HIGH ORDER DERIVATIVES AND TWO q-IDENTITIES RELATED TO PRODINGER'S FORMULA

QINGLUN YAN, YINGMEI ZHANG AND XIAONA FAN

College of Science, Nanjing University of Posts and Telecommunications, Nanjing 210046, P. R. China

ABSTRACT. By means of a q-binomial identity, we give two generalizations of Prodinger's formula, which is equivalent to the famous Dilcher's formula.

1. INTRODUCTION

For two complex x and q, the q-shifted factorial are defined by, respectively

$$(x;q)_0 \equiv 1 \quad \text{and} \quad (x;q)_n = x(1-xq)\dots(1-xq^{n-1}) \quad \text{for} \quad n = 1, 2, \dots.$$

In [3], Hernandez proved the following identity:

$$\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}\sum_{1\le k_1\le k_2\le\dots\le k_m=k}\frac{1}{k_1k_2\dots k_m}=\sum_{k=1}^{n}\frac{1}{k^m}.$$

Prodinger [7] pointed out that this formula is equivalent to the famous Dilcher's formula:

$$\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}\frac{1}{k^m}=\sum_{1\le k_1\le k_2\le\dots\le k_m=n}\frac{1}{k_1k_2\dots k_m}.$$

He also gave the following q-analogue:

$$\sum_{k=1}^{n}(-1)^{k-1}\begin{bmatrix}n\\k\end{bmatrix}q^{\binom{k}{2}-nk}\sum_{1\le k_1\le k_2\le\dots\le k_m=k}\prod_{i=1}^{m}\frac{q^{k_i}}{1-q^{k_i}}=\sum_{k=1}^{n}\frac{q^{k(m-1)}}{(1-q^k)^m}.$$

For more proofs and history about these formulae, please refer to Fu-Lascoux [2], Guo-Zhang [5], Ismail-Stanton [6], Zeng [8] and so on.

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Email addresses: *yanqinglun@126.com*.

Throughout the paper, we shall frequently appeal, without further explanation, to q-binomial inversions as follows:

$$\begin{aligned} a_n &= \sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix} b_k \\ b_n &= \sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2}} a_k. \end{aligned}$$

In this paper, by means of a q-binomial identity and high order derivatives, we give two generalizations of Prodinger's q-identity, which is equivalent to the famous Dilcher's formula.

2. THE FIRST GENERALIZATION OF PRODINGER'S q-IDENTITY

In this section, by means of q-Chu-Vandermonde convolution [4], we derive a binomial identity, from which we can derive a generalization of Prodinger's q-identity. The identity can also be proved by applying Partial fraction decomposition to the RHS.

Lemma 1. *With n being a natural number, there holds:*

$$\sum_{k=1}^{n}(-1)^{k-1} \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{k}{2}} \sum_{j=1}^{k} \frac{1}{1-xq^j} = \frac{(q;q)_{n-1}}{(xq;q)_n}. \tag{1}$$

Proof. With the help of

$$(-1)^j \begin{bmatrix} n \\ j \end{bmatrix} q^{\binom{j+1}{2}} = \sum_{k=j}^{n}(-1)^k \begin{bmatrix} n+1 \\ k+1 \end{bmatrix} q^{\binom{k+1}{2}},$$

and q-Chu-Vandermonde convolution, we have

$$\sum_{k=1}^{n}(-1)^{k-1} \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{k}{2}} \sum_{j=1}^{k} \frac{1}{1-xq^j} = \sum_{j=1}^{n} \frac{1}{1-xq^j} \sum_{k=j}^{n}(-1)^{k-1} \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{k}{2}}$$

$$= \sum_{j=1}^{n} \frac{(-1)^j}{1-xq^j} \begin{bmatrix} n-1 \\ j-1 \end{bmatrix} q^{\binom{j}{2}} = \frac{1}{1-xq}\, {}_2\varphi_1 \left[\begin{matrix} q^{-n+1}, & xq \\ & xq^2 \end{matrix} \,\middle|\, q; q^{n+1}\right] = \frac{(q;q)_{n-1}}{(xq;q)_n}.$$

□

Theorem 2. *For two natural numbers m, n and a complex x, there holds the following identity:*

$$\sum_{k=1}^{n}(-1)^{k-1} \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{k}{2}} \sum_{j=1}^{k} \frac{q^{jm}}{(1-xq^j)^{m+1}} = \frac{(q;q)_{n-1}}{(xq;q)_n} \sum_{1 \le k_1 \le k_2 \le \ldots \le k_m \le n} \prod_{i=1}^{m} \frac{q^{k_i}}{1-xq^{k_i}}.$$

Proof. Let $F_n(x) = \frac{(q;q)_{n-1}}{(xq;q)_n}$. It is not hard to see that the identity stated in the theorem is equivalent to the following formula on higher order derivatives of $F_n(x)$ with respect to x:

$$\frac{D_x^m F_n(x)}{F_n(x)} = m! \sum_{1\le k_1\le k_2\le\ldots\le k_m\le n} \prod_{i=1}^{m} \frac{q^{k_i}}{1-xq^{k_i}}. \tag{2}$$

It is trivial to see that when $m = 0$, both sides in (2) reduce to 1, that is to say, they admit the same initial condition. Then if we can check that both sides in (2) satisfy the same recurrence relation, we have confirmed the validity of (2) for all the natural numbers m.

Noting that

$$\frac{D_x^{m+1} F_n(x)}{F_n(x)} = \frac{D_x F_n(x)}{F_n(x)} \left\{ \frac{D_x^m F_n(x)}{F_n(x)} \right\},$$

we derive the following recurrence relation:

$$\frac{D_x^{m+1} F_n(x)}{F_n(x)} = \left\{ D_x - \sum_{k=0}^{n} \frac{q^k}{1-xq^k} \right\} \frac{D_x^m F_n(x)}{F_n(x)}.$$

We just need to check the RHS of (2) satisfies the same relation, i.e.

$$\begin{aligned}(m+1) \sum_{1\le k_1\le k_2\le\ldots\le k_{m+1}\le n} \prod_{i=1}^{m+1} u_{k_i} \;=\; & \sum_{1\le k_1\le k_2\le\ldots\le k_m\le n} \prod_{i=1}^{m} u_{k_i} \sum_{j=1}^{m} u_{k_j} \\ & + \sum_{1\le k_1\le k_2\le\ldots\le k_m\le n} \prod_{i=1}^{m} u_{k_i} \sum_{j=1}^{m} u_j, \end{aligned} \tag{3}$$

where $u_j = \frac{q^j}{1-xq^j}$.

It is not hard to derive that

$$\begin{aligned}&(m+1) \sum_{1\le k_1\le k_2\le\ldots\le k_{m+1}\le n} \prod_{i=1}^{m+1} u_{k_i} \\ =&[t^m] D_t \left\{ \frac{1}{(1-u_1t)(1-u_2t)\ldots(1-u_nt)} \right\} = (m+1) \sum_{r_1+\ldots+r_n=m+1} u_1^{r_1}\ldots u_n^{r_n}.\end{aligned}$$

On the other hand, we have

$$D_t \left\{ \frac{1}{(1-u_1t)(1-u_2t)\ldots(1-u_nt)} \right\} = \frac{1}{(1-u_1t)\ldots(1-u_nt)} \sum_{i=1}^{n} \frac{u_i}{1-u_it}.$$

Recalling

$$[t^m] \frac{1}{(1-u_1t)\ldots(1-u_nt)} \frac{u_1}{(1-u_1t)} = \sum_{r_1+\ldots+r_n=m} (r_1+1) u_1^{r_1+1} u_2^{r_2} \ldots u_n^{r_n}$$

we derive the following relation

$$(m+1)\sum_{r_1+\ldots+r_n=m+1} u_1^{r_1}\ldots u_n^{r_n}$$

$$=\sum_{r_1+\ldots+r_n=m} u_1^{r_1}\ldots u_n^{r_n}[(r_1+1)u_1+\ldots+(r_n+1)u_n],$$

which is equivalent to the equation (3). This implies that both sides of (2) satisfy the same recurrence relation. We complete the proof. □

When $x=1$, the identity in Theorem 2 becomes the following equation:

$$\sum_{k=1}^{n}(-1)^{k-1}\begin{bmatrix}n\\k\end{bmatrix}q^{\binom{k}{2}}\sum_{j=1}^{k}\frac{1}{(1-q^j)^{m+1}}=\frac{1}{1-q^n}\sum_{1\le k_1\le k_2\le\ldots\le k_m\le n}\prod_{i=1}^{m}\frac{q^{k_i}}{1-q^{k_i}}.$$

Taking $k_{m+1}:=n$, we have

$$\sum_{k=1}^{n}(-1)^{k-1}\begin{bmatrix}n\\k\end{bmatrix}q^{\binom{k}{2}}\sum_{j=1}^{k}\frac{q^{mj}}{(1-q^j)^{m+1}}=q^{-n}\sum_{1\le k_1\le k_2\le\ldots\le k_m\le k_{m+1}=n}\prod_{i=1}^{m+1}\frac{q^{k_i}}{1-q^{k_i}}.$$

Replacing m by $m+1$ in the last identity and applying the q-binomial inversions, we derive Prodinger's q-identity.

3. THE SECOND GENERALIZATION OF PRODINGER'S q-IDENTITY

In this section, by restating the identity in Lemma 1, we derive another generalization of Prodinger's q-identity.

Rewrite (1) as

$$\frac{(q;q)_{n-1}}{(x;q)_n}=\sum_{k=1}^{n}(-1)^{k-1}\begin{bmatrix}n\\k\end{bmatrix}q^{\binom{k}{2}}\sum_{j=0}^{k-1}\frac{1}{1-xq^j}$$

$$=\sum_{k=1}^{n}(-1)^{k-1}\begin{bmatrix}n\\k\end{bmatrix}q^{\binom{k}{2}}\sum_{j=0}^{k-1}\left\{1-\frac{1-q^j}{1-xq^j}\right\}\frac{q^{-j}}{1-x}.$$

Then we derive

$$\frac{(q;q)_{n-1}}{(xq;q)_{n-1}}=\sum_{k=1}^{n}(-1)^{k-1}\begin{bmatrix}n\\k\end{bmatrix}q^{\binom{k}{2}}+\sum_{k=2}^{n}(-1)^{k-1}\begin{bmatrix}n\\k\end{bmatrix}q^{\binom{k}{2}}\sum_{j=1}^{k-1}\left\{1-\frac{1-q^j}{1-xq^j}\right\}q^{-j},$$

from which we derive the following q-identity:

Theorem 3. *For two natural numbers m, n and a complex x, there holds the following identity:*

$$\sum_{k=2}^{n}(-1)^{k-1}\begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{k}{2}}\sum_{j=1}^{k-1}\frac{(1-q^j)q^{j(m-1)}}{(1-xq^j)^{m+1}}$$
$$=\frac{(q;q)_{n-1}}{(xq;q)_{n-1}}\sum_{1\le k_1\le k_2\le\ldots\le k_m\le n-1}\prod_{i=1}^{m}\frac{q^{k_i}}{1-xq^{k_i}}.$$

This proof is similar to Theorem 2. We omit the detail here.

Letting $x = 1$ and replacing n by $n + 1$, the last identity reduces to Prodinger's q-identity.

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