3, 31)$. Paste an $S(2, 4, 37)$.

12. $\lambda = 5$, $v = 43$, $w = 6$. Let $V = \mathbb{Z}_{42} \cup \{\infty\}$ and $W = \{a_0, a_1, \ldots, a_5\}$. Embed an $S(2, 3, 43)$ on V into an $S_2(2, 4, 43)$. On $\mathbb{Z}_{42}$ take a resolvable 3-GDD of type 6^7 having groups $G_1, G_2, \ldots, G_7$ and parallel classes $\mathcal{R}_j$, $j = 0, 1, \ldots, 17$. For $i = 1, 2, \ldots, 5$, place an $S_2(2, 4, 7)$ on $G_i \cup \{\infty\}$. Paste an $S_4(2, 4, 7)$ on $W \cup \{\infty\}$. For $i = 0, 1, \ldots, 5$, construct the blocks $\{a_i, x, y, t\}$, for every $\{x, y, t\} \in \cup_{j=0}^{2} \mathcal{R}_{2i+j}$. Now on $V \cup W$ take the blocks of a 4-GDD of type 6^8 having groups $W, G_1, G_2, \ldots, G_7$. The result is an $S_4(2, 4, 49)$ which embeds an $S(2, 3, 31)$. Paste an $S(2, 4, 49)$.

13. A K_4-decomposition of $6(K_{15} \backslash K_3)$ having vertex set $\mathbb{Z}_{12} \cup \{a_0, a_1, a_2\}$ and hole $\{a_0, a_1, a_2\}$ which embeds a K_3-decomposition of $K_{15} \backslash$

K_3 having vertex set $\mathbb{Z}_{12} \cup \{a_0, a_1, a_2\}$ and hole $\{a_0, a_1, a_2\}$. Develop (mod 12) the following base blocks: $\{[1,4,6],2\}$, $\{1,3,4,9\}$, $\{a_0,1,2,3\}$,$\{a_0,1,5,9\}$, $\{a_1,1,2,4\}$, $\{a_1,1,4,8\}$, $\{a_2,1,2,7\}$, $\{a_2,1,6,11\}$. Now add the blocks (each 2-times repeated) $\{1+i, 4+i, 7+i, 10+i\}, i = 0,1,2\}$ (the sum is mod 12). Using the differences 1,6 we obtain three one-factors F_0, F_1, F_2. Construct the triples: $\{[a_i, x, y], (x,y) \in F_i, i = 0,1,2\}$, $\{[1+i, 5+i, 9+i], i = 0,1,2,3\}$. Since the above-mentioned triples appear in the previous blocks we obtain the result.

14. $\lambda = 6$, $v = 7$, $w = 0$. Base blocks: $\{[0,1,3],6\}$, $\{0,1,3,6\}$, $\{0,1,3,6\}$.

15. $\lambda = 6$, $v = 15$, $w = 0$. Develop (mod 15) the following base blocks: $\{[1,8,14],9\}$, $\{[1,4,5],14\}$, $\{1,3,4,5\}$, $\{1,4,5,10\}$, $\{1,5,9,13\}$, $\{1,2,8,14\}$. Note that the difference 5 is missing. Now construct the following blocks:

    (a) $\{[i, 5+i, 10+i], 3+i\}$, $i = 0,1,\ldots,4$.

    (b) $\{i, 5+i, 10+i, 3+i\}$, $i \in (Z_{15} \setminus Z_5)$, the sum is (mod 15).

16. $\lambda = 6$, $v = 27$, $w = 0$. Let $X = \{a_0, a_1, \ldots, a_6\}$ and $V = X \cup \mathbb{Z}_{20}$. Construct an $S_6(2,4,7)$ $(X, \mathcal{D})$ which embeds an $S(2,3,7)$ on X (see step 14). First develop (mod 20) the following blocks :

    (a) $\{[1,9,14],3\}$, $\{[1,4,10],6\}$.

    (b) $\{a_0,1,7,14\}$,$\{a_0,1,4,5\}$, $\{a_1,1,8,9\}$, $\{a_1,1,4,13\}$, $\{a_2,1,3,8\}$, $\{a_2,1,4,13\}$, $\{a_3,1,2,3\}$, $\{a_3,1,10,19\}$, $\{a_4,1,3,7\}$, $\{a_4,1,4,5\}$, $\{a_5,1,9,6\}$, $\{a_5,1,5,11\}$, $\{a_6,1,8,9\}$, $\{a_6,1,5,11\}$.

    Now add the blocks (each 2-times repeated) $\{\{i, 5+i, 10+i, 15+i\}, i = 1,\ldots,5\}$(the sum is mod 20). Using the differences 1, 2, 4, 10 we obtain seven one-factors $F_0, F_1, F_2, F_3, F_4, F_5, F_6$. Construct the triples: $\{[a_i, x, y], (x,y) \in F_i, i = 0,1,...,6\}$. Since the above-mentioned triples appear in the blocks (b) we obtain an $S_6(2,4,27)$ which embeds an $S(2,3,27)$.

17. $\lambda = 6$, $v = 39$, $w = 0$. Let $X = \{a_0, a_1, \ldots, a_6\}$ and $V = X \cup \mathbb{Z}_{32}$. Construct an $S_6(2,4,7)$ $(X, \mathcal{D})$ which embeds an $S(2,3,7)$ on X (see step 14). First develop (mod 32) the following blocks:

    (a) $\{[1,5,8],21\}$, $\{1,3,4,15\}$, $\{1,3,4,15\}$, $\{1,3,4,15\}$,$\{1,8,10,16\}$, $\{1,5,14,15\}$, $\{[1,6,16],8\}$, $\{[10,22,28],1\}$.

    (b) $\{a_0,1,7,11\}$,$\{a_0,1,13,14\}$, $\{a_1,1,14,15\}$, $\{[1,10,12],a_1\}$, $\{a_2,1,6,11\}$, $\{a_2,1,10,14\}$, $\{a_3,1,4,10\}$, $\{a_3,1,14,18\}$, $\{a_4,1,9,16\}$, $\{a_4,1,6,12\}$, $\{a_5,1,9,16\}$, $\{a_5,1,6,16\}$, $\{a_6,1,7,17/\}$, $\{a_6,1,4,8\}$.

Now add the blocks (each 2-times repeated) $\{\{i, 8+i, 16+i, 24+i\}, i = 0, 1, \ldots, 7\}$, (the sum is mod 32). Using the differences 1, 13, 8, 16 we obtain seven one-factors $F_0, F_1, F_2, F_3, F_4, F_5, F_6$. Construct the triples: $\{[a_i, x, y], (x, y) \in F_i, i = 0, 1, ..., 6\}$. Since the above-mentioned triples appear in the blocks (b) we obtain an $S_6(2, 4, 39)$ which embeds an $S(2, 3, 39)$.

18. $\lambda = 6$, $v = 51$, $w = 0$. Let $X = \{a_0, a_1, \ldots, a_{14}\}$ and $V = X \cup \mathbb{Z}_{36}$. Construct an $S_6(2, 4, 15)$ $(X, \mathcal{D})$ which embeds an $S(2, 3, 15)$ on X (see step 15). First develop ((mod 36)) the following blocks:

19. $\{1, 2, 7, 15\}$, $\{1, 11, 15, 20\}$, $\{a_0, 1, 16, 17\}$,$\{a_0, 1, 13, 25\}$, $\{a_1, 1, 3, 4\}$, $\{[1, 3, 9], a_1\}$, $\{a_2, 1, 3, 12\}$, $\{[1, 5, 8], a_2\}$, $\{a_3, 1, 6, 15\}$, $\{[1, 6, 17], a_3\}$, $\{a_4, 1, 5, 15\}$, $\{a_4, 1, 2, 5\}$, $\{a_5, 1, 8, 18\}$, $\{a_5, 1, 3, 14\}$, $\{a_6, 1, 16, 29\}$, $\{a_6, 1, 7, 17\}$, $\{a_7, 1, 7, 18\}$, $\{a_7, 1, 17, 24\}$, $\{a_8, 1, 5, 15\}$, $\{a_8, 1, 6, 9\}$, $\{a_9, 1, 6, 15\}$, $\{a_9, 1, 5, 13\}$, $\{a_{10}, 1, 16, 17\}$, $\{a_{10}, 1, 8, 19\}$, $\{a_{11}, 1, 16, 29\}$, $\{a_{11}, 1, 4, 16\}$, $\{a_{12}, 1, 8, 18\}$, $\{a_{12}, 1, 7, 18\}$, $\{a_{13}, 1, 3, 4\}$, $\{a_{13}, 1, 3, 18\}$, $\{a_{14}, 1, 19, 25\}$, $\{a_{14}, 1, 8, 21\}$.

    Now add the blocks (each 2-times repeated) $\{\{i, 9+i, 18+i, 27+i\}, i = 0, 1, \ldots, 8\}$, (the sum is mod 36). Using the differences 1, 9, 10, 13, 14, 15, 17, 18 we obtain fifteen one-factors $F_0, F_1, F_2, \ldots, F_{14}$. Construct the triples: $\{[a_i, x, y], (x, y) \in F_i, i = 0, 1, ..., 14\}$, $\{[a_0, x, y], (x, y) \in F\}$, $\{[i, 12+i, 24+i], i = 0, 1, .., 11\}$. Since the triples from above appear in the blocks (a) we obtain an $S_6(2, 4, 51)$ which embeds an $S(2, 3, 51)$.

20. $\lambda = 6$, $v = 75$, $w = 0$. Let $X = \{a_0, a_1, a_2\}$ and $V = X \cup \{Z_{36} \times \mathbb{Z}_2\}$. Take a 4-GDD $(Z_{36}, \mathcal{G}, \mathcal{B})$ of type 6^6, having groups G_i, $i = 1, 2, .., 6$, $|G_i| = 6$. For each block $\{a, b, c, d\} \in \mathcal{G}$ construct the set $U = \{a, b, c, d\} \times \mathbb{Z}_2$ and place on U a K_4-decomposition of $6K_{2,2,2,2}$ which embeds a K_3-decomposition of $K_{2,2,2,2}$ (see step 3 in Appendix). Let $H_i = X \cup \{Z_6 \times \mathbb{Z}_2\}$, $i = 1, 2, ...6$. For each $i = 2, ..., 6$ place on H_i a K_4-decomposition of $6(H_i \setminus X)$ having vertex set H_i and hole X which embeds a K_3-decomposition of $(H_i \setminus X)$ having vertex set H_i and hole X. Paste on H_1 an $S_6(2, 4, 15)$ which embeds an $S(2, 3, 15)$. The result is an $S_6(2, 4, 75)$ which embeds an $S(2, 3, 75)$.

21. $\lambda = 9$, $v = 19$, $w = 6$. Embed an $S(2, 3, 19)$ into an $S_3(2, 4, 20)$ on $\mathbb{Z}_{19} \cup \{a_0\}$. Paste an $S_9(2, 4, 5)$ on $\{a_0, a_1, a_2, a_3, a_4\}$. Develop (mod 20) the base blocks: $\{1, 2, 4, 10\}$, $\{1, 4, 9, 11\}$, $\{a_0, 1, 10, 11\}$, $\{a_0, 1, 11, 7\}$, $\{a_0, 1, 5, 10\}$, $\{a_1, 1, 5, 8\}$, $\{a_1, 1, 8, 9\}$, $\{a_1, 1, 8, 10\}$, $\{a_2, 1, 6, 7\}$, $\{a_2, 1, 6, 9\}$, $\{a_2, 1, 3, 9\}$, $\{a_3, 1, 5, 10\}$, $\{a_3, 1, 3, 9\}$, $\{a_3, 1, 7, 8\}$, $\{a_4, 1, 2, 4\}$, $\{a_4, 1, 4, 8\}$, $\{a_4, 1, 5, 10\}$. The result is an $S_9(2, 4, 25)$ which embeds an $S(2, 3, 19)$.