Sharp upper bounds for the largest Laplacian eigenvalue of mixed graphs

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Abstract

The paper presents two sharp upper bounds for the largest Laplacian eigenvalue of mixed graphs in terms of the degrees and the average 2-degrees, which improve and generalize the main results of Zhang and Li [Linear Algebra Appl. 353(2002)11-20], Pan [Linear Algebra Appl. 355(2002)287-295], respectively. Moreover, we also characterize some extreme graphs which attain these upper bounds. In last, some examples show that our bounds are improvement on some known bounds in some cases.

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1. Introduction

Let $G = (V, E)$ be a graph, where V is the vertex set and $E \subseteq V \times V \setminus \{(u, u) : u \in V\}$ is the edge set. G is said to be *simple* if $(u, v) \in E$ implies

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$(v,u) \in E$ and G is said to be *mixed*[1] if $(u,v) \in E$ does not always imply $(v,u) \in E$. In a mixed graph G, an edge $(u,v) \in E$ is an unoriented edge uv if $(v,u) \in E$. If $(u,v) \in E$ and $(v,u) \notin E$, then (u,v) is an oriented edge which is also denoted by $u \to v$, where u and v are called the positive and negative ends of $u \to v$, respectively. Hence, in a mixed graph, some edges are oriented, while the others are not. For a mixed graph $G = (V,E)$, let $u \sim v$ denote an oriented or unoriented edge between u and v. The unoriented graph $\tilde{G} = (\tilde{V}, \tilde{E})$ is called the *underlying graph*[23, 24] of a mixed graph G if the vertex set $\tilde{V} = V$ and the edge set satisfies: any unoriented edge $uv \in \tilde{E}$ if and only if $u \sim v$ in G. Notice that our mixed graphs do not contain multi-edges and loops, thus $\tilde{G}$ is a simple graph. Clearly, it allows for the possibilities for mixed graphs to have all edges oriented or unoriented as extreme cases (see[4, 9]).

Let $G = (V,E)$ be a mixed graph. The *incidence matrix*[1] of G is defined to be $M(G) = (m_{u,e})$, where $m_{u,e} = 1$ if u is incident to an unoriented edge e or u is the positive end of the oriented edge e; $m_{u,e} = -1$ if u is the negative end of the oriented edge e; $m_{u,e} = 0$, otherwise. $L(G) = M(G)M(G)^t = (l_{u,v})$ and $K(G) = M(G)^t M(G)$ are called the *Laplacian matrix*[1] and the *edge version of the Laplacian matrix*[2] of G, respectively, where $M(G)^t$ is the transpose of $M(G)$. Clearly, $L(G)$ is a real symmetric, positive semidefinite matrix. The eigenvalues of $L(G)$ can be denoted by $\lambda_1(L(G)) \geq \lambda_2(L(G)) \geq \ldots \geq \lambda_n(L(G)) \geq 0$, which are called the *Laplacian eigenvalues* of G. In particular, if G is a mixed graph which all edges are oriented, $|M(G)|$ and $L(G)$ are consistent with the incident matrix and the Laplacian matrix of a simple graph, respectively (see[13]). If G is a mixed graph which all edges are unoriented, $L(G)$ is consistent with the signless Laplacian matrix for simple graphs. The (signless) Laplacian matrix of a simple graph has been extensively investigated for a long time (see, for example, [5, 6, 7, 10, 11, 12, 13, 14, 15, 16, 17, 19, 20, 21, 22] and the references therein). For many properties of mixed graphs, readers may refer to [1, 2, 8, 18, 23, 24] and the references therein.

Throughout this paper, for a mixed graph $G = (V,E)$, d_u and m_u denote the degree of a vertex $u \in V$ and the average (arithmetic mean) of the degrees of the vertices adjacent to u, respectively. m_u is called the average 2-degrees of u. Also let Δ and δ be the largest and smallest degrees of vertices in G, respectively.

For the largest Laplacian eigenvalue of mixed graphs, there are many results on the upper bounds as follows:

$$\lambda_1(L(G)) \leq \max_{u \in V} \{d_u + m_u\}, \text{ [24]} \tag{1}$$

$$\lambda_1(L(G)) \leq \max_{u \sim v} \left\{ \frac{d_u + d_v + \sqrt{(d_u - d_v)^2 + 4m_u m_v}}{2} \right\}, \text{ [24]} \tag{2}$$

$$\lambda_1(L(G)) \leq \Delta + \sqrt{2m-(n-1)\delta + \Delta(\delta-1)}, \text{ [24]} \tag{3}$$

$$\lambda_1(L(G)) \leq \max_{u\sim v}\left\{\frac{d_u(d_u+m_u)+d_v(d_v+m_v)}{d_u+d_v}\right\}, \text{ [23]} \tag{4}$$

$$\lambda_1(L(G)) \leq \max_{u\sim v, v\sim w, u\neq w}\left\{2+\sqrt{(d_u+d_v-2)(d_v+d_w-2)}\right\}, \text{ [23]} \tag{5}$$

$$\lambda_1(L(G)) \leq \max_{u\sim v}\left\{2+\sqrt{d_u(d_u+m_u-4)+d_v(d_v+m_v-4)+4}\right\} \text{ [22]} \tag{6}$$

$$\lambda_1(L(G)) \leq \max_{u\in V}\left\{d_u+\sqrt{d_u m_u}\right\}. \text{ [22]} \tag{7}$$

In this paper, we firstly prove that the equality in (4) holds if and only if G is pre-bipartite and regular or semi-regular mixed graph. Then we present two sharp upper bounds for the largest Laplacian eigenvalue of mixed graphs in terms of the degrees and the average 2-degrees, which improve and generalize the main results of Zhang et al.[23] and Pan[17], respectively. In addition, we also characterize some extreme graphs which attain these upper bounds. Some examples show that these bounds are improvement on the above presented bounds in some cases.

2. Lemmas and main results

Let $G=(V,E)$ be a mixed graph and its *adjacency matrix*[23] $A(G)=(a_{u,v})$, where $a_{u,v}=1$, if uv is an unoriented edge; $a_{u,v}=-1$ if $u\rightarrow v$ or $v\rightarrow u$; $a_{u,v}=0$, otherwise; while $a_{u,u}=0$. The *line graph*[23] of G is defined to be $G^l=(V(G^l),E(G^l))$, where $V(G^l)=E(G)$. For $e_i,e_j\in V(G^l)$, e_ie_j is an unoriented edge in G^l if e_i,e_j are unoriented edges in G and have a common vertex, or one of e_i,e_j is oriented edge in G and their common vertex is the positive end of the oriented edge, or both e_i and e_j are oriented edges in G and their common vertex is their common positive (or negative) end; $e_i\rightarrow e_j$ is an oriented edge in G^l, where e_i and e_j are the positive and negative ends of $e_i\rightarrow e_j$, respectively, if e_i is an unoriented edge, e_j is an oriented edge in G and their common vertex is the negative end of e_j, or both e_i and e_j are oriented edges in G and their common vertex is the positive and negative ends of e_i and e_j, respectively.

Lemma 1[23]. *Let $G=(V,E)$ be a mixed graph and $D(G)$ be the degree diagonal matrix. Then $L(G)=D(G)+A(G)$ and $K(G)=2I_m+A(G^l)$, where I_m is the identity matrix and m is the number of edges in G.*

A mixed graph $G=(V,E)$ is called *pre-bipartite* if there exists a partition V_1,V_2 of V such that every edge between V_1 and V_2 is oriented and every edge within V_1 or V_2 is unoriented. Let $H=(h_{i,j})$ be a symmetric matrix. Denote by $\lambda_1(H)$ the largest eigenvalue of H. The absolute matrix of H is denoted by $|H|=(|h_{i,j}|)$.

Lemma 2[23]. *Let $G = (V, E)$ be a connected mixed graph. Then the following statements are equivalent: (i) G is pre-bipartite; (ii) G^l is pre-bipartite; (iii) $\lambda_1(L(G)) = \lambda_1(|L(G)|)$; (iv) $\lambda_1(A(G^l)) = \lambda_1(|A(G^l)|)$; (v) $L(G)$ is similar to $|L(G)|$.*

Lemma 3[23]. *Let $G = (V, E)$ be a connected mixed graph and G^l be the line graph of G. Then*

(i) G^l is regular if and only if G is regular or semi-regular;

(ii) G^l is semi-regular but not regular if and only if G is a path with four vertices.

Lemma 4[3]. *Let A be an irreducible matrix and $A \geq |C|$. Then, for every eigenvalue λ of C, $|\lambda| \leq \rho(A)$.*

Lemma 5[17]. *Let $G = (V, E)$ be a connected simple graph. Then*

$$\lambda_1(L(G)) \leq \max_{u \sim v} \left\{ \frac{d_u(d_u + m_u) + d_v(d_v + m_v)}{d_u + d_v} \right\}$$

with equality if and only if G is a regular bipartite graph or a semi-regular bipartite graph.

Theorem 6. *Let $G = (V, E)$ be a connected mixed graph. Then the inequality (4) holds. Moreover, the equality in (4) holds if and only if G is pre-bipartite and regular or semi-regular.*

Proof. Zhang and Li[23] have proved that the equality in (4) holds if G is pre-bipartite and regular or semi-regular mixed graph. Next, we just need to show that if the equality in (4) holds, then G is pre-bipartite and regular or semi-regular. By Lemmas 1 and 4, we have

$$\lambda_1(L(G)) \leq 2 + \lambda_1(|A(G^l)|) = \lambda_1(2I_m + |A(G^l)|). \tag{8}$$

Now assume that the equality in (4) holds. Then the equality in (8) holds. Notice that $L(G) = M(G)M(G)^t$ and $K(G) = M(G)^t M(G)$ have the same nonzero eigenvalues. Lemma 1 implies that

$$\lambda_1(L(G)) = \lambda_1(K(G)) = \lambda_1(2I_m + A(G^l)) = 2 + \lambda_1(A(G^l)). \tag{9}$$

Hence, $\lambda_1(A(G^l)) = \lambda_1(|A(G^l)|)$. By Lemma 2, G is pre-bipartite.

Now we shall show that G is either regular or semi-regular. Let $\tilde{G}$ be the underlying graph of G. $\tilde{G}$ is a connected simple graph since all edges of $\tilde{G}$ are unoriented. Let $L(\tilde{G})$ and $K(\tilde{G})$ be the Laplacian matrix and the edge version of the Laplacian matrix of $\tilde{G}$, respectively. Then $K(\tilde{G}) = 2I_m + |A(G^l)|$. Notice that $L(\tilde{G})$ and $K(\tilde{G})$ have the same nonzero eigenvalues. By Lemma 5, we have

$$\lambda_1(L(\tilde{G})) \leq \max_{u \sim v} \left\{ \frac{d_u(d_u + m_u) + d_v(d_v + m_v)}{d_u + d_v} \right\}$$

with equality if and only if $\tilde{G}$ is a regular bipartite graph or a semi-regular bipartite graph. Since the equality in (8) holds. Thus we have, from (9), $\lambda_1(L(G)) = \lambda_1(K(G)) = \lambda_1(K(\tilde{G})) = \lambda_1(L(\tilde{G}))$. Hence G is either regular or semi-regular. This completes the proof. □

Now we shall present an upper bound for the spectral radius of the adjacency matrix of a connected simple graph, which is used to obtain a sharp upper bound for the largest Laplacian eigenvalue of a mixed graph.

Lemma 7. *Let $G = (V, E)$ be a connected simple graph. Then*

$$\lambda_1(A(G)) \le \max_{u\sim v}\left\{\sqrt{\frac{d_u(m_u+2)d_v(m_v+2)}{(d_u+2)(d_v+2)}}\right\}. \tag{10}$$

with equality if and only if G has one of the following properties:

(i) There exists a constant τ such that, for each $u \in V$,

$$\frac{d_u(m_u+2)}{d_u+2} = \tau; \tag{11}$$

(ii) G is bipartite graph with a partition V_1, V_2 of V and there exist two constants τ_1, τ_2 such that

$$\frac{d_u(m_u+2)}{d_u+2} = \tau_1,\ \forall u \in V_1;\ \frac{d_u(m_u+2)}{d_u+2} = \tau_2,\ \forall u \in V_2. \tag{12}$$

In particular, if G is a regular or semi-regular bipartite graph, then the equality in (10) holds.

Proof. Let $D = \mathrm{diag}(d_1+2, d_2+2, \dots, d_n+2)$, where d_i is the degree of vertex $i \in V$. Then the (i,j)th element of $D^{-1}A(G)D$ is

$$\begin{cases} \frac{d_j+2}{d_i+2} & ij \in E, \\ 0 & \text{otherwise.} \end{cases} \tag{13}$$

Since G is connected, $A(G)$ is irreducible nonnegative matrix, so also is $D^{-1}A(G)D$. Let x be the Perron vector of $D^{-1}A(G)D$, that is,

$$D^{-1}A(G)Dx = \lambda_1(A(G))x. \tag{14}$$

Then the Perron-Frobenius Theorem implies that $x > 0$. Let $x_s = \max\{x_i : i \in V\}$ and $x_t = \max\{x_i : i \in N_s\}$, where N_s denotes the neighbors of the vertex $s \in V$. From (14), one has

$$\lambda_1(A(G))x_s = \sum_{i\in N_s} x_i \frac{d_i+2}{d_s+2} \le x_t \sum_{i\in N_s} \frac{d_i+2}{d_s+2} = \frac{d_s m_s + 2d_s}{d_s+2} x_t \tag{15}$$

and

$$\lambda_1(A(G))x_t = \sum_{i\in N_t} x_i \frac{d_i+2}{d_t+2} \le x_s \sum_{i\in N_t} \frac{d_i+2}{d_t+2} = \frac{d_t m_t + 2d_t}{d_t+2} x_s. \quad (16)$$

Multiplying (15) and (16), one gets the required result (10).

Suppose that the equality in (10) holds, then the above equalities in both (15) and (16) hold. Hence, for any $i \in N_s$, $x_i = x_t$ and for any $i \in N_t$, $x_i = x_s$. Since G is connected, by repeated using the equalities in both (15) and (16), it is easy to see that for any $u \in V$, $x_u = x_s$ or x_t when the distance between vertices u and s is even or odd, respectively. If $x_s = x_t$, then x is constant vector. Following from (14), we get that (11) holds.

If $x_s > x_t$. Let $V_1 = \{u : x_u = x_s\}$ and $V_2 = \{u : x_u = x_t\}$. Thus $V = V_1 \cup V_2$ and the subgraphs induced by V_1 and V_2 respectively are empty graphs. Hence G is bipartite. It follows from (14) that, for any $k, l \in V_1$,

$$\lambda_1(A(G))x_k = \sum_{i\in N_k} x_i \frac{d_i+2}{d_k+2} = x_t \frac{d_k(m_k+2)}{d_k+2}$$

and

$$\lambda_1(A(G))x_l = \sum_{i\in N_l} x_i \frac{d_i+2}{d_l+2} = x_t \frac{d_l(m_l+2)}{d_l+2}.$$

Hence $\frac{d_k(m_k+2)}{d_k+2} = \frac{d_l(m_l+2)}{d_l+2} = \tau_1$, where τ_1 is a constant. Similarly, $\frac{d_u(m_u+2)}{d_u+2} = \tau_2$ for every $u \in V_2$, where τ_2 is also a constant.

Conversely, if (11) holds, then $D^{-1}A(G)De_n = \tau e_n$, where e_n is the vector of all ones. By the Perron-Frobenius Theorem, one has $\lambda_1(A(G)) = \tau$, which implies that the equality in (10) holds. Now suppose that G is bipartite and there exists a partition V_1, V_2 of V such that (12) holding. Without loss of generality, we assume that $D^{-1}A(G)D = \begin{pmatrix} 0_{n_1\times n_1} & B_{n_1\times n_2} \\ C_{n_2\times n_1} & 0_{n_2\times n_2} \end{pmatrix}$, where $0_{n_1\times n_1}$ is a $n_1 \times n_1$ matrix with all entries zeros and $|V_1| = n_1$, $|V_2| = n_2$. Note that the row sums of $B_{n_1\times n_2}$ and $C_{n_2\times n_1}$ are τ_1 and τ_2, respectively. Let $x = (\sqrt{\tau_1}e_{n_1}^t, \sqrt{\tau_2}e_{n_2}^t)^t$. Then $D^{-1}A(G)Dx = \sqrt{\tau_1\tau_2}x$, which implies $\lambda_1(A(G)) = \sqrt{\tau_1\tau_2}$. Hence the equality in (10) holds.

Moreover, if G is d-regular, then $\frac{d_u(m_u+2)}{d_u+2} = d$, $\forall u \in V$. If G is (p,q)-semi-regular bipartite graph, then there exists a partition V_1, V_2 of V such that

$$\frac{d_u(m_u+2)}{d_u+2} = \frac{p(q+2)}{p+2},\ \forall u \in V_1;\quad \frac{d_u(m_u+2)}{d_u+2} = \frac{q(p+2)}{q+2},\ \forall u \in V_2.$$

By the above argument, we get that the equality in (10) holds. □

Given a graph $G = (V, E)$, for convenience, define

$$f_{x,y} = \frac{d_x(d_x + m_x) + d_y(d_y + m_y)}{d_x + d_y}, \ x \sim y \in E.$$

Lemma 8. *Let $G = (V, E)$ be a connected mixed graph and $G^l = (V(G^l), E(G^l))$ be the line graph of G. Then*

$$\lambda_1(A(G^l)) \leq \max_{u\sim v, v\sim w, u\neq w} \left\{ \sqrt{(f_{u,v} - 2)(f_{v,w} - 2)} \right\}. \tag{17}$$

Moreover, the equality in (17) holds if G^l is pre-bipartite and regular or semi-regular.

Proof. From Lemma 4, one gets $\lambda_1(A(G^l)) \leq \lambda_1(|A(G^l)|)$, where $|A(G^l)|$ may be regarded as the adjacency matrix of the underlying graph of G^l. Noting that $e = (u, v)$ is a vertex of the line graph G^l if $e = (u, v)$ is an edge of G. For the line graph G^l, the degree of a vertex $e = (u, v) \in V(G^l)$ is $d_e = d_u + d_v - 2$ and

$$\begin{aligned} d_e m_e &= \sum_{x\in N_u, x\neq v} (d_x + d_u - 2) + \sum_{y\in N_v, y\neq u} (d_y + d_v - 2) \\ &= \sum_{x\in N_u} (d_x + d_u - 2) + \sum_{y\in N_v} (d_y + d_v - 2) - 2(d_u + d_v - 2) \\ &= d_u(d_u + m_u) + d_v(d_v + m_v) - 2(d_u + d_v) - 2(d_u + d_v - 2). \end{aligned}$$

Thus $\sqrt{\frac{d_e(m_e+2)}{d_e+2}} = \sqrt{f_{u,v} - 2}$. Similarly, we obtain, for a vertex $e' = (v, w) \in V(G^l)$, $\sqrt{\frac{d_{e'}(m_{e'}+2)}{d_{e'}+2}} = \sqrt{f_{v,w} - 2}$. It follows from Lemma 7 that the inequality (17) holds.

Now assume that G^l is pre-bipartite and regular or semi-regular. Then Lemma 2 implies that $\lambda_1(A(G^l)) = \lambda_1(|A(G^l)|)$. If G^l is regular, then from Lemma 7, the equality in (17) holds. If G^l is pre-bipartite and semi-regular, there exists a partition $V_1(G^l), V_2(G^l)$ of $V(G^l)$ such that every edge between $V_1(G^l)$ and $V_2(G^l)$ is oriented and every edge within $V_1(G^l)$ or $V_2(G^l)$ is unoriented. Fixed a vertex $e_i \in V_1(G^l)$. Let $V_{11}(G^l) = \{e_j \in V_1(G^l) : d(e_i, e_j) \text{ is even}\}$, $V_{12}(G^l) = \{e_j \in V_1(G^l) : d(e_i, e_j) \text{ is odd}\}$, $V_{21}(G^l) = \{e_j \in V_2(G^l) : d(e_i, e_j) \text{ is even}\}$, $V_{22}(G^l) = \{e_j \in V_2(G^l) : d(e_i, e_j) \text{ is odd}\}$, where $d(e_i, e_j)$ denotes the distance between vertices e_i and e_j of G^l. Since G^l is semi-regular,there exist no edges between $V_{11}(G^l)$ and $V_{21}(G^l)$, or between $V_{12}(G^l)$ and $V_{22}(G^l)$. Hence $V_{11}(G^l) \cup V_{21}(G^l)$ and $V_{12}(G^l) \cup V_{22}(G^l)$ is a bipartite partition of $V(G^l)$ and the degrees of vertices in each partition are the same, respectively. Thus the underlying graph of G^l is a semi-regular bipartite graph. It follows from Lemma 7 that the equality in (17) holds. This completes the proof. □

Theorem 9. *Let $G = (V, E)$ be a connected mixed graph. Then*

$$\lambda_1(L(G)) \leq \max_{u\sim v, v\sim w, u\neq w} \left\{2 + \sqrt{(f_{u,v} - 2)(f_{v,w} - 2)}\right\}. \tag{18}$$

Moreover, if G has one of the following properties:

(i) G is pre-bipartite and regular;

(ii) G is pre-bipartite and semi-regular;

(iii) G is a path with four vertices.

Then the equality in (18) holds.

Proof. Since $L(G)$ and $K(G)$ have the same nonzero eigenvalues, then Lemma 1 implies $\lambda_1(L(G)) = \lambda_1(K(G)) = 2 + \lambda_1(A(G^l))$. By Lemma 8, one gets that the inequality (18) follows.

Now assume that G is pre-bipartite and regular or semi-regular, or a path with four vertices. By Lemmas 2 and 3, one gets that G^l is pre-bipartite and regular or semi-regular. Then Lemma 8 implies that the equality in (18) holds. This completes the proof. □

Remark 1. By a simple calculation, one can see that (18) is never worse than (1) and (4), which are exactly Corollary 3.5 in [24] and Theorem 4.5 in [23], respectively.

Corollary 10. *Let $G = (V, E)$ be a simple connected graph. Then*

$$\lambda_1(L(G)) \leq \max_{u\sim v, v\sim w, u\neq w} \left\{2 + \sqrt{(f_{u,v} - 2)(f_{v,w} - 2)}\right\}. \tag{19}$$

Moreover, if G is a regular bipartite graph, or semi-regular bipartite graph, or a path with four vertices, then the equality in (19) holds.

Remark 2. By a simple calculation, one can see that (17) is never worse than Theorem in [14](or Theorem 2.4 in [17]) and Theorem 2.10 in [17](or Theorem 3 in [12]).

Theorem 11. *Let $G = (V, E)$ be a connected mixed graph. Then*

$$\lambda_1(L(G)) \leq 2 + \sqrt{(a-2)(b-2)}, \tag{20}$$

where $a = \max\{f_{u,v} : u \sim v \in E\}$ and suppose $x \sim y \in E$ satisfies $f_{x,y} = a$, $b = \max\{f_{u,v} : u \sim v \in E \setminus \{x \sim y\}\}$. Moreover, the equality in (20) holds if and only if G has one of the following properties:

(i) G is pre-bipartite and regular;

(ii) G is pre-bipartite and semi-regular;

(iii) G is a path with four vertices.

Proof. Let $\tilde{G}$ be the underlying graph of G. Then $\tilde{G}$ is a connected mixed graph which all edges are unoriented, that is, $\tilde{G}$ is a simple graph. Let $L(\tilde{G})$ and $K(\tilde{G})$ be the Laplacian matrix and the edge version of the Laplacian matrix of $\tilde{G}$, respectively. Then $K(\tilde{G}) = 2I_m + |A(G^l)|$. Note that $L(\tilde{G})$ and $K(\tilde{G})$ have the same nonzero eigenvalues. By Theorem 2.11 in [17],

$$\lambda_1(L(\tilde{G})) \leq 2 + \sqrt{(a-2)(b-2)}. \tag{21}$$

From Lemma 1, $K(G) = 2I_m + A(G^l)$. Lemma 4 implies that $\lambda_1(K(G)) \leq \lambda_1(K(\tilde{G}))$. Since $L(G)$ and $K(G)$ also have the same nonzero eigenvalues. Then $\lambda_1(L(G)) \leq \lambda_1(L(\tilde{G}))$, which yields the desired result (20).

Now suppose that the equality in (20) holds. Then all inequalities in the above argument must be equalities. Since $\lambda_1(A(G^l)) = \lambda_1(|A(G^l)|)$, by Lemma 2, we get that G is pre-bipartite . By Theorem 2.11 in [17], we obtain that the equality in (21) holds if and only if $\tilde{G}$ is a regular bipartite graph or a semi-regular bipartite graph, or a path with four vertices. Hence G is pre-bipartite and regular or semi-regular, or a path with four vertices.

Conversely, by a similar argument of Lemma 8, it is easy to verify that $\tilde{G}$ is a regular bipartite graph or semi-regular bipartite graph, or a path with four vertices. Hence the equality in (20) holds. □

Remark 3. It is easy to see that (20) is also never worse than both (1) and (4), but (18) and (20) are incomparable. In addition, upper bounds (18) and (20) are better than those presented bounds in Introduction in some cases. For example, let G_1 and G_2 be the two connected mixed graphs in Fig.1. Values of $\lambda_1 = \lambda_1(L(G))$ and of the various bounds for the two graphs are given (to four decimal places) in Table 1. It's worth mentioning that the proof of Theorem 11 implies $\lambda_1(L(G)) \leq \lambda_1(L(\tilde{G}))$ for a mixed graph G. Since the underlying graph $\tilde{G}$ of G is a simple graph, then some known upper bounds of the largest Laplacian eigenvalue of simple graphs are valid for mixed graphs. For example, Theorem 2.10, Theorem 2.11 in [17], Theorem 3.2 in [11] and Theorem 2.14 in [7] etc.

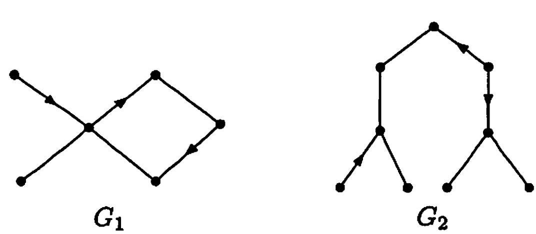

Fig.1.

	λ_1	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(18)	(20)
G_1	5.2361	5.5000	5.3729	6.6458	5.4000	5.4641	5.4641	6.4495	5.4000	5.3665
G_2	4.3028	4.5000	4.3930	5.8284	4.4000	4.4495	4.4495	5.0000	4.3238	4.4000

Table 1: Examples showing that (18) and (20) are, in some cases, best.

Let G be a mixed graph and $\tilde{G}$ be the underlying graph of G. Since $\tilde{G}$ is a simple graph, the Laplacian matrix $L(\tilde{G})$ of $\tilde{G}$ may be regarded as the Laplacian matrix of a mixed graph which all edges are oriented. By the above argument, one get that $\lambda_1(L(G)) \leq \lambda_1(L(\tilde{G}))$. Thus the question arises: given a mixed graph G, construct a new mixed graph $\bar{G}$ by replacing some unoriented edges with oriented edges, can we get that $\lambda_1(L(G)) \leq \lambda_1(L(\bar{G}))$? The following example shows that the answer is negative. Consider the following mixed graphs G_3, G_4 in Fig. 2

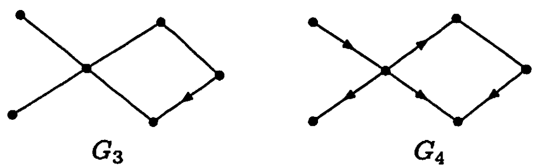

Fig.2.

and G_1, as showed in Fig. 1. Using MATLAB calculation, we obtain that $\lambda_1(L(G_3)) = 5.1249$, $\lambda_1(L(G_1)) = 5.2361$, $\lambda_1(L(G_4)) = 5.1249$. Hence $\lambda_1(L(G_3)) \leq \lambda_1(L(G_1))$, but $\lambda_1(L(G_1)) \nleq \lambda_1(L(G_4))$.

Remark 4. In [16], Oliveira et al. pointed out that some upper bounds for the largest Laplacian eigenvalue yield valid for the largest signless Laplacian eigenvalue of simple graphs. In fact, modifying slightly the proof of Theorem 9 and utilizing the properties of the signless Laplacian eigenvalue of simple graphs in [5, 6], we may obtain that the main results in this paper are also valid for the largest signless Laplacian eigenvalue for simple graphs.

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