Different duality theorems *

Abderrahim Boussaïri[†] Pierre Ille[‡]

Abstract

Given a (directed) graph $G = (V, A)$, the induced subgraph of G by a subset X of V is denoted by $G[X]$. A graph $G = (V, A)$ is a tournament if for any distinct vertices x and y of G, $G[\{x, y\}]$ possesses a single arc. With each graph $G = (V, A)$ associate its dual $G^* = (V, A^*)$ defined as follows: for $x, y \in V$, $(x, y) \in A^*$ if $(y, x) \in A$. Two graphs G and H are hemimorphic if G is isomorphic to H or to H^*. Moreover, let $k > 0$. Two graphs $G = (V, A)$ and $H = (V, B)$ are k-hemimorphic if for every $X \subseteq V$, with $|X| \leq k$, $G[X]$ and $H[X]$ are hemimorphic. A graph G is k-forced when G and G^* are the only graphs k-hemimorphic to G.

Given a graph $G = (V, A)$, a subset X of V is an interval of G provided that for $a, b \in X$ and $x \in V \setminus X$, $(a, x) \in A$ if and only if $(b, x) \in A$, and similarly for (x, a) and (x, b). For example, $\emptyset$, $\{x\}$, where $x \in V$, and V are intervals called trivial. A graph $G = (V, A)$ is indecomposable if all its intervals are trivial. Boussaïri, Ille, Lopez and Thomassé [2] established the following duality result. An indecomposable graph which does not contain the graph $(\{0, 1, 2\}, \{(0, 1), (1, 0), (1, 2)\})$ and its dual as induced subgraphs is 3-forced. A simpler proof of this theorem is provided in the case of tournaments and also in the general case. The 3-forced graphs are then characterized.

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1 Introduction

A *(directed) graph* G consists of a finite and nonempty *vertex* set V and an *arc* set A, where an arc is an ordered pair of distinct vertices. Such a

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[†]Faculté des Sciences Aïn Chock, Département de Mathématiques et Informatique, Km 8 route d'El Jadida, BP 5366 Maarif, Casablanca, Maroc; aboussairi@hotmail.com.

[‡]Institut de Mathématiques de Luminy, CNRS – UMR 6206, 163 avenue de Luminy, Case 907, 13288 Marseille Cedex 09, France; ille@iml.univ-mrs.fr.

graph is denoted by (V, A). For example, given a finite and nonempty set V, $(V, \emptyset)$ is the *empty* graph on V whereas $(V, (V \times V) \setminus \{(x,x);\ x \in V\})$ is the *complete* graph on V. Given a graph $G = (V, A)$, with each nonempty subset X of V associate the *subgraph* $G[X] = (X, A \cap (X \times X))$ of G induced by X. For convenience, given a proper subset X of V, $G[V \setminus X]$ is also denoted by $G - X$, and by $G - x$ whenever $X = \{x\}$.

Given graphs $G = (V, A)$ and $G' = (V', A')$, a bijection f from V onto V' is an *isomorphism* from G onto G' provided that for any $x, y \in V$, $(x,y) \in A$ if and only if $(f(x), f(y)) \in A'$. Two graphs G and G' are then *isomorphic* if there exists an isomorphism from one onto the other. This is denoted by $G \simeq G'$. Given a graph $G = (V, A)$, consider a bijection from V onto a set S. We denote by $f(G)$ the unique graph defined on S such that f realizes an isomorphism from G onto $f(G)$. Finally, a graph H *embeds* into a graph G if H is isomorphic to a subgraph of G.

With each graph $G = (V, A)$ associate its *dual* $G^\star = (V, A^\star)$ and its *complement* $\overline{G} = (V, \overline{A})$ defined as follows. Given $x \neq y \in V$, $(x,y) \in A^\star$ if $(y,x) \in A$, and $(x,y) \in \overline{A}$ if $(x,y) \notin A$. The graph $\overrightarrow{G} = (V, \overrightarrow{A})$ is obtained from G by deleting every arc (x,y) of G such that (y,x) is an arc of G as well. Formally, $\overrightarrow{A} = A \setminus A^\star$. Two graphs G and H are *hemimorphic* if G is isomorphic to H or $H^\star$. Given an integer $k > 0$, the graphs $G = (V, A)$ and $H = (V, B)$ are *k-hemimorphic* if for every nonempty subset X of V, with $|X| \leq k$, the subgraphs $G[X]$ and $H[X]$ are hemimorphic. A graph G is *k-forced* if G and $G^\star$ are the only graphs k-hemimorphic to G.

Given a graph $G = (V, A)$, a pair $\{x, y\}$ of vertices of G is *oriented* if $G[\{x,y\}]$ possesses a unique arc. A graph $G = (V, A)$ is *symmetric* if none of the pairs of its vertices is oriented or, equivalently, if $A = A^\star$. Given $n \geq 2$, the *path* $P_n = (\{0, \ldots, n-1\}, \{(i, i+1), (i+1, i)\}_{0 \leq i \leq n-2})$ and the *cycle* $C_n = (\{0, \ldots, n-1\}, \{(i, i+1), (i+1, i)\}_{0 \leq i \leq n-2} \cup \{(0, n-1), (n-1, 0)\})$ are symmetric graphs.

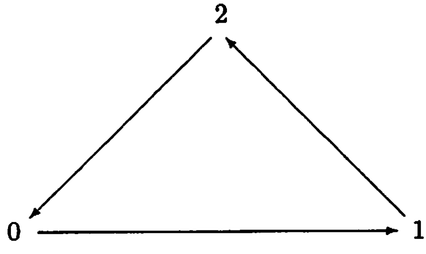

Figure 1: The tournament T_3.

A graph $G = (V, A)$ is a *tournament* if $A^\star = \overline{A}$ or, equivalently, if for any $x \neq y \in V$, $\{x, y\}$ is oriented. For instance, $T_3 = (\{0, 1, 2\}, \{(0, 1), (1, 2), (2, 0)\})$ is a tournament (see Figure 1). A graph $G = (V, A)$ is *asymmetric* if $A \cap A^\star = \emptyset$, that is, if $A = \overrightarrow{A}$. A graph $G = (V, A)$ is a *poset* provided that for any $x, y, z \in V$, if $(x, y) \in A$ and if $(y, z) \in A$, then $(x, z) \in A$. In particular, a poset is asymmetric. Finally, a *total order* is both a tournament and a poset. Given $n > 1$, the graph $O_n = (\{0, \ldots, n-1\}, \{(i, j) : i < j \in \{0, \ldots, n-1\}\})$ is a total order.

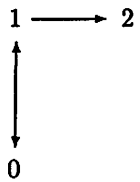

Figure 2: A flag F.

The graph $F = (\{0, 1, 2\}, \{(0, 1), (1, 0), (1, 2)\})$ is neither asymmetric nor symmetric (see Figure 2). The graphs F and $F^\star$ are called *flags*.

Some notations are needed. Let $G = (V, A)$ be a graph. For $x \neq y \in V$, $x \longrightarrow y$ means $(x, y) \in A$ and $(y, x) \notin A$, $x \longleftrightarrow y$ means $(x, y), (y, x) \in A$ and $x \cdots y$ means $(x, y), (y, x) \notin A$. For $x \in V$ and $Y \subseteq V$, $x \longrightarrow Y$ signifies that for every $y \in Y$, $x \longrightarrow y$. For $X, Y \subseteq V$, $X \longrightarrow Y$ signifies that for every $x \in X$, $x \longrightarrow Y$. For $x \in V$ and for $X, Y \subseteq V$, $x \longleftarrow Y$, $x \longleftrightarrow Y$, $x \cdots Y$, $X \longleftrightarrow Y$ and $X \cdots Y$ are defined in the same way.

Consider graphs $G = (V, A)$ and $G' = (V', A')$. Given distinct vertices x, y of G and x', y' of G', $(x, y)_G \equiv (x', y')_{G'}$ signifies that the function $\{x, y\} \longrightarrow \{x', y'\}$, defined by $x \mapsto x'$ and $y \mapsto y'$, is an isomorphism from $G[\{x, y\}]$ onto $G'[\{x', y'\}]$. The negation is denoted by $(x, y)_G \not\equiv (x', y')_{G'}$. When $G = G'$, $(x, y)_G \equiv (x', y')_G$ is denoted by $(x, y) \equiv_G (x', y')$ or by $(x, y) \equiv (x', y')$ if no confusion is possible. For $X \subset V$ and for $x \in V \setminus X$, $x \sim X$ signifies that $(x, y) \equiv (x, y')$ for any $y, y' \in X$. The negation is denoted by $x \not\sim X$. Given $X, Y \subseteq V$, such that $X \cap Y = \emptyset$, $X \sim Y$ means $(x, y) \equiv (x', y')$ for $x, x' \in X$ and $y, y' \in Y$.

Given a graph $G = (V, A)$, a subset X of V is an *interval* [4, 10] (or an *autonomous* set [8, 11, 12] or a *clan* [6] or a *homogeneous* set [3, 9] or a *module* [14]) of G if for every $x \in V \setminus X$, $x \sim X$. For instance, $\emptyset$, V and $\{x\}$, where $x \in V$, are intervals of G called *trivial* intervals. A graph is *indecomposable* [4, 10, 13] (or *prime* [3] or *primitive* [6]) if all its intervals are trivial. Otherwise it is *decomposable*. For example, the tournament

T_3 and the flags F and F^* are indecomposable. On the other hand, all tournaments on 4 vertices and all total orders on at least 3 vertices are decomposable. We recall a well-known property of the intervals. Given a graph $G = (V, A)$, if X and Y are disjoint intervals of G, then $X \sim Y$. Given this property, we consider *interval partitions* of G, that is, partitions of V, all the elements of which are intervals of G. The elements of such a partition P become the vertices of the *quotient* $G/P = (P, A/P)$ of G by P defined as follows: given $X \neq Y \in P$, $(X,Y) \in A/P$ if $(x, y) \in A$ for $x \in X$ and $y \in Y$.

A graph $G = (V, A)$ is *connected* provided that for any distinct vertices x and y of G there is a sequence $x_0 = x, \ldots, x_n = y$ of vertices of G such that $G[\{x_i, x_{i+1}\}]$ is not empty for $0 \leq i \leq n-1$. A nonempty subset X of V is a *connected component* of G if $G[X]$ is connected and if $X \cdots (V \setminus X)$ when $X \neq V$. A vertex x of a graph G is *isolated* if $\{x\}$ constitutes a connected component of G. Obviously, a graph $G = (V, A)$ is connected if V is a connected component of G. Clearly, every connected component of G is an interval of G. Therefore, a non connected graph with at least 3 vertices is decomposable.

Now, we recall the first duality theorem.

Theorem 1 (Gallai [8, 12]). *An indecomposable poset is 3-forced.*

This result is true for tournaments as well.

Theorem 2 (Boussaïri et al. [2]). *An indecomposable tournament is 3-forced.*

Theorems 1 and 2 are generalized as follows.

Theorem 3 (Boussaïri et al. [2]). *An indecomposable graph into which the flags do not embed is 3-forced.*

In Section 3, we present a new approach to establish Theorem 2 by considering the indecomposable tournaments which are minimal for two vertices (see Subsection 2.2) and by using the following classical result on indecomposable graphs.

Lemma 1 (Ehrenfeucht, Rozenberg [6]). *Given a graph $G = (V, A)$, consider a subset X of V such that $|X| \geq 3$, $|V \setminus X| \geq 2$ and $G[X]$ is indecomposable. If G is indecomposable, then there exist distinct elements x and y of $V \setminus X$ such that $G[X \cup \{x, y\}]$ is indecomposable.*

A new proof of Theorem 3 using the critical graphs (see Subsection 2.1) is provided in Section 4. The verification of Theorem 3 for critical graphs (see Section 7) is somewhat tedious. However, once done, the new proof is direct.

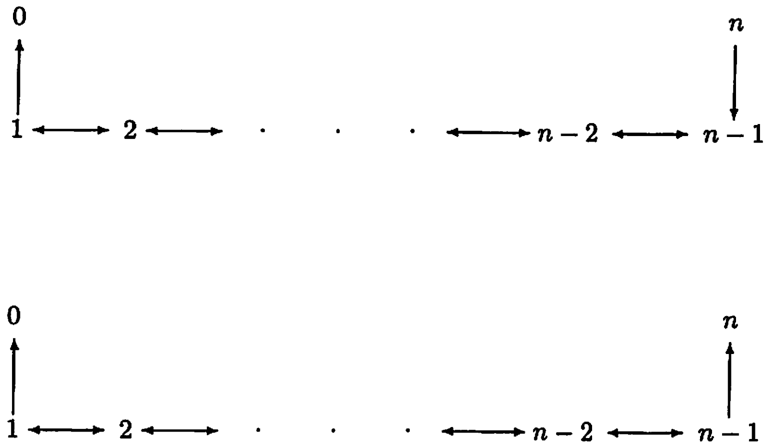

Figure 3: Non 5-forced and indecomposable graphs, where $n \geq 5$.

Theorem 3 does not hold for any indecomposable graph. On the other hand, by considering a symmetric graph, it is easy to observe that a decomposable graph may be 3-forced as well (see Remark 3). In Section 6, we characterize the 3-forced graphs. Furthermore, as shown in Figure 3, there exist 5-hemimorphic and indecomposable graphs which are neither equal nor dual one of the other. In Section 5, we obtain an alternate proof of the following by adapting the new proof of Theorem 3.

Theorem 4 ([1, 5]). *An indecomposable graph is 6-forced.*

The remark below is useful to verify Theorem 3 for particular graphs. Given a graph $G = (V, A)$, consider subsets $X_1, \ldots, X_n$ of V, where $n \geq 2$. The sequence $(X_1, \ldots, X_n)$ is called a *propagation sequence* of G if the following is satisfied:

1. for $1 \leq i \leq n$, $G[X_i]$ is 3-forced;

2. for $1 \leq i \leq n-1$, $X_i \cap X_{i+1}$ contains an oriented pair.

A propagation sequence $(X_1, \ldots, X_n)$ of G is then said to be *covering* if each oriented pair of G is included in a X_i.

Remark 1. Let $(X_1, \ldots, X_n)$ be a propagation sequence of a graph $G = (V, A)$. Denote by A_i the arc set of $G[X_i]$ for $1 \leq i \leq n$. Clearly, the graph

$(V, \cup_{1 \le i \le n} A_i)$ is 3-forced. Consequently, G is 3-forced when $(X_1, \dots, X_n)$ is covering.

2 Critical and minimal graphs

2.1 Critical graphs

An indecomposable graph $G = (V, A)$, with $|V| \geq 4$, is *critical* if $G - x$ is decomposable for every $x \in V$. The six graphs below are used to characterize the critical graphs.

The tournaments T_{2n+1}, U_{2n+1} and V_{2n+1} are defined on $\{0, \dots, 2n\}$, where $n \geq 2$, as follows (see Figures 4, 5 and 6).

- $T_{2n+1}[\{0, \dots, n\}] = U_{2n+1}[\{0, \dots, n\}] = O_{n+1}$, $T_{2n+1}[\{n+1, \dots, 2n\}] = (U_{2n+1})^*[\{n+1, \dots, 2n\}]$ is the usual total order on $\{n+1, \dots, 2n\}$ and for every $i \in \{0, \dots, n-1\}$, $\{i+1, \dots, n\} \longrightarrow i+n+1 \longrightarrow \{0, \dots, i\}$ in T_{2n+1} and in U_{2n+1}.

- $V_{2n+1}[\{0, \dots, 2n-1\}] = O_{2n}$ and $\{1, 3, \dots, 2n-1\} \longrightarrow 2n \longrightarrow \{0, 2, \dots, 2n-2\}$ in V_{2n+1}.

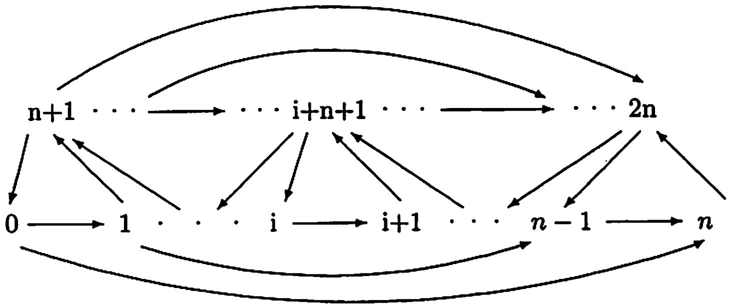

Figure 4: The tournament T_{2n+1}.

The posets Q_{2n} and R_{2n} are defined on $\{0, \dots, 2n-1\}$, where $n \geq 2$, as follows (see Figures 7 and 8).

- For any $x \neq y \in \{0, \dots, 2n-1\}$, (x, y) is an arc of Q_{2n} if there are $0 \leq i \leq j \leq n-1$ such that $(x, y) = (2i, 2j+1)$.

- For any $x \neq y \in \{0, \dots, 2n-1\}$, (x, y) is an arc of R_{2n} if $x < y$ and if x is odd or y is even.

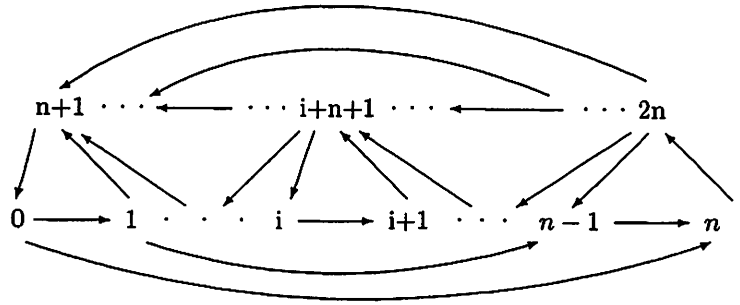

Figure 5: The tournament U_{2n+1}.

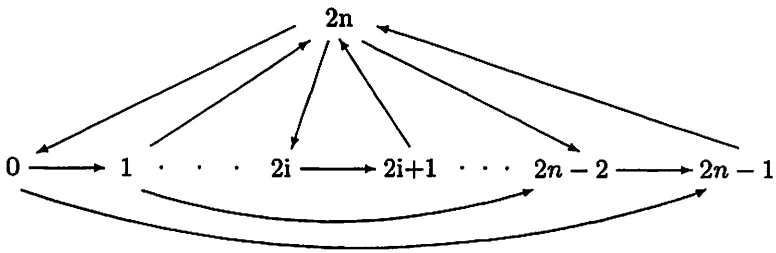

Figure 6: The tournament V_{2n+1}.

Lastly, we consider the graph H_{2n+1}, defined on $\{0, \ldots, 2n\}$, where $n \geq 2$, which is obtained from O_{2n+1} by removing the arcs $(2i, 2j)$ for $0 \leq i < j \leq n$ (see Figure 9). For convenience, the families $\{H_{2n+1}; n \geq 2\}$, $\{Q_{2n}; n \geq 2\}$, $\{R_{2n}; n \geq 2\}$, $\{T_{2n+1}; n \geq 2\}$, $\{U_{2n+1}; n \geq 2\}$ and $\{W_{2n+1}; n \geq 2\}$ are denoted by $\mathcal{H}$, $\mathcal{Q}$, $\mathcal{R}$, $\mathcal{T}$, $\mathcal{U}$ and $\mathcal{W}$ respectively. For $\mathcal{X} = \mathcal{H}, \mathcal{Q}$ or $\mathcal{R}$, set $\overline{\mathcal{X}} = \{\overline{G}; G \in \mathcal{X}\}$.

Theorem 5 (Schmerl, Trotter [13]). *Up to isomorphism, the only critical and non symmetric graphs are the elements of* $\mathcal{H} \cup \mathcal{Q} \cup \mathcal{R} \cup \mathcal{T} \cup \mathcal{U} \cup \mathcal{W} \cup \overline{\mathcal{H}} \cup \overline{\mathcal{Q}} \cup \overline{\mathcal{R}}$.

The indecomposability graph is useful to describe the critical graphs. With each indecomposable graph $G = (V, A)$ associate its *indecomposability graph* $\mathbb{I}(G)$ defined on V by: given $x \neq y \in V$, (x, y) is an arc of $\mathbb{I}(G)$ if $G - \{x, y\}$ is indecomposable.

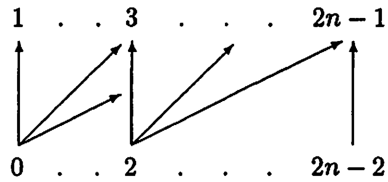

Figure 7: The poset Q_{2n}.

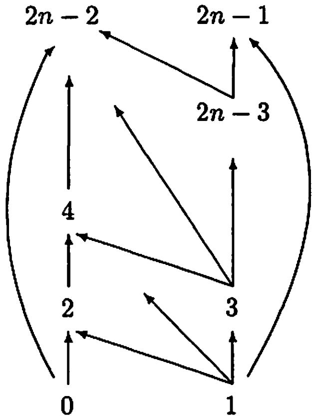

Figure 8: The poset R_{2n}.

Remark 2. Given $n \geq 2$, φ_{2n+1} denotes the permutation of $\mathbb{Z}_{2n+1}$ defined by $i \mapsto (n+1) \times i$ (modulo $2n+1$) for every $i \in \mathbb{Z}_{2n+1}$.

1. For $n \geq 2$, $\mathbb{I}((\varphi_{2n+1})^{-1}(T_{2n+1})) = C_{2n+1}$.
2. For $n \geq 2$, $\mathbb{I}((\varphi_{2n+1})^{-1}(U_{2n+1})) = \mathbb{I}(H_{2n+1}) = \mathbb{I}(\overline{H_{2n+1}}) = P_{2n+1}$.
3. For $n \geq 3$, $\mathbb{I}(Q_{2n}) = \mathbb{I}(\overline{Q_{2n}}) = \mathbb{I}(R_{2n}) = \mathbb{I}(\overline{R_{2n}}) = P_{2n}$.
4. For $n \geq 2$, $2n$ is an isolated vertex of $\mathbb{I}(V_{2n+1})$ and $\mathbb{I}(V_{2n+1}) - (2n) = P_{2n}$.

Furthermore, let $\mathcal{X}$ be one of the families $\mathcal{H}$, $\mathcal{Q}$, $\mathcal{R}$, $\mathcal{T}$, $\mathcal{U}$, $\mathcal{W}$, $\overline{\mathcal{H}}$, $\overline{\mathcal{Q}}$ or $\overline{\mathcal{R}}$. For every element $G = (V, A)$ of $\mathcal{X}$, with $|V| \geq 6$, we have: given $x \neq y \in V$, (x, y) is an arc of $\mathbb{I}(G)$ if and only if $G - \{x, y\} \in \mathcal{X}$.

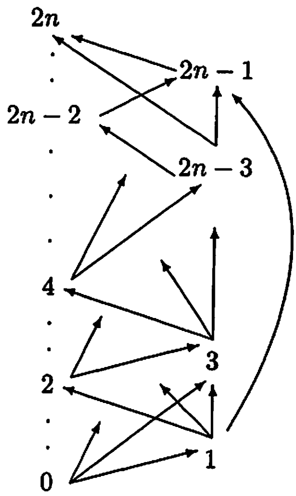

Figure 9: The graph H_{2n+1}.

In the last section, we prove the following proposition, that is, Theorem 3 for critical graphs, by using propagation sequences and Remark 2.

Proposition 1. *A critical graph is 3-forced.*

2.2 Minimal tournaments

Let $G = (V, A)$ be an indecomposable graph, with $|V| \geq 3$. Consider distinct vertices $x_1, \ldots, x_l$ of G. The graph G is *minimal* for $x_1, \ldots, x_l$ provided that for every subset X of V, we have: if $x_1, \ldots, x_l \in X$, if $|X| \geq 3$ and if $G[X]$ is indecomposable, then $X = V$. The next two tournaments are used in the characterization of the minimal tournaments for two vertices.

For $k \geq 2$, M_k is the tournament defined on $\{0, \ldots, k-1\}$ by: given $i \neq j \in \{0, \ldots, k-1\}$, (i, j) is an arc of M_k if either $j = i + 1$ or $j < i - 1$ (see Figure 10).

For $k \geq 5$, N_k is the tournament defined on $\{0, \ldots, k-1\}$ in the following way (see Figure 11):

1. $N_k - \{k-2, k-1\} = M_{k-2}$;
2. $\{0, \ldots, k-4\} \longrightarrow k-2 \longrightarrow k-3$;
3. $\{0, \ldots, k-3\} \longrightarrow k-1 \longrightarrow k-2$.

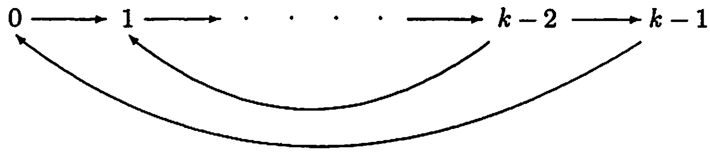

Figure 10: The tournament M_k.

Theorem 6 (Cournier, Ille [4]). *Let $T = (V, A)$ be an indecomposable tournament, with $|V| \geq 6$. Given distinct vertices x and y of T, T is minimal for x and y if and only if there exists an isomorphism f from T onto M_k, N_k or $N_k^\star$ such that $f(\{x, y\}) = \{0, k-1\}$.*

In the last section, we show the following proposition, that is, Theorem 2 for minimal tournaments for two vertices, by using propagation sequences.

Proposition 2. *Let $T = (V, A)$ be an indecomposable tournament, with $|V| \geq 6$. If T is minimal for two of its vertices, then it is 3-forced.*

The following three remarks complete the section.

Remark 3. Let $G = (V, A)$ be a symmetric graph. Clearly, the only graph 2-hemimorphic to G is G. Therefore, G is 2-forced and hence k-forced for $k \geq 2$.

Remark 4. Given graphs G and H, $G \simeq H$ implies $G^\star \simeq H^\star$ and $\overline{G} \simeq \overline{H}$. We also have $\overline{(G^\star)} = (\overline{G})^\star$. Given an integer $k \geq 2$ and a graph G, it follows that the three assertions below are equivalent:

1. G is k-forced;
2. $G^\star$ is k-forced;
3. $\overline{G}$ is k-forced.

Remark 5. Since T_3 and its dual are the only indecomposable tournaments on $\{0, 1, 2\}$, T_3 is 3-forced. Now, we verify that $H_5[\{0, 1, 2\}]$ is 3-forced also. Since $\{0, 2\}$ is the only non oriented pair of $H_5[\{0, 1, 2\}]$, the only possible non trivial interval of $H_5[\{0, 1, 2\}]$ is $\{0, 2\}$. But, $\{0, 2\}$ is not an interval of $H_5[\{0, 1, 2\}]$ because $0 \longrightarrow 1 \longrightarrow 2$. Thus, $H_5[\{0, 1, 2\}]$ is indecomposable. Let H' be a graph 3-hemimorphic to $H_5[\{0, 1, 2\}]$. As H' and $H_5[\{0, 1, 2\}]$ are 2-hemimorphic, we obtain that $0 \cdots 2$ in H' and the

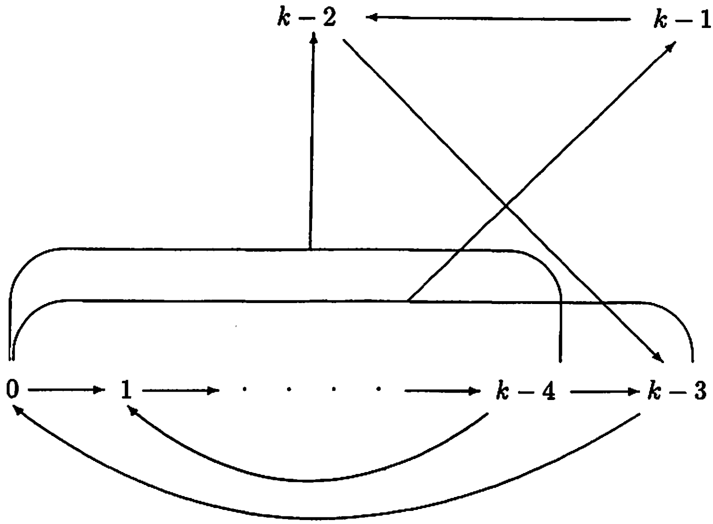

Figure 11: The tournament N_k.

pairs $\{0,1\}$ and $\{1,2\}$ are oriented in H'. Furthermore, since H' is isomorphic to $H_5[\{0,1,2\}]$ or to its dual, H' is indecomposable. In particular, $\{0,2\}$ is not an interval of H'. Therefore, either $0 \longrightarrow 1 \longrightarrow 2$ in H' and $H' = H_5[\{0,1,2\}]$ or $2 \longrightarrow 1 \longrightarrow 0$ in H' and $H' = (H_5[\{0,1,2\}])^*$. Consequently, $H_5[\{0,1,2\}]$ is 3-forced. Clearly, $\{1,3\}$ is an interval of $Q_4[\{0,1,3\}]$ and hence $Q_4[\{0,1,3\}]$ is decomposable. Consider a graph H 3-hemimorphic to $Q_4[\{0,1,3\}]$. We obtain that $1 \cdots 3$ in H and the pairs $\{0,1\}$ and $\{0,3\}$ are oriented in H. Moreover, H is decomposable. If $1 \longrightarrow 0 \longrightarrow 3$ in H or if $3 \longrightarrow 0 \longrightarrow 1$ in H, then H would be isomorphic to $H_5[\{0,1,2\}]$ and hence indecomposable. Thus, either $0 \longrightarrow \{1,3\}$ in H and $H = Q_4[\{0,1,3\}]$ or $0 \longleftarrow \{1,3\}$ in H and $H = (Q_4[\{0,1,3\}])^*$. Therefore, $Q_4[\{0,1,3\}]$ is 3-forced.

By Remark 3, a symmetric graph on 3 vertices is 3-forced. Now, consider a graph G on 3 vertices which admits a single oriented pair. Clearly, G and G^* are the only graphs 3-hemimorphic to G. Thus, G is 3-forced. Concerning the graphs on 3 vertices which admit exactly two oriented pairs, we verified that $H_5[\{0,1,2\}]$ and $Q_4[\{0,1,3\}]$ are 3-forced. By Remark 4, $\overline{H_5[\{0,1,2\}]}$ and $\overline{Q_4[\{0,1,3\}]}$ are also. Therefore, a graph on 3 vertices which admits exactly two oriented pairs is 3-forced. Concerning the tournaments on 3 vertices, we observed that T_3 is 3-forced. On the other hand, O_3 is not because O_3 and $(\{0,1,2\},\{(0,1),(0,2),(2,1)\})$ are 3-hemimorphic,

$(\{0,1,2\},\{(0,1),(0,2),(2,1)\}) \neq O_3$ and $(\{0,1,2\},\{(0,1),(0,2),(2,1)\}) \neq (O_3)^\star$. It follows that, up to isomorphism, the only non 3-forced graph on 3 vertices is O_3.

Lastly, concerning O_3 and T_3, we notice the following. Given distinct vertices a, b and c of 3-hemimorphic tournaments G and H, if $a \longrightarrow b \longrightarrow c$ in G and H, then $G[\{a,c\}] = H[\{a,c\}]$.

3 A new proof of Theorem 2

Let $T = (V, A)$ be an indecomposable tournament. We proceed by induction on $|V|$ to prove that T is 3-forced. If $|V| \leq 2$, then T is 3-forced. If $|V| = 3$, then T is isomorphic to T_3 and hence is 3-forced by Remark 5. Recall that all the tournaments defined on 4 vertices are decomposable. So assume that $|V| \geq 5$. When $|V| = 5$, we also deduce that T is critical and hence 3-forced by Proposition1. Thus, assume that $|V| \geq 6$. Proposition 2 permits us to conclude when T is minimal for two of its vertices. Therefore, for any distinct vertices x and y of T, assume that there exists a proper subset $X_{\{x,y\}}$ of V satisfying $x, y \in X_{\{x,y\}}$, $|X_{\{x,y\}}| \geq 3$ and $T[X_{\{x,y\}}]$ is indecomposable. By Lemma 1 applied several times from $X_{\{x,y\}}$, we obtain a subset $Y_{\{x,y\}}$ of V such that $x, y \in Y_{\{x,y\}}$, $|V \setminus Y_{\{x,y\}}| = 1$ or 2 and $T[Y_{\{x,y\}}]$ is indecomposable.

Let H be a 3-hemimorphic graph to T. Consider distinct vertices u and v of T. By induction hypothesis, $T[Y_{\{u,v\}}]$ is 3-forced. For instance, assume that $H[Y_{\{u,v\}}] = T[Y_{\{u,v\}}]$. Now, consider any distinct vertices x and y of T. We have $|Y_{\{x,y\}} \cap Y_{\{u,v\}}| = |Y_{\{x,y\}}| + |Y_{\{u,v\}}| - |Y_{\{x,y\}} \cup Y_{\{u,v\}}|$. Therefore, $|Y_{\{x,y\}} \cap Y_{\{u,v\}}| \geq (|V| - 2) + (|V| - 2) - |V| = |V| - 4 \geq 2$. Once again, $T[Y_{\{x,y\}}]$ is 3-forced by induction hypothesis. Consequently, $H[Y_{\{x,y\}}] = T[Y_{\{x,y\}}]$ or $(T[Y_{\{x,y\}}])^\star$. Since $H[Y_{\{u,v\}}] = T[Y_{\{u,v\}}]$ and since $|Y_{\{x,y\}} \cap Y_{\{u,v\}}| \geq 2$, we obtain that $H[Y_{\{x,y\}}] = T[Y_{\{x,y\}}]$. In particular, $H[\{x,y\}] = T[\{x,y\}]$. It ensues that $H = T$. □

4 A new proof of Theorem 3

Let $G = (V, A)$ be an indecomposable graph, into which the flags do not embed. We proceed by induction on $|V|$ to prove that G is 3-forced. If $|V| \leq 2$, then G is 3-forced. If $|V| = 3$, then G is isomorphic to T_3, $H_5[\{0,1,2\}]$ or $\overline{H_5[\{0,1,2\}]}$. It follows from Remark 5 that G is 3-forced. Thus assume that $|V| \geq 4$ and consider a graph H 3-hemimorphic to G. When G is critical, it suffices to apply Proposition 1. So assume that there exists $x \in V$ such that $G - x$ is indecomposable. By induction hypothesis,

$G-x$ is 3-forced. As $G-x$ and $H-x$ are 3-hemimorphic, we have $G-x = H-x$ or $(H-x)^\star$. By interchanging H and $H^\star$, assume that $G-x = H-x$.

Denote by D the family of the elements y of $V \setminus \{x\}$ such that $G[\{x,y\}] \neq H[\{x,y\}]$. Evidently, if $D = \emptyset$, then $G = H$. Assume that $D \neq \emptyset$. Firstly, we prove that $V = D \cup \{x\}$. Otherwise, since G is indecomposable, $D \cup \{x\}$ is not an interval of G. There exist $y \in V \setminus (D \cup \{x\})$ and $d \in D$ such that $(d,y) \not\equiv (x,y)$. Since the flags do not embed into G, at least one of the pairs $\{y,d\}$ or $\{y,x\}$ is oriented. Furthermore, we have $G[\{d,x\}] \neq H[\{d,x\}]$, $G[\{d,y\}] = H[\{d,y\}]$ and $G[\{x,y\}] = H[\{x,y\}]$. It is easy to verify that one of the graphs $G[\{d,x,y\}]$ or $H[\{d,x,y\}]$ is indecomposable whereas the other is not. Consequently, $G[\{d,x,y\}]$ and $H[\{d,x,y\}]$ would not be hemimorphic. It follows that $V = D \cup \{x\}$. Secondly, we establish that $G-x$ is symmetric. It suffices to show that each connected component C of $\overrightarrow{G} - x$ is an interval of G. Let c and c' be distinct elements of C such that $\{c,c'\}$ is oriented. We have $G[\{c,c'\}] = H[\{c,c'\}]$. As $\{c,x\}$ and $\{c',x\}$ are oriented pairs on which G and H differ and as $G[\{c,c',x\}]$ and $H[\{c,c',x\}]$ are hemimorphic, we obtain that $(c,x) \equiv (c',x)$. Now, consider $d \in D \setminus C$. Since C is a connected component of $\overrightarrow{G} - x$, the pairs $\{c,d\}$ and $\{c',d\}$ are not oriented. As the flags do not embed in G, we have $(c,d) \equiv (c',d)$. Because $\overrightarrow{G}[C]$ is connected, we obtain that $y \sim C$ for every $y \in V \setminus C$. Consequently, $G-x$ is symmetric and hence $G = H^\star$. □

5 An alternate proof of Theorem 4

We begin with an immediate consequence of Theorem 5.

Corollary 1. *The flags do not embed into a critical graph.*

Before demonstrating Theorem 4, we prove the following three results.

Lemma 2. *Let G and H be 4-hemimorphic graphs. Consider distinct vertices a,b,c and d of G such that $G[\{a,b,c\}]$ is a flag and $G[\{a,b,d\}]$ is a tournament. If $G[\{a,b\}] = H[\{a,b\}]$, then $G[\{a,d\}] = H[\{a,d\}]$ or $G[\{b,d\}] = H[\{b,d\}]$.*

Proof. Let f be an isomorphism from $G[\{a,b,c,d\}]$ onto $H[\{a,b,c,d\}]$ or $(H[\{a,b,c,d\}])^\star$. Since c is the unique element of $\{a,b,c,d\}$ which is contained in at least two non oriented pairs of $G[\{a,b,c,d\}]$, we have $f(c) = c$. As $(c,a) \not\equiv_G (c,b)$, there is $x \in \{a,b\}$ such that $(c,d) \not\equiv_G (c,x)$. For instance, assume that $x = a$. We obtain that $f(a) = a$. To conclude, we distinguish the following two cases. Firstly, assume that $f(b) = b$ and $f(d) = d$. Because $G[\{a,b\}] = H[\{a,b\}]$, f is an isomorphism from $G[\{a,b,c,d\}]$ onto $H[\{a,b,c,d\}]$. Therefore, $G[\{a,d\}] = H[\{a,d\}]$ and $G[\{b,d\}] = H[\{b,d\}]$.

Secondly, asssume that $f(b) = d$ and $f(d) = b$. If f is an isomorphism from $G[\{a,b,c,d\}]$ onto $(H[\{a,b,c,d\}])^\star$, then $G[\{b,d\}] = H[\{b,d\}]$. Otherwise, $G[\{a,d\}] = H[\{a,d\}]$ because $G[\{a,b\}] = H[\{a,b\}]$. □

Lemma 3. *Let G be an indecomposable graph defined on 4 vertices and which possesses exactly 2 oriented pairs. If these pairs are disjoint, then G is 4-forced.*

Proof. Denote by a, b, c and d the vertices of G and assume that $\{a,b\}$ and $\{c,d\}$ are the only oriented pairs of G. More precisely, assume that $a \longrightarrow b$ and $c \longrightarrow d$ in G. By contradiction, suppose that there exists a graph H 4-hemimorphic to G such that $a \longrightarrow b$ and $d \longrightarrow c$ in H. Consider an isomorphism f from G onto H or $H^\star$. If f is an isomorphism from G onto H, then $f(\{a,c\}) = \{a,d\}$ and $f(\{b,d\}) = \{b,c\}$. Since $\{a,c\}, \{a,d\}, \{b,c\}$ and $\{b,d\}$ are not oriented, we obtain that $(a,c)_G \equiv (a,d)_H \equiv (a,d)_G$ and $(b,d)_G \equiv (b,c)_H \equiv (b,c)_G$. Therefore, $\{c,d\}$ would be an interval of G. Similarly, if f is an isomorphism from G onto $H^\star$, then $\{a,b\}$ would be an interval of G. □

Proposition 3. *Given an indecomposable graph G, consider a graph H 6-hemimorphic to G. Let C and C' be distinct connected components of $\overrightarrow{G}$. If $H[C] = G[C]$ or $(G[C])^\star$ and if $H[C'] = G[C']$ or $(G[C'])^\star$, then $H[C \cup C'] = G[C \cup C']$ or $(G[C \cup C'])^\star$.*

Proof. We can assume that $|C| \geq 2$ and $|C'| \geq 2$. By interchanging G and $G^\star$, assume that $H[C] = G[C]$. We have to prove that $H[C'] = G[C']$. Denote by $S(C)$ (resp. $S(C')$) the elements x of $V \setminus C$ (resp. $V \setminus C'$) such that $x \not\leftarrow C$ (resp. $x \not\leftarrow C'$). Because G is indecomposable, $S(C)$ and $S(C')$ are nonempty.

Firstly, assume that C and C' are intervals of $G[C \cup C']$ and that $S(C) \cap S(C') \neq \emptyset$. Let $x \in S(C) \cap S(C')$. As C and C' are intervals of $G[C \cup C']$, we have $x \notin C \cup C'$. Since $\overrightarrow{G}[C]$ and $\overrightarrow{G}[C']$ are connected, there are an oriented pair $\{a,b\}$ of $G[C]$ and an oriented pair $\{a',b'\}$ of $G[C']$ such that $G[\{a,b,x\}]$ and $G[\{a',b',x\}]$ are flags. As $\{a,b\}$ and $\{a',b'\}$ are intervals of $G[\{a,b,a',b'\}]$, $G[\{a,b,x\}]$ and $G[\{a',b',x\}]$ are the only flags of $G[\{a,b,a',b',x\}]$. Moreover, since $G[\{a,b,a',b',x\}]$ and $H[\{a,b,a',b',x\}]$ are hemimorphic, we obtain that $G[\{a,b,x\}] \simeq G[\{a',b',x\}]$ if and only if $H[\{a,b,x\}] \simeq H[\{a',b',x\}]$. As there are only two flags, dual one of the other, we deduce that $G[\{a,b,x\}] \simeq H[\{a,b,x\}]$ if and only if $G[\{a',b',x\}] \simeq H[\{a',b',x\}]$. We have $G[\{a,b,x\}] \simeq H[\{a,b,x\}]$ because $G[C] = H[C]$. Therefore, $G[\{a',b',x\}] \simeq H[\{a',b',x\}]$ and thus $G[\{a',b'\}] = H[\{a',b'\}]$. Finally, we have $G[C'] = H[C']$ because $H[C'] = G[C']$ or $(G[C'])^\star$.

Secondly, assume that C and C' are intervals of $G[C \cup C']$ and that $S(C) \cap S(C') = \emptyset$. Let $x \in S(C)$ and $x' \in S(C')$. As C and C' are intervals of $G[C \cup C']$, we have $x, x' \notin C \cup C'$. Since $\overrightarrow{G}[C]$ and $\overrightarrow{G}[C']$ are connected, there are an oriented pair $\{a, b\}$ of $G[C]$ and an oriented pair $\{a', b'\}$ of $G[C']$ such that $G[\{a, b, x\}]$ and $G[\{a', b', x'\}]$ are flags. Assume that $\{x, x'\}$ is oriented. As $x, x' \notin C \cup C'$, $\{a, b\}$ and $\{x, x'\}$ are the only oriented pairs of $G[\{a, b, x, x'\}]$. Since $G[\{a, b, x\}]$ is a flag, $\{a, b\}$ is not an interval of $G[\{a, b, x, x'\}]$. Moreover, $\{x, x'\}$ is not an interval of $G[\{a, b, x, x'\}]$ because $x' \notin S(C)$. It follows that $G[\{a, b, x, x'\}]$ is indecomposable. By Lemma 3, $G[\{a, b, x, x'\}]$ is 4-forced and hence $G[\{a, b, x, x'\}] = H[\{a, b, x, x'\}]$ or $(H[\{a, b, x, x'\}])^*$. As $G[C] = H[C]$, $G[\{a, b\}] = H[\{a, b\}]$ and hence $G[\{a, b, x, x'\}] = H[\{a, b, x, x'\}]$. In particular, we obtain that $G[\{x, x'\}] = H[\{x, x'\}]$. Similarly, $G[\{a', b', x, x'\}]$ is indecomposable and thus 4-forced by Lemma 3. Since $G[\{x, x'\}] = H[\{x, x'\}]$, we have $G[\{a', b'\}] = H[\{a', b'\}]$. Therefore $H[C'] = G[C']$ because $H[C'] = G[C']$ or $(G[C'])^*$. Now, assume that $\{x, x'\}$ is not oriented. It follows that $G[\{a, b, x\}]$ and $G[\{a', b', x'\}]$ are the only flags of $G[\{a, b, a', b', x, x'\}]$. We conclude as in the first case because $G[\{a, b, a', b', x, x'\}]$ and $H[\{a, b, a', b', x, x'\}]$ are hemimorphic.

Thirdly, assume for example that C is not an interval of $G[C \cup C']$. There exists $x' \in S(C) \cap C'$. Since $\overrightarrow{G}[C]$ is connected, there are $a, b \in C$ such that $G[\{a, b, x'\}]$ is a flag, the oriented pair of which is $\{a, b\}$. Denote by D' the set of $y' \in C'$ such that $\{x', y'\}$ is an interval of $G[\{a, b, x', y'\}]$. Notice that $D' \neq \emptyset$ because $x' \in D'$. Assume that $D' \subset C'$. As $\overrightarrow{G}[C']$ is connected, there are $y' \in D'$ and $z' \in C' \setminus D'$ such that $\{y', z'\}$ is oriented. Since $y' \in D'$ and since $G[\{a, b, x'\}]$ is a flag, $G[\{a, b, y'\}]$ is also. Therefore, $\{a, b\}$ is not an interval of $G[\{a, b, y', z'\}]$. Furthermore, $\{y', z'\}$ is not an interval of $G[\{a, b, y', z'\}]$ because $y' \in D'$ and $z' \notin D'$. Consequently, $G[\{a, b, y', z'\}]$ is indecomposable and we conclude by applying Lemma 3 as previously. Lastly, assume that $D' = C'$. We have C' is an interval of $G[\{a, b\} \cup C']$ and $G[\{a, b, c'\}]$ is a flag for every $c' \in D'$. As C' is not an interval of G, there exists $x \in S(C') \setminus \{a, b\}$. Since $\overrightarrow{G}[C']$ is connected, there are $a', b' \in C'$ such that $G[\{a', b', x\}]$ is a flag, the oriented pair of which is $\{a', b'\}$. Consider an isomorphism f from $G[\{a, b, a', b', x\}]$ onto $H[\{a, b, a', b', x\}]$ or $(H[\{a, b, a', b', x\}])^*$. If $\{a, x\}$ and $\{b, x\}$ are not oriented, then the connected components of $\overrightarrow{G[\{a, b, a', b', x\}]}$ and of $\overrightarrow{H[\{a, b, a', b', x\}]}$ are $\{a, b\}$, $\{a', b'\}$ and $\{x\}$. Thus $f(x) = x$. If $\{a, x\}$ is oriented or if $\{b, x\}$ is oriented, then the connected components of $\overrightarrow{G[\{a, b, a', b', x\}]}$ and of $\overrightarrow{H[\{a, b, a', b', x\}]}$ are $\{a, b, x\}$ and $\{a', b'\}$. Thus $f(\{a', b'\}) = \{a', b'\}$. But, x is the unique element of $\{a, b, x\}$ such that $G[\{a', b', x\}]$ is a flag. Consequently, we obtain that $f(x) = x$ in both cases. Therefore, $f(\{a, b, a', b'\}) = \{a, b, a', b'\}$. Since $\{a', b'\}$ is the only non trivial interval of $G[\{a, b, a', b'\}]$ and of $H[\{a, b, a', b'\}]$, we have $f(\{a', b'\}) = \{a', b'\}$. It follows that $f(a') = a'$ and $f(b') = b'$ because $f(x) = x$ and $G[\{a', b', x\}]$ is a flag. Similarly, we have $f(a) = a$

and $f(b) = b$ because $f(a') = a'$ and $G[\{a, b, a'\}]$ is a flag. As $G[C] = H[C]$, we have $G[\{a, b\}] = H[\{a, b\}]$. Thus, f is an isomorphism from $G[\{a, b, a', b', x\}]$ onto $H[\{a, b, a', b', x\}]$. We obtain that $G[\{a', b'\}] = H[\{a', b'\}]$ and hence $G[C'] = H[C']$. □

Proof of Theorem 4. Let $G = (V, A)$ be an indecomposable graph. We proceed by induction on $|V|$ to prove that G is 6-forced. This is clear when $|V| = 1$ or 2. Assume that $|V| \geq 3$. If the flags do not embed into G, then we conclude by Theorem 3. So assume that a flag embeds into G. Thus, if $|V| = 3$, then G is a flag, which is 3-forced. Therefore assume that $|V| \geq 4$ and consider a graph H 6-hemimorphic to G. As a flag embeds into G, G is not critical by Corollary 1. Consider $x \in V$ such that $G - x$ is indecomposable. By induction hypothesis, we have $H - x = G - x$ or $(G - x)^\star$. For instance, assume that $H - x = G - x$.

Denote by D the set of $y \in V \setminus \{x\}$ such that $G[\{x, y\}] \neq H[\{x, y\}]$. Obviously, if $D = \emptyset$, then $G = H$. We will establish that if $D \neq \emptyset$, then $G = H^\star$. We begin with an easy observation, where $v \in V \setminus (D \cup \{x\})$. We have either for every $u \in D \cup \{x\}$, $\{u, v\}$ is oriented or for every $u \in D \cup \{x\}$, $\{u, v\}$ is not oriented. Furthermore, if $\{u, v\}$ is oriented for every $u \in D \cup \{x\}$, then $v \sim D \cup \{x\}$ in G and H. For every $d \in D$, it suffices to recall that $G[\{d, v, x\}]$ and $H[\{d, v, x\}]$ are hemimorphic and that $G[\{v, x\}] = H[\{v, x\}]$, $G[\{d, v\}] = H[\{d, v\}]$ and $G[\{d, x\}] \neq H[\{d, x\}]$. Consequently, if $\{v, x\}$ is oriented, then $\{d, v\}$ is also for every $d \in D$. Conversely, if $\{d, v\}$ is oriented for some $d \in D$, then $\{v, x\}$ is as well and hence $\{d', v\}$ is oriented for every $d' \in D$. Now, consider $d \in D$ such that $G[\{d, v, x\}]$ is a tournament. Once again, $G[\{v, x\}] = H[\{v, x\}]$, $G[\{d, v\}] = H[\{d, v\}]$ and $G[\{d, x\}] \neq H[\{d, x\}]$. For a contradiction, suppose that $(v, x)_G \not\equiv (v, d)_G$ and hence that $(v, x)_H \not\equiv (v, d)_H$. We obtain that one of the subgraphs $G[\{d, v, x\}]$ or $H[\{d, v, x\}]$ is isomorphic to T_3 whereas the other is isomorphic to O_3.

Firstly, we show that $D \cup \{x\}$ is a connected component of $\overrightarrow{G}$. For a contradiction, suppose that there exist $d' \in D \cup \{x\}$ and $v \in V \setminus (D \cup \{x\})$ such that $\{d', v\}$ is oriented. By the preceding observation, $\{u, v\}$ is oriented for every $u \in D \cup \{x\}$ and $v \sim D \cup \{x\}$. Since $D \cup \{x\}$ is not an interval of G, there is $w \in V \setminus (D \cup \{x\})$ such that $w \not\sim D \cup \{x\}$ in G. There is $d \in D$ such that $(w, x)_G \not\equiv (w, d)_G$. By the previous observation, $\{w, x\}$ and $\{d, w\}$ are not oriented so that $G[\{d, w, x\}]$ is a flag. We obtain a contradiction by applying Lemma 2 to $G[\{d, v, w, x\}]$ and $(H[\{d, v, w, x\}])^\star$. It follows from the observation above that for any $u \in D \cup \{x\}$ and $v \in V \setminus (D \cup \{x\})$, $\{u, v\}$ is not oriented. Since $\overrightarrow{G[D \cup \{x\}]}$ is connected, $D \cup \{x\}$ is a connected component of $\overrightarrow{G}$.

Secondly, we prove that $G[D]$ is symmetric. Because G is indecomposable, it suffices to verify that each connected component C of $\overrightarrow{G[D]}$

is an interval of G. Otherwise, there is $v \in V \setminus C$ such that $v \not\sim C$ in G. Since $\overrightarrow{G[C]}$ is connected, there exist $c, c' \in C$ such that $\{c, c'\}$ is oriented and $(v, c)_G \not\equiv (v, c')_G$. We obtain that $G[\{c, c', x\}]$ and $H[\{c, c', x\}]$ are 3-hemimorphic tournaments. As $G[\{c, c'\}] = H[\{c, c'\}]$, $G[\{c, x\}] \neq H[\{c, x\}]$ and $G[\{c', x\}] \neq H[\{c', x\}]$, we have $(x, c)_G \equiv (x, c')_G$. Thus $v \in (D \setminus C) \cup (V \setminus (D \cup \{x\}))$. Since C is a connected component of $\overrightarrow{G[D]}$ and since $D \cup \{x\}$ is a connected component of $\overrightarrow{G}$, $\{c, v\}$ and $\{c', v\}$ are non oriented so that $G[\{c, c', v\}]$ is a flag. We obtain a contradiction by applying Lemma 2 to $G[\{c, c', v, x\}]$ and $H[\{c, c', v, x\}]$.

Finally, we establish that $G = H^\star$. Consider a connected component D' of $\overrightarrow{G}$ such that $D' \neq D \cup \{x\}$. Clearly, $G[D'] = H[D']$ because $G - x = H - x$. Since $G[D]$ is symmetric, we have $G[D \cup \{x\}] = (H[D \cup \{x\}])^\star$. It follows from Proposition 3 that $G[(D \cup \{x\}) \cup D'] = H[(D \cup \{x\}) \cup D']$ or $(H[(D \cup \{x\}) \cup D'])^\star$. As $|D \cup \{x\}| \geq 2$, we obtain that $|D'| = 1$. It follows that $G - x = H - x$ is symmetric because $G[D]$ is symmetric. Since $D \cup \{x\}$ is a connected component of $\overrightarrow{G}$, $\{v, x\}$ is not oriented for every $v \in V \setminus (D \cup \{x\})$. Consequently, $G = H^\star$. □

6 A characterization of the 3-forced graphs

We use the Gallai decomposition theorem below. To state it, we need the following strengthening of the notion of interval. Given a graph $G = (V, A)$, a subset X of V is a *strong interval* [8, 12] of G provided that X is an interval of G and for each interval Y of G, we have: if $X \cap Y \neq \emptyset$, then $X \subseteq Y$ or $Y \subseteq X$. The family of the maximal strong intervals under inclusion which are distinct from V is denoted by $P(G)$.

Theorem 7 (Gallai [8, 12]). *Given a graph $G = (V, A)$, with $|V| \geq 2$, the family $P(G)$ constitutes an interval partition of G. Moreover, the corresponding quotient $G/P(G)$ is a complete graph, an empty graph, a total order or an indecomposable graph with $|P(G)| \geq 3$.*

We also use the next result.

Proposition 4 (Boussaïri et al. [2]). *If G and H are 3-hemimorphic graphs into which the flags do not embed, then $P(G) = P(H)$.*

Now, we recall the following characterization in the case of posets.

Theorem 8 (Filippov, Shevrin [7]). *Given a connected poset Q, Q is 3-forced if and only if each non trivial interval of Q induces an empty subgraph of Q.*

As shown by the following, the characterization of the 3-forced graphs reduces to the asymmetric case.

Proposition 5. *Given a graph G, G is 3-forced if and only if $\overrightarrow{G}$ is 3-forced*

Proof. Firstly, assume that G is 3-forced and consider a graph H' 3-hemimorphic to $\overrightarrow{G}$. There exists a unique graph H 2-hemimorphic to G and such that $\overrightarrow{H} = H'$. It is easy to verify that G and H are 3-hemimorphic. Therefore, $H = G$ or $G^\star$. Because $\overrightarrow{H} = H'$, we obtain that $H' = \overrightarrow{G}$ or $\overrightarrow{(G^\star)}$. As $\overrightarrow{(G^\star)} = (\overrightarrow{G})^\star$, it follows that $H' = \overrightarrow{G}$ or $(\overrightarrow{G})^\star$. Consequently, $\overrightarrow{G}$ is 3-forced.

Conversely, assume that $\overrightarrow{G}$ is 3-forced and consider a graph H 3-hemimorphic to G. Obviously, $\overrightarrow{G}$ and $\overrightarrow{H}$ are 3-hemimorphic. Therefore, $\overrightarrow{H} = \overrightarrow{G}$ or $(\overrightarrow{G})^\star$, that is, $\overrightarrow{(G^\star)}$. Since G and H are 2-hemimorphic, we obtain that $H = G$ or $G^\star$. □

Given a graph G with an isolated vertex x, we clearly have: G is 3-forced if and only if $G - x$ is. Consequently, we have only to characterize the 3-forced graphs which are asymmetric and without isolated vertices.

Theorem 9. *Given an asymmetric graph $G = (V, A)$, with $|V| \geq 2$ and without isolated vertices, the three assertions below are equivalent:*

1. *G is 3-forced;*
2. *each non trivial interval of G induces an empty subgraph of G;*
3. *$G/P(G)$ is indecomposable and $G[X]$ is empty for every $X \in P(G)$.*

Proof. To begin, assume that G is 3-forced and consider a non trivial interval X of G. Let H be the graph obtained from G by reversing all the arcs of $G[X]$. It is easy to verify that the graphs G and H are 3-hemimorphic so that $H = G$ or $G^\star$. Let $x \in V \setminus X$. Since x is not an isolated vertex of G, there exists $y \in V \setminus \{x\}$ such that $\{x, y\}$ is oriented. As $\{x, y\} \setminus X \neq \emptyset$, we have $H[\{x, y\}] = G[\{x, y\}]$. Therefore, $H = G$ and necessarily $G[X]$ is empty.

To continue, assume that each non trivial interval of G induces an empty subgraph of G. By Theorem 7, the following three observations suffice to obtain the third assertion. Firstly, $G/P(G)$ is not complete because G and hence $G/P(G)$ are asymmetric. Secondly, if $G/P(G)$ is empty, then G is also since $G[X]$ is empty for every $X \in P(G)$. Thirdly, if $G/P(G)$ is a total order and if $|P(G)| \geq 3$, then $V \setminus S$ is a non trivial interval of G such that $G - S$ is not empty, where S denotes the smallest element of $G/P(G)$. Consequently, if $G/P(G)$ is a total order, then $|P(G)| = 2$ and hence $G/P(G)$ is indecomposable.

Lastly, assume that $P(G)$ satisfies the third assertion and consider a graph H 3-hemimorphic to G. The flags do not embed into G because it is asymmetric. It follows from Proposition 4 that $P(H) = P(G)$. Given distinct elements X, Y and Z of $P(G)$, we have $(G/P(G))[\{X,Y,Z\}] \simeq G[\{x,y,z\}]$ and $(H/P(G))[\{X,Y,Z\}] \simeq H[\{x,y,z\}]$ for any $x \in X$, $y \in Y$ and $z \in Z$. Therefore, $G/P(G)$ and $H/P(G)$ are 3-hemimorphic graphs into which the flags do not embed. As $G/P(G)$ is indecomposable, it follows from Theorem 3 that $H/P(G) = G/P(G)$ or $(G/P(G))^{\star}$. Finally, since G and H are 2-hemimorphic, $H[X]$ is empty for every $X \in P(G)$. Consequently, $H = G$ or $G^{\star}$. □

7 Proof of Propositions 1 and 2

Before establishing Proposition 1, we verify it for critical graphs defined on 4 or 5 vertices, and for the poset R_6.

Lemma 4. *Every critical graph defined on 4 vertices is 3-forced.*

Proof. Let $G = (V, A)$ be a critical graph with $|V| = 4$. By Remark 3, assume that G is not symmetric. By Theorem 5, G is isomorphic to Q_4, R_4, $\overline{Q_4}$ or $\overline{R_4}$. Because $Q_4 \simeq R_4$, we have $G \simeq Q_4$ or $\overline{Q_4}$. Lastly, by Remark 4, assume that $G = Q_4$. By Remark 5, $Q_4[\{0,1,3\}]$ and $Q_4[\{0,2,3\}]$ are 3-forced. Consequently, $(\{0,1,3\},\{0,2,3\})$ constitutes a covering propagation sequence of Q_4. By Remark 1, Q_4 is 3-forced. □

Lemma 5. *Every critical graph defined on 5 vertices is 3-forced.*

Proof. Let $G = (V, A)$ be a critical graph with $|V| = 5$. Remark 3, Theorem 5 and Remark 4 allow us to assume that $G = H_5, T_5, U_5$ or V_5, as in the preceding proof. So we distinguish the four cases below where H is a graph 3-hemimorphic to G.

Firstly, assume that $G = H_5$. By Remark 5, $H_5[\{0,1,2\}]$ and $H_5[\{0,1,4\}]$ are 3-forced. Thus, $(\{0,1,2\},\{0,1,4\})$ is a covering propagation sequence of $H_5 - 3$ and $H_5 - 3$ is 3-forced by Remark 1. By interchanging H and $H^{\star}$, assume that $H - 3 = H_5 - 3$. By Remark 5, $H_5[\{0,3,4\}]$ and $H_5[\{2,3,4\}]$ are 3-forced so that $(\{0,3,4\},\{2,3,4\})$ is a covering propagation sequence of $H_5 - 1$. By Remark 1, $H_5 - 1$ is 3-forced and hence $H - 1 = H_5 - 1$ or $(H_5 - 1)^{\star}$. For a contradiction, suppose that $H - 1 = (H_5 - 1)^{\star}$. Since $H[\{0,1,3\}] \simeq H_5[\{0,1,3\}]$ or $(H_5[\{0,1,3\}])^{\star}$ and since $H_5[\{0,1,3\}]$ is a total order, $H[\{0,1,3\}]$ is a total order. As $3 \longrightarrow 0 \longrightarrow 1$ in H, we have $3 \longrightarrow 1$ in H. However, by considering the total orders $H_5[\{1,3,4\}]$ and $H[\{1,3,4\}]$, we should obtain $1 \longrightarrow 3$ in H. Consequently, $H - 1 = H_5 - 1$. It follows that for any $x \neq y \in \{0,1,2,3,4\}$, we have $H[\{x,y\}] = H_5[\{x,y\}]$

if $\{x,y\} \neq \{1,3\}$. In particular, we have $1 \longrightarrow 2 \longrightarrow 3$ in H_5 and H. Since $1 \longrightarrow 3$ in H_5, we obtain $1 \longrightarrow 3$ in H by Remark 5. Thus, $H = H_5$.

Secondly, assume that $G = T_5$ or U_5. For any $x \neq y \in \{0,1,2,3,4\}$, we have $T_5[\{x,y\}] = U_5[\{x,y\}]$ if $\{x,y\} \neq \{3,4\}$. By Remark 5, $G[\{0,1,3\}]$, $G[\{0,2,3\}]$ and $G[\{0,2,4\}]$ and $G[\{1,2,4\}]$ are 3-forced so that $(\{0,1,3\}, \{0,2,3\}, \{0,2,4\}, \{1,2,4\})$ constitutes a propagation sequence of G. For instance, assume that $0 \longrightarrow 1$ in H. It follows that for any $x \neq y \in \{0,1,2,3,4\}$, we have $H[\{x,y\}] = G[\{x,y\}]$ if $\{x,y\} \neq \{3,4\}$. By Remark 5, since $4 \longrightarrow 1 \longrightarrow 3$ in G and H, we have $H[\{3,4\}] = G[\{3,4\}]$ and hence $H = G$.

Thirdly, assume that $G = V_5$. By Remark 5, $V_5[\{0,1,4\}]$, $V_5[\{0,3,4\}]$ and $V_5[\{2,3,4\}]$ are 3-forced and hence $(\{0,1,4\}, \{0,3,4\}, \{2,3,4\})$ is a propagation sequence of V_5. For example, assume that $0 \longrightarrow 1$ in H. For any $x \neq y \in \{0,1,2,3,4\}$, we have $H[\{x,y\}] = V_5[\{x,y\}]$ if $\{x,y\} \neq \{0,2\}, \{1,2\}$ and $\{1,3\}$. To conclude, we apply successively Remark 5 as follows. Since $1 \longrightarrow 4 \longrightarrow 2$ in V_5 and H and since $1 \longrightarrow 2$ in V_5, we have $1 \longrightarrow 2$ in H. Thus, we have $0 \longrightarrow 1 \longrightarrow 2$ and $1 \longrightarrow 2 \longrightarrow 3$ in V_5 and H. As $0 \longrightarrow 2$ and $1 \longrightarrow 3$ in V_5, we obtain $0 \longrightarrow 2$ and $1 \longrightarrow 3$ in H. Consequently, $H = V_5$. □

Lemma 6. *The poset R_6 is 3-forced*

Proof. We have $R_6[X] \simeq Q_4$ for $X = \{0,1,2,3\}, \{0,1,2,5\}, \{0,1,4,5\}, \{0,3,4,5\}$ and $\{2,3,4,5\}$. By Lemma 4, Q_4 is 3-forced. It follows that $(\{0,1,2,3\}, \{0,1,2,5\}, \{0,1,4,5\}, \{0,3,4,5\}, \{2,3,4,5\})$ is a covering propagation sequence of R_6. By Remark 1, R_6 is 3-forced. □

Proof of Proposition 1. Let $G = (V, A)$ be a critical graph. We proceed by induction on $|V| \geq 4$ to demonstrate that G is 3-forced. By Remark 3, assume that G is not symmetric. By Theorem 5, G is isomorphic to T_{2n+1}, U_{2n+1}, V_{2n+1}, Q_{2n}, R_{2n}, H_{2n+1}, $\overline{Q_{2n}}$, $\overline{R_{2n}}$ or $\overline{H_{2n+1}}$, where $n \geq 2$. By Remark 4, assume that G is isomorphic to T_{2n+1}, U_{2n+1}, V_{2n+1}, Q_{2n}, R_{2n} or H_{2n+1}, where $n \geq 2$. Lemmas 4 and 5 permit to conclude when $n = 2$. Furthermore, R_6 is 3-forced by Lemma 6. So, assume that $G = Q_{2n}, H_{2n+1}, V_{2n+1}, (\varphi_{2n+1})^{-1}(T_{2n+1}), (\varphi_{2n+1})^{-1}(U_{2n+1})$ or R_{2n+2}, with $n \geq 3$. It follows from the first part of Remark 2 that $\mathbb{I}(G)[\{0,\ldots,5\}] = P_6$ whatever G. Moreover, by the second part of Remark 2 and by induction hypothesis, $G - \{0,1\}$, $G - \{2,3\}$ and $G - \{4,5\}$ are 3-forced. If $G \neq R_{2n+2}$, then $\{4,5\}$ is an oriented pair of $G - \{0,1\}$ and $G - \{2,3\}$, and $\{0,1\}$ is an oriented pair of $G - \{2,3\}$ and $G - \{4,5\}$. If $G = R_{2n+2}$, then $\{5,6\}$ is an oriented pair of $G - \{0,1\}$ and $G - \{2,3\}$, and $\{1,6\}$ is an oriented pair of $G - \{2,3\}$ and $G - \{4,5\}$. Consequently, $(V \setminus \{0,1\}, V \setminus \{2,3\}, V \setminus \{4,5\})$ is a covering propagation sequence of G. By Remark 1, G is 3-forced. □

Proof of Proposition 2. By Theorem 6 and Remark 4, it suffices to prove that M_k and N_k are 3-forced for $k \geq 6$.

Firstly, we verify that M_k is 3-forced by induction on $k \geq 5$. The permutation (0 2 4 1 3) of $\{0, \ldots, 4\}$ realizes an isomorphism from M_5 onto V_5. By Lemma 5, M_5 is 3-forced. Assume that $k \geq 6$ and consider a graph H 3-hemimorphic to M_k. Clearly, $M_k - 0$ and $M_k - (k-1)$ are isomorphic to M_{k-1}. Therefore, $(\{1, \ldots, k-1\}, \{0, \ldots, k-2\})$ is a propagation sequence of M_k. For instance, assume that $M_k - 0 = H - 0$. We obtain that $M_k - (k-1) = H - (k-1)$. Consequently, for any $x \neq y \in \{0, \ldots, k-1\}$, we have $M_k[\{x, y\}] = H[\{x, y\}]$ if $\{x, y\} \neq \{0, k-1\}$. In particular, we have $k - 1 \longrightarrow 2 \longrightarrow 0$ in M_k and H. Since $k - 1 \longrightarrow 0$ in M_k, we obtain $k - 1 \longrightarrow 0$ in H by Remark 5. Thus, $H = M_k$.

Secondly, we show that $N_k - (k-1)$ is 3-forced by induction on $k \geq 6$. The permutation (0 1 2 4 3) of $\{0, \ldots, 4\}$ realizes an isomorphism from $N_6 - 5$ onto U_5. By Lemma 5, $N_6 - 5$ is 3-forced. Assume that $k \geq 7$ and consider a graph H 3-hemimorphic to $N_k - (k-1)$. Clearly, $(N_k - (k-1)) - 0$ is isomorphic to $N_{k-1} - (k-2)$ and $(N_k - (k-1)) - (k-2)$ is isomorphic to M_{k-2}. By induction hypothesis and by the foregoing, $(N_k - (k-1)) - 0$ and $(N_k - (k-1)) - (k-2)$ are 3-forced. Therefore, $(\{1, \ldots, k-2\}, \{0, \ldots, k-3\})$ is a propagation sequence of $N_k - (k-1)$. For example, assume that $H - 0 = (N_k - (k-1)) - 0$. We obtain that $H - (k-2) = (N_k - (k-1)) - (k-2)$. Consequently, for any $x \neq y \in \{0, \ldots, k-2\}$, we have $(N_k - (k-1))[\{x, y\}] = H[\{x, y\}]$ if $\{x, y\} \neq \{0, k-2\}$. In particular, we have $0 \longrightarrow 1 \longrightarrow k-2$ in $N_k - (k-1)$ and H. As $0 \longrightarrow k-2$ in $N_k - (k-1)$, we obtain $0 \longrightarrow k-2$ in H by Remark 5. Thus, $H = N_k - (k-1)$.

Finally, we establish that N_k is 3-forced by induction on $k \geq 5$. The transposition (2 4) of $\{0, \ldots, 4\}$ realizes an isomorphism from N_5 onto V_5. By Lemma 5, N_5 is 3-forced. Assume that $k \geq 6$ and consider a graph H 3-hemimorphic to N_k. As previously shown, $N_k - (k-1)$ is 3-forced. Furthermore, $N_k - 0$ is isomorphic to N_{k-1}. By induction hypothesis, $N_k - 0$ is 3-forced. Therefore, $(\{1, \ldots, k-1\}, \{0, \ldots, k-2\})$ is a propagation sequence of N_k. For example, assume that $H - 0 = N_k - 0$. We obtain that $H - (k-1) = N_k - (k-1)$. Consequently, for any $x \neq y \in \{0, \ldots, k-1\}$, we have $N_k[\{x, y\}] = H[\{x, y\}]$ if $\{x, y\} \neq \{0, k-1\}$. In particular, we have $0 \longrightarrow 1 \longrightarrow k-1$ in N_k and H. As $0 \longrightarrow k-1$ in N_k, we obtain $0 \longrightarrow k-1$ in H by Remark 5. Thus, $H = N_k$. □

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