UMBRAL CALCULUS AND EULER POLYNOMIALS

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ABSTRACT. In this paper, we study some properties of Euler polynomials arising from umbral calculus. Finally, we give some interesting identities of Euler polynomials using our results. Recently, D. S. Kim and T. Kim have studied some identities of Frobenius-Euler polynomials arising from umbral calculus (*see*[6]).

1. Introduction

We recall that the Euler polynomials are defined by the generating function to be

$$\frac{2}{e^t+1}e^{xt} = e^{E(x)t} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}, \quad (see[2,3,9-18]). \tag{1}$$

with the usual convention about replacing $E^n(x)$ by $E_n(x)$.
In the special case, $x = 0$, $E_n(0) = E_n$ are called the n-th Euler numbers. From (1), we have

$$E_n(x) = \sum_{l=0}^{n} \binom{n}{l} x^l E_{n-l} = (E+x)^n, \quad (see[4,5,7,8,19]). \tag{2}$$

Thus, by (1) and (2), we get

$$E_0 = 1, \quad (E+1)^n + E_n = E_n(1) + E_n = 2\delta_{0,n}, \tag{3}$$

where $\delta_{k,n}$ is the Kronecker's symbol (*see*[2,3,8]). Note that $E_n(x)$ is a monic polynomials of degree n and $E_n'(x) = \frac{dE_n(x)}{dx} = nE_{n-1}(x)$.
Let $\mathbb{C}$ be the complex number field and let $\mathcal{F}$ be the set of all formal power series in the variable t over $\mathbb{C}$ with

$$\mathcal{F} = \{f(t) = \sum_{n=0}^{\infty} \frac{a_k}{k!}t^k \mid a_k \in \mathbb{C}\}. \tag{4}$$

Let $\mathbb{P} = \mathbb{C}[t]$ and let $\mathbb{P}^*$ be the vector space of all linear functionals on $\mathbb{P}$. Now we use the notation $< L \mid p(x) >$ to denote the action of a linear functional L on a polynomial $p(x)$ (*see*[6,13]). We remind that the vector space operations on $\mathbb{P}^*$ are defined by $< L+M \mid p(x) > = < L \mid p(x) > + <$

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$M \mid p(x)>$, $< cL \mid p(x) >= c < L \mid p(x) >$, $(see[5,6,13])$, where c is any constant in $\mathbb{C}$. The formal power series

$$(5) \qquad f(t)=\sum_{k=0}^{\infty}\frac{a_k}{k!}t^k \in \mathcal{F}, \quad (see[6,13]),$$

defines a linear functional on $\mathbb{P}$ by setting

$$(6) \qquad < f(t) \mid x^n >= a_n, \quad for \quad all \quad n \geq 0.$$

Thus, by (5) and (6), we have

$$(7) \qquad < t^k \mid x^n >= n!\delta_{n,k}, \quad (see[6,13]).$$

Let $f_L(t)=\sum_{k=0}^{\infty}\frac{< L \mid x^k >}{k!}t^k$. Then, from (5) and (7), we note that $< f_L(t) \mid x^n >=< L \mid x^n >$ and so as linear functionals $L=f_L(t)$ $(see[13])$. It is known in [6,13] that the map $L \mapsto f_L(t)$ is a vector space isomorphism from $\mathbb{P}^*$ onto $\mathcal{F}$. Henceforth, $\mathcal{F}$ will denote both the algebra of formal power series in t and the vector space of all linear functionals on $\mathbb{P}$ and so an element $f(t)$ of $\mathcal{F}$ will be thought of as both a formal power series and a linear functional. We shall call $\mathcal{F}$ the umbral algebra $(see[6,13])$. The umbral calculus is the study of umbral algebra and modern classical umbral calculus can be described as a systematic study of the class of Sheffer sequences. By (6), we see that

$$< e^{yt} \mid x^n >= y^n \quad and \quad so \quad < e^{yt} \mid p(x) >= p(y) \quad (see[5,6,13]).$$

Note that for all $f(t)$ in $\mathcal{F}$

$$(8) \qquad f(t)=\sum_{k=0}^{\infty}\frac{< f(t) \mid x^k >}{k!}t^k,$$

and for all polynomials $p(x)$

$$(9) \qquad p(x)=\sum_{k=0}^{\infty}\frac{< t^k \mid p(x) >}{k!}x^k, \quad (see[6,13]).$$

For $f_1(t), f_2(t), \ldots f_m(t) \in \mathcal{F}$, we have

$$(10) \qquad < f_1(t) \times f_2(t) \times \ldots \times f_m(t) \mid x^n >$$
$$=\sum \binom{n}{i_1, i_2, \ldots i_m} < f_1(t) \mid x^{i_1} > \ldots < f_m(t) \mid x^{i_m} >,$$

where the sum is over all non-negative integers $i_1, i_2, \ldots i_m$ such that $i_1 + \ldots + i_m = n$ $(see[6,13])$. The order $O(f(t))$ of the power series $f(t) \neq 0$ is the smallest integer k for which a_k does not vanish.
Now we set $O(f(t))=\infty$ if $f(t)=0$. From the definition of order,we note that $O(f(t)g(t))=O(f(t)+O(g(t))$ and $O(f(t)+g(t)) \geq min\{O(f(t), O(g(t))\}$. The series $f(t)$ has a multiplicative inverse, denoted by $f(t)^{-1}$ or $\frac{1}{f(t)}$, if

and only if $O(f(t)) = 0$. Such a series is called an invertible series. A series $f(t)$ for which $O(f(t)) = 1$ is called a delta series (*see*[6, 13]). let $f(t), g(t) \in \mathcal{F}$. Then, we easily that

$$< f(t)g(t) \mid p(x) >=< f(t) \mid g(t)p(x) >=< g(t) \mid f(t)p(x) > .$$

By (9), we set

$$(11) \quad p^{(k)}(x) = \frac{d^k p(x)}{dx^k} = \sum_{l=k}^{\infty} \frac{1}{l!} < t^l \mid p(x) > l(l-1)...(l-k+1)x^{l-k}$$
$$= \sum_{l=k}^{\infty} \frac{1}{(l-k)!} < t^k \mid p(x) > x^{l-k}.$$

Thus, from (11), we have

$$(12) \qquad p^{(k)}(0) =< t^k \mid p(x) > \quad and < 1 \mid p^{(k)}(x) >= p^{(k)}(0).$$

By (12), we get

$$(13) \qquad t^k p(x) = p^{(k)}(x) = \frac{d^k p(x)}{dx^k} \quad (see[6,13]).$$

Thus, by (13) and Taylor expansion, we get

$$(14) \qquad e^{yt} p(x) = p(x+y) \quad (see[6,13]).$$

Let $S_n(x)$ be polynomials with deg $S_n(x) = n$ and let $f(t)$ be a delta series and $g(t)$ be an invertible series. Then there exists a unique sequence $S_n(x)$ of polynomials with $< g(t)f(t)^k \mid S_n(x) >= n!\delta_{n,k}, (n, k \geq 0)$. The sequence $S_n(x)$ is called the Sheffer sequence for $(g(t), f(t))$, which is denoted by $S_n(x) \sim (g(t), f(t))$ (*see*[13]). If $S_n(x) \sim (1, f(t))$, then $S_n(x)$ is called the associated sequence for $f(t)$, or $S_n(x)$ is associated to $f(t)$. If $S_n(x) \sim (g(t), t)$, then $S_n(x)$ is called the Appell sequence for $g(t)$ or $S_n(x)$ is Applell for $g(t)$ (*see*[13]). For $p(x) \in \mathbb{P}$, it is known in [13] that

$$(15) \qquad < \frac{e^{yt}-1}{t} \mid p(x) >= \int_0^y p(u)du,$$

$$(16) \qquad < f(t) \mid xp(x) >=< \partial_t f(t) \mid p(x) >=< f'(t) \mid p(x) >,$$

and

(17)

$$< f(t) \mid p(\alpha x) >=< f(\alpha t) \mid p(x) >, < e^{yt} - 1 \mid p(x) >= p(y) - p(0),$$

where α is a complex constant (*see*[6, 13]).

Suppose that $S_n(x)$ is the Sheffer sequence for $(g(t), f(t))$. Then we have the following equations:

$$(18) \qquad h(t) = \sum_{k=0}^{\infty} \frac{< h(t) \mid S_k(x) >}{k!} g(t)f(t)^k, \quad h(t) \in \mathcal{F},$$

$$p(x)=\sum_{k=0}^{\infty}\frac{<g(t)f(t)^k \mid p(x)>}{k!}S_k(x), \quad p(x)\in \mathbb{P}, \tag{19}$$

$$f(t)S_n(x)=nS_{n-1}(x), \tag{20}$$

and

$$\frac{1}{g(\overline{f}(t))}e^{y\overline{f}(t)}=\sum_{k=0}^{\infty}\frac{S_k(y)}{k!}t^k, \quad for \quad all \quad y\in\mathbb{C}, \tag{21}$$

where $\overline{f}(t)$ is the compositional inverse of $f(t)$, (*see*[13]).
Recently, D. S. Kim and T. Kim have studied some identities of Frobenius-Euler polynomials arising from umbral calculus (*see*[6]). In this paper, we study properties of Euler polynomials arising from umbral calculus. From our investigation, we derive some interesting identities of Euler polynomials.

2. Umbral calculus and Euler polynomials

Let $S_n(x)$ be an Appell sequence for $g(t)$. From (21), we note that

$$\frac{1}{g(t)}x^n=S_n(x)\Leftrightarrow x^n=g(t)S_n(x), \quad (n\geq 0). \tag{22}$$

Let us take $g(t)=\frac{e^t+1}{2}\in\mathcal{F}$. Then, we note that $g(t)$ is an invertible series. By (1), we get

$$\sum_{k=0}^{\infty}E_k(x)\frac{t^k}{k!}=\frac{1}{g(t)}e^{xt}, \tag{23}$$

and, from (23), we have

$$\frac{1}{g(t)}x^n=E_n(x),(n\geq 0), \quad tE_n(x)=n\frac{1}{g(t)}x^{n-1}=nE_{n-1}(x). \tag{24}$$

Thus, by (21) and (24), we see that $E_n(x)$ is the Appell sequence for $(\frac{e^t+1}{2},t)$. Indeed,

$$\begin{aligned} <\frac{e^t+1}{2}t^k \mid E_n(x)> &= \frac{k!\binom{n}{k}}{2}<e^t+1 \mid E_{n-k}(x)> \\ &= \frac{k!\binom{n}{k}}{2}(E_{n-k}(1)+E_{n-k}), \quad (n,k\geq 0). \end{aligned} \tag{25}$$

By (3) and (25), we get

$$<\frac{1+e^t}{2}t^k \mid E_n(x)>=n!\delta_{n,k}, \quad (n,k\geq 0). \tag{26}$$

Therefore, by (24) and (26), we obtain the following lemma.

Lemma 1 . For $n\geq 0$, $E_n(x)$ is the Appell sequence for $(\frac{1+e^t}{2},t)$.

Remark . By (1), we get

$$< \frac{2}{e^t+1} \mid x^n >= \sum_{l=0}^{n} \frac{E_l}{l!} < t^l \mid x^n >= \sum_{l=0}^{n} \frac{E_n}{l!} n!\delta_{n,l}. \tag{27}$$

Thus, by (27), we have

$$< \frac{2}{e^t+1} \mid x^n >= E_n, \quad (n \geq 0).$$

From Lemma 1, we note that

$$\sum_{k=0}^{\infty} \frac{E_k(x)}{k!} t^k = \frac{1}{g(t)} e^{xt}, \tag{28}$$

where $g(t) = \frac{e^t+1}{2} \in \mathcal{F}$.
Let us take the first derivative with respect to t on both sides in (28). Then we have

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{E_k(x)}{k!} k t^{k-1} &= \frac{x g(t) e^{xt} - g'(t) e^{xt}}{g(t)^2} \\ &= \sum_{k=0}^{\infty} \{ x \frac{1}{g(t)} x^k - \frac{g'(t)}{g(t)} \frac{1}{g(t)} x^k \} \frac{t^k}{k!}. \end{aligned} \tag{29}$$

Thus, by (28) and (29), we get

$$E_{k+1}(x) = \left(x - \frac{g'(t)}{g(t)} \right) E_k(x), \quad (k \geq 0).$$

From (2) and (24), we get

$$\begin{aligned} \int_x^{x+y} E_n(u) du &= \frac{1}{n+1} \{ E_{n+1}(x+y) - E_{n+1}(x) \} \\ &= \frac{1}{n+1} \sum_{k=1}^{\infty} \binom{n+1}{k} E_{n+1-k}(x) y^k \\ &= \sum_{k=1}^{\infty} (n)_{k-1} E_{n+1-k}(x) \frac{y^k}{k!} \\ &= \frac{1}{t} \left(\sum_{k=0}^{\infty} \frac{y^k}{k!} t^k - 1 \right) E_n(x) = \frac{e^{yt}-1}{t} E_n(x). \end{aligned} \tag{30}$$

Therefore, by (30), we obtain the following lemma.

Lemma 2 . For $n \geq 0$, we have

$$\int_x^{x+y} E_n(u) du = \frac{e^{yt}-1}{t} E_n(x).$$

By (24), we get

$$E_n(x) = t\frac{1}{n+1}E_{n+1}(x). \tag{31}$$

Thus, by (31), we get

$$\begin{aligned} &< \frac{e^{yt}-1}{t} \mid E_n(x) > = < \frac{e^{yt}-1}{t} \mid t\frac{1}{n+1}E_{n+1}(x) > \\ &= < e^{yt}-1 \mid \frac{E_{n+1}(x)}{n+1} > = \frac{1}{n+1}\{E_{n+1}(y) - E_{n+1}(0)\} \\ &= \int_0^y E_n(u)du. \end{aligned} \tag{32}$$

Therefore, by (30) and (32), we obtain the following proposition.

Proposition 3 . For $n \geq 0$, we have

$$\int_0^y E_n(u)du = < \frac{e^{yt}-1}{t} \mid E_n(x) > .$$

Let

$$\mathbb{P}_n = \{p(x) \in \mathbb{Q}[x] \mid \deg p(x) \leq n\}. \tag{33}$$

Then $\mathbb{P}_n$ is a $(n+1)$-dimensional vector space over $\mathbb{Q}$ and $\{E_0(x), E_1(x), \ldots, E_n(x)\}$ is a basis for $\mathbb{P}_n$. For $p(x) \in \mathbb{P}_n$, we write it as

$$p(x) = \sum_{k=0}^{n} b_k E_k(x). \tag{34}$$

By Lemma 1, we get

$$< \frac{e^t+1}{2}t^k \mid E_n(x) > = n!\delta_{n,k}, \quad (n, k \geq 0). \tag{35}$$

From (34) and (35), we have

$$\begin{aligned} < \frac{e^t+1}{2}t^k \mid p(x) > &= \sum_{l=0}^{n} b_l < \frac{e^t+1}{2}t^k \mid E_l(x) > \\ &= \sum_{l=0}^{n} b_l l! \delta_{l,k} = k! b_k. \end{aligned} \tag{36}$$

Thus, by (36), we get

$$\begin{aligned} b_k &= \frac{1}{k!} < \frac{e^t+1}{2}t^k \mid p(x) > = \frac{1}{2k!} < (e^t+1)t^k \mid p(x) > \\ &= \frac{1}{2k!} < e^t+1 \mid p^{(k)}(x) > = \frac{1}{2k!}\{p^{(k)}(1) + p^{(k)}(0)\}. \end{aligned} \tag{37}$$

Therefore, by (37), we obtain the following theorem.

Theorem 4 . For $p(x) \in \mathbb{P}_n$, let

$$p(x) = \sum_{k=0}^{n} b_k E_k(x).$$

Then, we have

$$b_k = \frac{1}{2k!} < (e^t + 1)t^k \mid p(x) > .$$

In other words,

$$b_k = \frac{1}{2k!} \{p^{(k)}(1) + p^{(k)}(0)\}.$$

Let us take $p(x) = B_n(x) \in \mathbb{P}_n$. Then $B_n(x)$ can be written as a linear combination of $E_0(x), E_1(x), ..E_n(x)$ as follows:

(38) $$B_n(x) = \sum_{k=0}^{n} b_k E_k(x), \quad (n \geq 0).$$

By Theorem 4, we get

$$\begin{aligned} b_k &= \frac{1}{2k!} < (e^t + 1)t^k \mid B_n(x) > \\ &= \frac{1}{2k!} < e^t + 1 \mid n(n-1)\cdots(n-k+1)B_{n-k}(x) > \\ &= \frac{1}{2k!} \binom{n}{k} k! < e^t + 1 \mid B_{n-k}(x) > . \end{aligned}$$

From (17) and (38), we have

(39) $$b_k = \frac{1}{2k!} \binom{n}{k} k! \{B_{n-k}(1) + B_{n-k}\}.$$

As is well known, the recurrence of Bernoulli numbers is given by

(40) $$B_0 = 1, B_n(1) - B_n = \delta_{1,n}, \quad (see[1, 8]).$$

By (38),(39) and (40), we get

(41) $$\begin{aligned} B_n(x) &= b_n E_n(x) + b_{n-1} E_{n-1}(x) + \sum_{k=0}^{n-2} b_k E_k(x) \\ &= E_n(x) + \sum_{k=0}^{n-2} \binom{n}{k} B_{n-k} E_k(x). \end{aligned}$$

Therefore, we obtain the following corollary.

Corollary 5 . For $n \geq 0$, we have

$$B_n(x) = E_n(x) + \sum_{k=0}^{n-2} \binom{n}{k} B_{n-k} E_k(x).$$

For $r \in \mathbb{Z}_+ = \mathbb{N} \cup \{0\}$, the Euler polynomials $E_n^{(r)}(x)$ of order r are defined by the generating function to be

$$(42) \qquad \underbrace{\left(\frac{2}{e^t+1}\right) \times \left(\frac{2}{e^t+1}\right) \times \dots \times \left(\frac{2}{e^t+1}\right)}_{r-times} e^{xt} = \sum_{n=0}^{\infty} E_n^{(r)}(x) \frac{t^n}{n!}.$$

In the special case, $x = 0$, $E_n^{(r)}(0) = E_n^{(r)}$ are called the n-th Euler numbers of order r (*see*[9, 10]). Let us take

$$(43) \qquad g^r(t) = \frac{1}{2^r} \underbrace{(e^t+1) \times \dots \times (e^t+1)}_{r-times} = \left(\frac{e^t+1}{2}\right)^r.$$

Then, we note that $g^r(t)$ is an invertible series.
From (42) and (43), we note that

$$(44) \qquad \sum_{n=0}^{\infty} E_n^{(r)}(x) \frac{t^n}{n!} = \frac{1}{g^r(t)} e^{xt} = \sum_{n=0}^{\infty} \frac{1}{g^r(t)} x^n \frac{t^n}{n!}.$$

By (44), we get

$$(45) \qquad E_n^{(r)}(x) = \frac{1}{g^r(t)} x^n, \quad tE_n^{(r)}(x) = \frac{n}{g^r(t)} x^{n-1} = nE_{n-1}^{(r)}(x).$$

From (21) and (45), we note that $E_n^{(r)}(x)$ is the Appell sequence for $\left(\frac{e^t+1}{2}\right)^r$.
By Appell identity, we also get

$$(46) \qquad E_n^{(r)}(x+y) = \sum_{k=0}^{n} \binom{n}{k} E_{n-k}^{(r)}(x) y^k.$$

It is easy to show that

$$(47) \qquad < \underbrace{\frac{2}{e^t+1} \times \dots \times \frac{2}{e^t+1}}_{r-times} \mid x^n >=< \left(\frac{2}{e^t+1}\right)^r \mid x^n >= E_n^{(r)}.$$

By (47), we get

$$(48) \qquad < \left(\frac{2}{e^t+1}\right)^r e^{yt} \mid x^n >= \sum_{l=0}^{n} \binom{n}{l} y^l E_{n-l}^{(r)} = E_n^{(r)}(y), \quad (n \geq 0),$$

and

$$(49)\qquad <\left(\frac{2}{e^t+1}\right)^r e^{yt}\mid x^n>=<\left(\frac{2}{e^t+1}\right)^r e^{(\frac{y}{r})rt}\mid x^n>$$
$$=\sum_{i_1+\cdots+i_r=n}\binom{n}{i_1,\ldots,i_r}E_{i_1}(\frac{y}{r})\ldots E_{i_r}(\frac{y}{r}).$$

From (48) and (49), we have

$$(50)\qquad E_n^{(r)}(x)=\sum_{i_1+\cdots+i_r=n}\binom{n}{i_1,\ldots,i_r}E_{i_1}(\frac{x}{r})\ldots E_{i_r}(\frac{x}{r}).$$

Therefore, we obtain the following theorem.

Theorem 6. For $n\geq 0$, $E_n^{(r)}(x)$ is the Appell sequence for $\left(\frac{e^t+1}{2}\right)^r$. In addition,

$$E_n^{(r)}(x)=\sum_{i_1+\cdots+i_r=n}\binom{n}{i_1,\ldots,i_r}E_{i_1}(\frac{x}{r})\ldots E_{i_r}(\frac{x}{r}).$$

From (46) and (50), we note that $E_n^{(r)}(x)$ is a monic polynomials of degree n.

Let $p(x)=E_n^{(r)}(x)\in\mathbb{P}_n$. Then $E_n^{(r)}(x)$ can be written as a linear combination of $E_0(x),\ldots,E_n(x)$ as follows:

$$(51)\qquad p(x)=E_n^{(r)}(x)=\sum_{k=0}^{n}b_kE_k(x),\quad (n\geq 0).$$

By Theorem 4, b_k is given by

$$(52)\qquad b_k=\frac{1}{2k!}<(e^t+1)t^k\mid p(x)>=\frac{1}{2k!}<(e^t+1)t^k\mid E_n^{(r)}(x)>$$
$$=\frac{\binom{n}{k}}{2}<e^t+1\mid E_{n-k}^{(r)}(x)>=\frac{\binom{n}{k}}{2}\{E_{n-k}^{(r)}(1)+E_{n-k}^{(r)}\}.$$

From (52), we have

$$(53)\qquad \sum_{n=0}^{\infty}\{E_n^{(r)}(x+1)+E_n^{(r)}(x)\}\frac{t^n}{n!}=\left(\frac{2}{e^t+1}\right)^r(e^t+1)e^{xt}$$
$$=2\left(\frac{2}{e^t+1}\right)^{r-1}e^{xt}=2\sum_{n=0}^{\infty}E_n^{(r-1)}(x)\frac{t^n}{n!}.$$

By comparing coefficients of both sides of (52), we get

$$(54)\qquad E_n^{(r)}(x+1)+E_n^{(r)}(x)=2E_n^{(r-1)}(x),\quad (n\geq 0).$$

Thus, from (51)and (54), we have

$$b_k = \binom{n}{k} E_{n-k}^{(r-1)}. \tag{55}$$

Thus, by (51) and (55), we obtain the following corollary.

Corollary 7. For $n \geq 0$, we have

$$E_n^{(r)}(x) = \sum_{k=0}^{n} \binom{n}{k} E_{n-k}^{(r-1)} E_k(x).$$

It is not difficult to show that $\{E_0^{(r)}(x), E_1^{(r)}(x), \cdots, E_n^{(r)}(x)\}$ is a basis for $\mathbb{P}_n$. Let $p(x) \in \mathbb{P}_n$. Then $p(x)$ can be written as a linear combination of $E_0^{(r)}(x), E_1^{(r)}(x), \cdots, E_n^{(r)}(x)$ as follows:

$$p(x) = \sum_{k=0}^{n} b_k^r E_k^{(r)}(x). \tag{56}$$

By Theorem 6, we see that

$$< \underbrace{\frac{e^t+1}{2} \times \dots \times \frac{e^t+1}{2}}_{r-times} t^k \mid E_n^{(r)}(x) >= n!\delta_{n,k}, \quad (n, k \geq 0). \tag{57}$$

From (56) and (57), we have

$$\begin{aligned} < \left(\frac{e^t+1}{2}\right)^r t^k \mid p(x) > &= \sum_{l=0}^{n} b_l^r < \left(\frac{e^t+1}{2}\right)^r t^k \mid E_l^{(r)}(x) > \\ &= \sum_{l=0}^{n} b_l^r l! \delta_{l,k} = k! b_k^r. \end{aligned} \tag{58}$$

By (58), we get

(59)
$$b_k^r = \frac{1}{k!} < \underbrace{\frac{e^t+1}{2} \times \dots \times \frac{e^t+1}{2}}_{r-times} t^k \mid p(x) >$$
$$= \frac{1}{2^r k!} < (e^t+1)^r t^k \mid p(x) >$$
$$= \frac{1}{2^r k!} \sum_{l=0}^{r} \binom{r}{l} < e^{lt} \mid t^k p(x) >$$
$$= \frac{1}{2^r k!} \sum_{l=0}^{r} \binom{r}{l} < e^{lt} \mid p^{(k)}(x) >$$
$$= \frac{1}{2^r k!} \sum_{l=0}^{r} \binom{r}{l} p^{(k)}(l),$$

where $p^{(k)}(l) = \frac{d^k p(x)}{dx^k}|_{x=l}$. Therefore, by (56) and (59), we obtain the following theorem.

Theorem 8. For $p(x) \in \mathbb{P}_n$, let

$$p(x) = \sum_{k=0}^{n} b_k^r E_k^{(r)}(x).$$

Then, we have

$$b_k^r = \frac{1}{2^r k!} < (e^t+1)^r t^k \mid p(x) > .$$

In other words,
$b_k^r = \frac{1}{2^r k!} \sum_{l=0}^{r} \binom{r}{l} p^{(k)}(l)$, where $p^{(k)}(l) = \frac{d^k p(x)}{dx^k}|_{x=l}$.

Let us take $p(x) = E_n(x) \in \mathbb{P}_n$, $(n \geq 0)$.
From Theorem 8, we have

(60)
$$E_n(x) = \sum_{k=0}^{n} b_k^r E_k^{(r)}(x),$$

where

(61)
$$b_k^r = \frac{1}{2^r k!} < (e^t+1)^r t^k \mid E_n(x) >$$
$$= \frac{\binom{n}{k}}{2^r} < (e^t+1)^r \mid E_{n-k}(x) >$$
$$= \frac{\binom{n}{k}}{2^r} \sum_{l=0}^{r} \binom{r}{l} E_{n-k}(l).$$

By (60) and (61), we get

$$E_n(x) = \frac{1}{2^r}\sum_{k=0}^{n}\binom{n}{k}\left(\sum_{l=0}^{r}\binom{r}{l}E_{n-k}(l)\right)E_k^{(r)}(x).$$

Let $\alpha(\neq 0) \in \mathbb{C}$. Then we have

$$E_n(\alpha x) = \alpha^n \frac{g(t)}{g(\frac{t}{\alpha})}E_n(x),$$

where $g(t) = \frac{1}{2}(e^t + 1)$. Let us consider the Bernoulli polynomials of order s with

$$p(x) = B_n^{(s)}(x) \in \mathbb{P}_n, \quad (n \geq 0).$$

By Theorem 8, we get

$$B_n^{(s)}(x) = \sum_{k=0}^{n} b_k^r E_k^{(r)}(x), \tag{62}$$

where

$$\begin{aligned} b_k^r &= \frac{1}{2^r k!} < (e^t+1)^r t^k \mid B_n^{(s)}(x) > \\ &= \frac{\binom{n}{k}}{2^r} < (e^t+1)^r \mid B_{n-k}^{(s)}(x) > \\ &= \frac{\binom{n}{k}}{2^r}\sum_{l=0}^{r}\binom{r}{l} < e^{lt} \mid B_{n-k}^{(s)}(x) > \\ &= \frac{\binom{n}{k}}{2^r}\sum_{l=0}^{r}\binom{r}{l}B_{n-k}^{(s)}(l). \end{aligned} \tag{63}$$

Therefore, by (62) and (63), we get

$$B_n^{(s)}(x) = \frac{1}{2^r}\sum_{k=0}^{n}\binom{n}{k}\{\sum_{l=0}^{r}\binom{r}{l}B_{n-k}^{(s)}(l)\}E_k^{(r)}(x).$$

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