MINIMAL ZERO-SUM SEQUENCES OF LENGTH FIVE OVER FINITE CYCLIC GROUPS

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ABSTRACT. Let G be a finite cyclic group. Every sequence S of length l over G can be written in the form $S = (n_1 g) \cdot \ldots \cdot (n_l g)$ where $g \in G$ and $n_1, \ldots, n_l \in [1, \mathrm{ord}(g)]$, and the index $\mathrm{ind}(S)$ of S is defined to be the minimum of $(n_1 + \cdots + n_l)/\mathrm{ord}(g)$ over all possible $g \in G$ such that $\langle g \rangle = G$. In this paper, we determine the index of any minimal zero-sum sequence S of length 5 when $G = \langle g \rangle$ is a cyclic group of a prime order and S has the form $S = g^2(n_2 g)(n_3 g)(n_4 g)$. It is shown that if $G = \langle g \rangle$ is a cyclic group of prime order $p \geq 31$, then every minimal zero-sum sequence S of the above mentioned form has index 1 except in the case that $S = g^2(\frac{p-1}{2}g)(\frac{p+3}{2}g)((p-3)g)$.

1. INTRODUCTION

Throughout the paper G is assumed to be a finite cyclic group of order n written additively. Denote by $\mathcal{F}(G)$, the free abelian monoid with basis G and elements of $\mathcal{F}(G)$ are called *sequences* over G. A sequence of length l of not necessarily distinct elements from G can be written in the form $S = (n_1 g) \cdot \ldots \cdot (n_l g)$ for some $g \in G$. Call S a *zero-sum sequence* if the sum of S is zero (i.e. $\sum_{i=1}^{l} n_i g = 0$). If S is a zero-sum sequence, but no proper nontrivial subsequence of S has sum zero, then S is called a *minimal zero-sum sequence.* Recall that the index of a sequence S over G is defined as follows.

Definition 1.1. For a sequence over G

$$S = (n_1 g) \cdot \ldots \cdot (n_l g), \text{ where } 1 \leq n_1, \ldots, n_l \leq \mathrm{ord}(g),$$

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the index of S is defined by $\operatorname{ind}(S) = \min\{\|S\|_g \mid g \in G \text{ with } G = \langle g\rangle\}$ where

$$\|S\|_g = \frac{n_1 + \cdots + n_l}{\operatorname{ord}(g)}.$$

Clearly, S has sum zero if and only if $\operatorname{ind}(S)$ is an integer. There are also slightly different definitions of the index in the literature, but they are all equivalent (see Lemma 5.1.2 in [7]).

The index of a sequence is a crucial invariant in the investigation of (minimal) zero-sum sequences (resp. of zero-sum free sequences) over cyclic groups. It was first addressed by Kleitman-Lemke (in the conjecture [9, page 344]), used as a key tool by Geroldinger ([6, page 736]), and then investigated by Gao [3] in a systematical way. Since then it has received a great deal of attention (see for example [1, 2, 4, 5, 7, 8, 11, 12, 13, 14, 15]).

A main focus of the investigation of index is to determine minimal zero-sum sequences of index 1. If S is a minimal zero-sum sequence of length $|S|$ such that $|S| \leq 3$ or $|S| \geq \lfloor \frac{n}{2} \rfloor + 2$, then ind(S) = 1 (see [1, 13, 15]). In contrast to that, it was shown that for each l with $5 \leq l \leq \lfloor \frac{n}{2} \rfloor + 1$, there is a minimal zero-sum sequence S of length $|S| = l$ with $\operatorname{ind}(S) \geq 2$ ([13, 15]) and that the same is true for $l = 4$ and $\gcd(n, 6) \neq 1$ ([12]). In two recent papers [11, 10], the authors proved that $\operatorname{ind}(S) = 1$ if $|S| = 4$ and $\gcd(n, 6) = 1$ when n is a prime power or a product of two prime powers with some restriction. However, the general case is still open.

Let $S = (n_1 g) \cdot \ldots \cdot (n_l g)$ be a minimal zero-sum sequence of length l over G. Suppose that there exist an element $ag \in S$ and two elements $xg, yg \in G$ such that $xg + yg = ag$ and $T = S(ag)^{-1}(xg)(yg)$ is a minimal zero-sum sequence of length $l+1$. Clearly $\operatorname{ind}(S) \leq \operatorname{ind}(T)$ as $||S||_g \leq ||T||_g$ for all $g \in G$ with $G = \langle g \rangle$. In this case, the investigation of the index of a minimal zero-sum sequence of length 4 can be transformed into the investigation of the index of a minimal zero-sum sequence of length 5. In order to further investigate the index of a general minimal zero-sum sequence of length 4, it is helpful to determine the index of certain minimal zero-sum sequences of length 5. Little is known about the index of a minimal zero-sum sequence over G of length 5. It is routine to check that if S is a minimal zero-sum sequence over G of length 5, then $1 \leq \operatorname{ind}(S) \leq 2$. Let $\mathsf{h}(S)$ be the maximal repetition of an element in S. Suppose that $|G|$ is a prime. It is shown in Proposition 2.1 that if $\mathsf{h}(S) \geq 3$, then $\operatorname{ind}(S) = 1$. If $\mathsf{h}(S) = 2$, there exist minimal zero-sum sequences S of length 5 with $\operatorname{ind}(S) = 2$ (see Propositions 2.2 and 2.3 below for details). The main purpose of the present paper is to determine the index of a minimal zero-sum sequence S over G of length 5 with $\mathsf{h}(S) \geq 2$. Our main result is as follows.

Theorem 1.2. *Let G be a cyclic group of order p for some prime $p \geq 31$, and let $S \in \mathcal{F}(G)$ be a minimal zero-sum sequence of length $|S| = 5$ with*

$\mathsf{h}(S) \ge 2$. *Then* $\operatorname{ind}(S) \in \{1,2\}$, *and* $\operatorname{ind}(S) = 2$ *if and only if* $S = g^2(\frac{p-1}{2}g)(\frac{p+3}{2}g)((p-3)g)$ *for some* $g \in G$.

We remark that Theorem 1.2 together with Propositions 2.1 and 2.3 determines completely the index of every minimal zero-sum sequence S of length 5 with $\mathsf{h}(S) \ge 2$. However, the remaining case when $\mathsf{h}(S) = 1$ is much more complicated and $\operatorname{ind}(S)$ is not yet determined.

2. Preliminaries

We first prove some preliminary results which will be needed in the next section. Let G be a cyclic group of order n. Suppose that $S = (n_1 g) \cdot \ldots \cdot (n_l g)$ for some $g \in G$. Let $\|S\|'_g = \operatorname{ord}(g)\|S\|_g = \sum_{i=1}^{l} n_i \in \mathbb{N}_0$ and denote by $|x|_n$ the least positive residue of x modulo n, where $n \in \mathbb{N}$ and $x \in \mathbb{Z}$. Let mS denote the sequence $(mn_1 g) \cdot \ldots \cdot (mn_l g)$. If $\operatorname{ord}(g) = n$, then $mS = (|mn_1|_n g) \cdot \ldots \cdot (|mn_l|_n g)$. We note that if $\gcd(n,m) = 1$, then the multiplication by m is a group automorphism of G and hence $\operatorname{ind}(S) = \operatorname{ind}(mS)$.

Proposition 2.1. *Let G be a cyclic group of prime order p and $S \in \mathcal{F}(G)$ be a minimal zero-sum sequence of length* 5. *If* $\mathsf{h}(S) \ge 3$, *then* $\operatorname{ind}(S) = 1$.

Proof. Suppose that $S = (n_1 g) \cdot \ldots \cdot (n_5 g)$ for some $g \in G$ and $1 \le n_1 \le \cdots \le n_5 < p$. Since $\mathsf{h}(S) \ge 3$, without loss of generality we may assume that $n_1 = n_2 = n_3 = 1$. Since S is a minimal zero-sum sequence, we have that $\|S\|'_g = 3 + n_4 + n_5 < 2p$. Therefore $\operatorname{ind}(S) = 1$. □

Proposition 2.2. *Let G be a cyclic group of prime order $p \ge 5$. If* $S = g^2 \cdot (\frac{p-1}{2}g) \cdot (\frac{p+3}{2}g) \cdot ((p-3)g) \in \mathcal{F}(G)$, *then* $\operatorname{ind}(S) = 2$.

Proof. Since $\|S\|'_g = 2p$, it suffices to show for any $m \in [1, p-1]$, we have $\|mS\|'_g > p$. Then $\operatorname{ind}(S) = 2$.

First assume that $m = 2k$. Then $|m(\frac{p-1}{2})|_p = |kp - k|_p = p - k$. Note that $|m(\frac{p+3}{2})|_p \ge 1$ and $|m(p-3)|_p \ge 1$. Therefore, $\|mS\|'_g \ge 2k + 2k + (p-k) + 1 + 1 > p$ and we are done.

Next suppose that $m = 2k+1$, then $2k+1 \le p-2$ and thus $k \le \frac{p-3}{2}$. Hence

$$|(2k+1)(\frac{p-1}{2})|_p = |kp - k + \frac{p-1}{2}|_p = \frac{p-1}{2} - k.$$

If $k < \frac{p-3}{6}$, then $|(2k+1)(\frac{p+3}{2})|_p = \frac{p+3}{2} + 3k$, $|(2k+1)(p-3)|_p = p - 6k - 3$. Therefore, $\|mS\|'_g = (2k+1) + (2k+1) + (\frac{p-1}{2} - k) + (\frac{p+3}{2} + 3k) + (p - 6k - 3) = 2p > p$.

If $\frac{p-3}{6} < k < \frac{2p-3}{6}$, then $|(2k+1)(\frac{p+3}{2})|_p = 3k - \frac{p-3}{2}$, $|(2k+1)(p-3)|_p = 2p - 6k - 3$, so $\|mS\|'_g = 4k + 2 + (\frac{p-1}{2} - k) + (3k - \frac{p-3}{2}) + (2p - 6k - 3) = 2p > p$.

If $\frac{2p-3}{6} \le k \le \frac{p-3}{2}$, then $|(2k+1)(\frac{p+3}{2})|_p = 3k - \frac{p-3}{2}$, $|(2k+1)(p-3)|_p = 3p-6k-3$, so $\|mS\|'_g = 4k+2+(\frac{p-1}{2}-k)+(3k-\frac{p-3}{2})+(3p-6k-3) = 3p > p$.

This completes the proof. □

Proposition 2.3. *Let $G = \langle g \rangle$ be a cyclic group of order p for some prime $p \in [5, 59]$, and let $S = g^2(x_1g)(x_2g)(x_3g)$ be a minimal zero-sum sequence over G, where $2 \le x_1 \le x_2 \le x_3 \le p-3$. Then $\mathrm{ind}(S) = 2$ if and only if one of the following conditions holds.*

(1). $x_1 = \frac{p-1}{2}, x_2 = \frac{p+3}{2}, x_3 = p-3$.
(2). $p = 17$ *and* $x_1 = 8, x_2 = 11, x_3 = 13$.
(3). $p = 19$ *and* $x_1 = 6, x_2 = 14, x_3 = 16$.
(4). $p = 19$ *and* $x_1 = 9, x_2 = 12, x_3 = 15$.
(5). $p = 23$ *and* $x_1 = 11, x_2 = 15, x_3 = 18$.
(6). $p = 23$ *and* $x_1 = 9, x_2 = 15, x_3 = 20$.
(7). $p = 29$ *and* $x_1 = 14, x_2 = 19, x_3 = 23$.

Proof. It is routine to check the proposition holds and we omit the proof here. □

Lemma 2.4. *Let $G = \langle g \rangle$ be a cyclic group of prime order $p \ge 5$, and let $S = g^2(cg)((p-b)g)((p-a)g)$ be a minimal zero-sum sequence over G with $2 + c = a + b$ and $2 < a \le b < c < \frac{p}{2}$. Then $\mathrm{ind}(S) = 1$ if one of the following conditions holds.*

(1). $a = 4,\ b = 6,\ c = 8$ *and* $p > 17$.
(2). $a = 4,\ b = 7,\ c = 9$ *and* $p > 19$.
(3). $a = 3,\ b = 4,\ c = 5$ *and* $p > 15$.
(4). $a = 3,\ b = 5,\ c = 6$ *and* $p > 24$.

Proof. (1). Suppose $p = 6m + t$, where $1 \le t \le 5$. Then $\gcd(m, p) = 1$ and $\|mS\|'_g = \frac{p-t}{6} + \frac{p-t}{6} + \frac{2p-8t}{6} + t + \frac{2p+4t}{6} = p$. Therefore, $\mathrm{ind}(S) = 1$.

(2). Suppose $p = 7m + t$, where $1 \le t \le 6$. Then $\gcd(m, p) = 1$ and $\|mS\|'_g = \frac{p-t}{7} + \frac{p-t}{7} + \frac{2p-9t}{7} + t + \frac{3p+4t}{7} = p$. Therefore, $\mathrm{ind}(S) = 1$.

(3). Suppose $p = 4m + t$, where $1 \le t \le 3$. Then $\gcd(m, p) = 1$ and $\|mS\|'_g = \frac{p-t}{4} + \frac{p-t}{4} + \frac{p-5t}{4} + t + \frac{p+3t}{4} = p$. Therefore, $\mathrm{ind}(S) = 1$.

(4). Suppose $p = 5m + t$, where $1 \le t \le 4$. Then $\gcd(m, p) = 1$ and $\|mS\|'_g = \frac{p-t}{5} + \frac{p-t}{5} + \frac{p-6t}{5} + t + \frac{2p+3t}{5} = p$. Therefore, $\mathrm{ind}(S) = 1$. □

3. Proof of main theorem

In this section we determine the index of every minimal zero-sum sequence S of length 5 over a cyclic group of a prime order with $\mathsf{h}(S) \ge 2$.

Let $G=$ be a cyclic group of prime order $p\geq 31$ and $S\in\mathcal{F}(G)$ be a minimal zero-sum sequence of length 5. We will show that $\text{ind}(S)=1$ except in the case that $S=g^2(\frac{p-1}{2}g)(\frac{p+3}{2}g)((p-3)g)$ for some $g\in G$.

According to Proposition 2.1, we may always assume that $\mathsf{h}(S)=2$. Since p is a prime, there exists $g\in G$ such that $S=g^2(x_1g)(x_2g)(x_3g)$, where $1<x_1\leq x_2\leq x_3<p-2$. This implies that $1+1+x_2+x_2+x_3<3p$. If $1+1+x_1+x_2+x_3=p$, then $\text{ind}(S)=1$. So we may assume that $1+1+x_1+x_2+x_3=2p$. If $x_3>x_2>x_1>\frac{p}{2}$, then $\|2S\|_g'=2+2+(2x_1-p)+(2x_2-p)+(2x_3-p)=p$, and hence $\text{ind}(S)=1$. So we may assume that $x_1<\frac{p}{2}$. Clearly $x_2>\frac{p}{2}$, otherwise $1+1+x_1+x_2+x_3<1+1+\frac{p}{2}+\frac{p}{2}+x_3<2p$, yielding a contradiction. Let $c=x_1, b=p-x_2$, and $a=p-x_3$. Then we can write S in the form

$$S=g^2(cg)((p-b)g)((p-a)g), \tag{3.1}$$

where $2+c=a+b$ and $2<a\leq b<c<\frac{p}{2}$.

By Proposition 2.2, it suffices to show that if $a\neq 3$ or $c\neq\frac{p-1}{2}$, then $\text{ind}(S)=1$. To do so, we will find k and m such that

$$\frac{kp}{c}\leq m<\frac{kp}{b},\ \gcd(m,p)=1,\ 1\leq k\leq b,\ \text{and}\ ma<p. \tag{3.2}$$

Then $\|mS\|_g'\leq m+m+(mc-kp)+(kp-mb)+(p-ma)=p$, and thus $\text{ind}(S)=1$.

Let k_1 be the largest positive integer such that $\lceil\frac{(k_1-1)p}{c}\rceil=\lceil\frac{(k_1-1)p}{b}\rceil$ and $\frac{k_1p}{c}\leq m_1<\frac{k_1p}{b}$. Since $\frac{bp}{c}\leq p-1<p=\frac{bp}{b}$ and $\frac{tp}{b}-\frac{tp}{c}=\frac{t(c-b)p}{bc}>2$ for all $t\geq b$, such integer k_1 always exists and $k_1\leq b$. Since $\lceil\frac{(k_1-1)p}{c}\rceil=\lceil\frac{(k_1-1)p}{b}\rceil$, we have

$$1>\frac{(k_1-1)p}{b}-\frac{(k_1-1)p}{c}=\frac{(k_1-1)p(c-b)}{bc}=\frac{(k_1-1)p(a-2)}{bc}. \tag{3.3}$$

Throughout this section we always assume that S and k_1 are defined as above. We first handle some special cases, and then provide a proof of the main theorem.

In terms of Proposition 2.3, from now on we may always assume that $p\geq 31$.

Lemma 3.1. *If S is a minimal zero-sum sequence such that $k_1\geq 2$, $3<\frac{p}{c}<\frac{p}{b}<4$, $a=3$, $b=3k_1-1$ and $c=3k_1$, then* $\text{ind}(S)=1$.

Proof. Suppose that $p=3b+b_0=9k_1-3+b_0$. Then $b_0\not\equiv 0\pmod 3$. Since $\frac{p}{c}>3$, we infer that $3<b_0<b=3k_1-1$. By (3.3) we have $1>\frac{(k_1-1)(9k_1-3+b_0)}{(3k_1-1)(3k_1)}$. Hence $b_0k_1-9k_1+3-b_0<0$. If $b_0\geq 15$, then $0>b_0(k_1-1)-9k_1+3\geq 15k_1-15-9k_1+3\geq 0$, yielding a contradiction. Hence we must have $4\leq b_0\leq 14$ and $\gcd(b_0,3)=1$.

If $11 \le b_0 \le 14$, then $0 > b_0(k_1-1)-9k_1+3 = 11k_1-11-9k_1+3 = 2k_1-8$ and thus $k_1 \le 3$. Since $11 \le b_0 < 3k_1 - 1$, we infer that $k_1 > 4$, a contradiction.

If $b_0 = 10$, then $0 > b_0(k_1 - 1) - 9k_1 + 3 = 10k_1 - 10 - 9k_1 + 3 = k_1 - 7$ and thus $k_1 < 7$. Since $10 = b_0 < 3k_1 - 1$, we infer that $k_1 \ge 4$. If $k_1 \le 5$, then $p \le 52$, the result follows from Lemma 2.3. If $k_1 = 6$, then $p = 61$. Since $\frac{2p}{c} < 7 < \frac{2p}{b}$ and $7a = 21 < p$, Equation (3.2) holds and we are done.

If $b_0 = 8$, then $\frac{p}{c} = 3+\frac{5}{3k_1}$ and $\frac{p}{b} = 3+\frac{8}{3k_1-1}$. By the definition of k_1, we have $\lceil\frac{(k_1-1)p}{c}\rceil = \lceil\frac{(k_1-1)p}{b}\rceil$. Since $\frac{(k_1-1)p}{c} = 3k_1-3+\frac{5(k_1-1)}{3k_1} < 3k_1-3+2$, we have $\frac{(k_1-1)p}{b} = 3k_1 - 3 + \frac{8(k_1-1)}{3k_1-1} < 3k_1 - 3 + 2$, then $k_1 = 2$. But $8 = b_0 < 3k_1 - 1 = 5$, yielding a contradiction.

If $b_0 = 7$, then $\frac{p}{c} = 3+\frac{4}{3k_1}$ and $\frac{p}{b} = 3+\frac{7}{3k_1-1}$. As above since $\frac{(k_1-1)p}{c} = 3k_1 - 3 + \frac{4(k_1-1)}{3k_1} < 3k_1 - 3 + 2$, we have $\frac{(k_1-1)p}{b} = 3k_1 - 3 + \frac{7(k_1-1)}{3k_1-1} < 3k_1 - 3 + 2$, so $k_1 \le 4$. Since $7 = b_0 < 3k_1 - 1$, we infer that $k_1 \ge 3$. If $k_1 = 3$, then $p = 31$, the lemma follows from Lemma 2.3. If $k_1 = 4$, then $p = 40$, a contradiction to that p is prime.

If $b_0 = 5$, then $\frac{p}{c} = 3+\frac{2}{3k_1}$ and $\frac{p}{b} = 3+\frac{5}{3k_1-1}$. As above since $\frac{(k_1-1)p}{c} = 3k_1 - 3 + \frac{2(k_1-1)}{3k_1} < 3k_1 - 3 + 1$, we have $\frac{(k_1-1)p}{b} = 3k_1 - 3 + \frac{5(k_1-1)}{3k_1-1} < 3k_1 - 3 + 1$, so $k_1 < 2$, yielding a contradiction.

If $b_0 = 4$, then $\frac{p}{c} = 3+\frac{1}{3k_1}$ and $\frac{p}{b} = 3+\frac{4}{3k_1-1}$. As above since $\frac{(k_1-1)p}{c} = 3k_1-3+\frac{k_1-1}{3k_1} < 3k_1-3+1$, we have $\frac{(k_1-1)p}{b} = 3k_1-3+\frac{4(k_1-1)}{3k_1-1} < 3k_1-3+1$, so $k_1 = 2$. Therefore $p = 19 < 31$, yielding a contradiction. □

Lemma 3.2. *There exists no minimal zero-sum sequence S such that $k_1 \ge 2$, $3 < \frac{p}{c} < \frac{p}{b} < 4$, $a = 3$, $b = 3k_1 - 2$ and $c = 3k_1 - 1$.*

Proof. Assume to the contrary that such S exists. Suppose $p = 3b + b_0 = 9k_1 - 6 + b_0$. Then $b_0 \not\equiv 0 \pmod 3$. Since $\frac{p}{c} > 3$, we infer that $3 < b_0 < 3k_1-2$. By (3.3) we have $1 > \frac{(k_1-1)(9k_1-6+b_0)}{(3k_1-2)(3k_1-1)}$. Hence $b_0k_1-6k_1+4-b_0 < 0$. If $b_0 \ge 8$, then $0 > b_0(k_1 - 1) - 6k_1 + 4 \ge 8k_1 - 8 - 6k_1 + 4 \ge 0$, yielding a contradiction. Hence we must have $4 \le b_0 \le 7$.

If $b_0 = 7$, then $0 > b_0(k_1 - 1) - 6k_1 + 4 = 7k_1 - 7 - 6k_1 + 4 = k_1 - 3$ and thus $k_1 = 2$. Since $7 = b_0 < 3k_1 - 2$, we infer that $k_1 > 3$, a contradiction.

If $b_0 = 5$, then $\frac{p}{c} = 3 + \frac{2}{3k_1-1}$ and $\frac{p}{b} = 3 + \frac{5}{3k_1-2}$. By the definition of k_1, we have $\lceil\frac{(k_1-1)p}{c}\rceil = \lceil\frac{(k_1-1)p}{b}\rceil$. But $\frac{(k_1-1)p}{c} = 3k_1 - 3 + \frac{2(k_1-1)}{3k_1-1} < 3k_1 - 3 + 1 < 3k_1 - 3 + \frac{5(k_1-1)}{3k_1-2} = \frac{(k_1-1)p}{b}$, yielding a contradiction.

If $b_0 = 4$, then $\frac{p}{c} = 3 + \frac{1}{3k_1-1}$ and $\frac{p}{b} = 3 + \frac{4}{3k_1-2}$. As above we have $\frac{(k_1-1)p}{c} = 3k_1 - 3 + \frac{k_1-1}{3k_1-1} < 3k_1 - 3 + 1 < 3k_1 - 3 + \frac{4(k_1-1)}{3k_1-2} = \frac{(k_1-1)p}{b}$, yielding a contradiction.

In all cases, we have found contradictions. Thus such sequence S does not exist. □

Lemma 3.3. *If S is a minimal zero-sum sequence such that $k_1 \geq 5$, $2 < \frac{p}{c} < \frac{p}{b} < 3$, $a = 4$, $b = 4k_1 - 1$ and $c = 4k_1 + 1$, then* $\mathrm{ind}(S) = 1$.

Proof. Suppose $p = 2b + b_0 = 8k_1 - 2 + b_0$. Then $b_0 \equiv 1 \pmod 2$. Since $\frac{p}{c} > 2$, we infer that $4 < b_0 < 4k_1 - 1$. By (3.3) we have $1 > \frac{2(k_1-1)(8k_1-2+b_0)}{(4k_1-1)(4k_1+1)}$. Hence $2b_0k_1 - 20k_1 + 5 - 2b_0 < 0$. If $b_0 \geq 12$, then $0 > b_0(2k_1-2) - 20k_1 + 5 \geq 24k_1 - 24 - 20k_1 + 5 \geq 0$, yielding a contradiction. Hence we must have $5 \leq b_0 \leq 11$.

If $b_0 = 11$, then $0 > b_0(2k_1-2) - 20k_1 + 5 = 22k_1 - 22 - 20k_1 + 5 = 2k_1 - 17$ and thus $k_1 \leq 8$. If $k_1 = 8$, then $p = 73,\ b = 31,\ c = 33$. Since $\frac{4p}{c} < 9 < \frac{4p}{b}$ and $9a = 36 < p$, we are done. If $k_1 = 7$, then $p = 67,\ b = 27,\ c = 29$. Since $\frac{3p}{c} < 7 < \frac{3p}{b}$ and $7a = 28 < p$, we are done. If $k_1 = 6$, then $p = 57$, a contradiction to p is prime. If $k_1 = 5$, then $p = 49$, a contradiction again.

If $b_0 = 9$, then $\frac{p}{c} = 2 + \frac{5}{4k_1+1}$ and $\frac{p}{b} = 2 + \frac{9}{4k_1-1}$. By the definition of k_1, we have $\lceil \frac{(k_1-1)p}{c} \rceil = \lceil \frac{(k_1-1)p}{b} \rceil$. Since $\frac{(k_1-1)p}{c} = 2k_1 - 2 + \frac{5(k_1-1)}{4k_1+1} < 2k_1 - 2 + 2$, we have $\frac{(k_1-1)p}{b} = 2k_1 - 2 + \frac{9(k_1-1)}{4k_1-1} < 2k_1 - 2 + 2$, then $k_1 < 7$. If $k_1 = 6$, then $p = 55$, a contradiction to that p is prime. If $k_1 = 5$, then $p = 47,\ b = 19,\ c = 21$, the result follows from Lemma 2.3.

If $b_0 = 7$, then $\frac{p}{c} = 2 + \frac{3}{4k_1+1}$ and $\frac{p}{b} = 2 + \frac{7}{4k_1-1}$. By the definition of k_1, we have $\lceil \frac{(k_1-1)p}{c} \rceil = \lceil \frac{(k_1-1)p}{b} \rceil$. But $\frac{(k_1-1)p}{c} = 2k_1 - 2 + \frac{3(k_1-1)}{4k_1+1} < 2k_1 - 2 + 1 < 2k_1 - 2 + \frac{7(k_1-1)}{4k_1-1} = \frac{(k_1-1)p}{b}$, yielding a contradiction.

If $b_0 = 5$, then $\frac{p}{c} = 2 + \frac{1}{4k_1+1}$ and $\frac{p}{b} = 2 + \frac{5}{4k_1-1}$. As above we have $\frac{(k_1-1)p}{c} = 2k_1 - 2 + \frac{(k_1-1)}{4k_1+1} < 2k_1 - 2 + 1 < 2k_1 - 2 + \frac{5(k_1-1)}{4k_1-1} = \frac{(k_1-1)p}{b}$, yielding a contradiction. □

Lemma 3.4. *There exists no minimal zero-sum sequence S such that $k_1 \geq 5$, $2 < \frac{p}{c} < \frac{p}{b} < 3$, $a = 4$, $b = 4k_1 - 2$ and $c = 4k_1$.*

Proof. Assume to the contrary that such S exists. Suppose $p = 2b + b_0 = 8k_1 - 4 + b_0$. Then $b_0 \equiv 1 \pmod 2$. Since $\frac{p}{c} > 2$, we infer that $4 < b_0 < 4k_1 - 2$. By (3.3) we have $1 > \frac{2(k_1-1)(8k_1-4+b_0)}{(4k_1-2)(4k_1)}$. Hence $b_0k_1 - 8k_1 + 4 - b_0 < 0$. If $b_0 \geq 9$, then $0 > b_0(k_1 - 1) - 8k_1 + 4 \geq 9k_1 - 9 - 8k_1 + 4 \geq 0$, yielding a contradiction. Hence we must have $5 \leq b_0 \leq 7$.

If $b_0 = 7$, then $\frac{p}{c} = 2 + \frac{3}{4k_1}$ and $\frac{p}{b} = 2 + \frac{7}{4k_1-2}$. By the definition of k_1, we have $\lceil \frac{(k_1-1)p}{c} \rceil = \lceil \frac{(k_1-1)p}{b} \rceil$. But $\frac{(k_1-1)p}{c} = 2k_1 - 2 + \frac{3(k_1-1)}{4k_1} < 2k_1 - 2 + 1 < 2k_1 - 2 + \frac{7(k_1-1)}{4k_1-1} = \frac{(k_1-1)p}{b}$, yielding a contradiction.

If $b_0 = 5$, then $\frac{p}{c} = 2 + \frac{1}{4k_1}$ and $\frac{p}{b} = 2 + \frac{5}{4k_1-2}$. As above $\frac{(k_1-1)p}{c} = 2k_1 - 2 + \frac{(k_1-1)}{4k_1} < 2k_1 - 2 + 1 < 2k_1 - 2 + \frac{5(k_1-1)}{4k_1-2} = \frac{(k_1-1)p}{b}$, yielding a contradiction. □

Lemma 3.5. *There exists no minimal zero-sum sequence S such that $k_1 \geq 5$, $2 < \frac{p}{c} < \frac{p}{b} < 3$, $a = 4$, $b = 4k_1 - 3$ and $c = 4k_1 - 1$.*

Proof. Assume to the contrary that such S exists. Suppose $p = 2b + b_0 = 8k_1 - 6 + b_0$. Then $b_0 \equiv 1 \pmod 2$. Since $\frac{p}{c} > 2$, we infer that $4 < b_0 < 4k_1 - 3$. By (3.3) we have $1 > \frac{2(k_1-1)(8k_1-6+b_0)}{(4k_1-3)(4k_1-1)}$. Hence $2b_0k_1 - 12k_1 + 9 - 2b_0 < 0$. If $b_0 \geq 7$, then $0 > b_0(2k_1 - 2) - 12k_1 + 9 \geq 14k_1 - 14 - 12k_1 + 9 \geq 0$, giving a contradiction. Hence we must have $b_0 = 5$.

If $b_0 = 5$, then $\frac{p}{c} = 2 + \frac{1}{4k_1-1}$ and $\frac{p}{b} = 2 + \frac{5}{4k_1-3}$. By the definition of k_1, we have $\lceil \frac{(k_1-1)p}{c} \rceil = \lceil \frac{(k_1-1)p}{b} \rceil$. But $\frac{(k_1-1)p}{c} = 2k_1 - 2 + \frac{(k_1-1)}{4k_1-1} < 2k_1 - 2 + 1 < 2k_1 - 2 + \frac{5(k_1-1)}{4k_1-3} = \frac{(k_1-1)p}{b}$, yielding a contradiction. □

Lemma 3.6. *If S is a minimal zero-sum sequence such that $k_1 \geq 5$, $2 < \frac{p}{c} < \frac{p}{b} < 3$, $a = 3$, $b = 2k_1 + k_0$ and $c = 2k_1 + k_0 + 1 < \frac{p-1}{2}$, where $0 \leq k_0 \leq k_1 - 1$, then* $\operatorname{ind}(S) = 1$.

Proof. We will show that there exist $x, y \in [1, \lfloor \frac{b}{3} \rfloor]$ such that $\frac{p}{c} < 2 + \frac{x}{y} < \frac{p}{b}$. Then $(2y + x)a < \frac{yp}{b} \times 3 \leq p$ and we are done.

Suppose $p = 2b + b_0$, where $1 \leq b_0 \leq b - 1$. Since p is prime, we infer that $b_0 \equiv 1 \pmod 2$. Note that $c = b + 1$. It suffices to show there exist $x, y \in [1, \lfloor \frac{b}{3} \rfloor]$ such that $\frac{b_0-2}{b+1} < \frac{x}{y} < \frac{b_0}{b}$.

Case 1. $b \equiv 0 \pmod 3$. Since p is prime, we infer that $b_0 \not\equiv 0 \pmod 3$. Suppose $b = 3s$.

If $b_0 = 3t + 1$, then let $x = t$ and $y = s$. We infer that $\frac{3t-1}{3s+1} < \frac{t}{s} < \frac{3t+1}{3s}$, and we are done.

If $b_0 = 3t + 2$, then let $x = t$ and $y = s$. We infer that $\frac{3t}{3s+1} < \frac{t}{s} < \frac{3t+2}{3s}$, and we are done.

Case 2. $b \equiv 1 \pmod 3$. Since p is prime, we infer that $b_0 \not\equiv 1 \pmod 3$. Suppose $b = 3s + 1$.

First assume that $b_0 = 3t \equiv 1 \pmod 2$. Since $c = b+1 < \frac{p-1}{2} = b + \frac{b_0-1}{2}$, we infer that $b_0 > 3$ and thus $t \geq 3$. If $s < 2t - 2$, then let $x = t - 1$ and $y = s$. We infer that $\frac{3t-2}{3s+2} < \frac{t-1}{s} < \frac{3t}{3s+1}$, and we are done. Next assume that $s \geq 2t - 2$. Choose $y = s - \lceil \frac{s-2t+3}{3t-2} \rceil$ and $x = t - 1$. We will show that $\frac{3t-2}{3s+2} < \frac{t-1}{y} < \frac{3t}{3s+1}$. Since $y = s - \lceil \frac{s-2t+3}{3t-2} \rceil \leq s - \frac{s-2t+3}{3t-2} = \frac{3st-3s+2t-3}{3t-2} < \frac{(t-1)(3s+2)}{3t-2}$, we have $\frac{3t-2}{3s+2} < \frac{t-1}{y}$. Since $t \geq 3$ and $s \geq 2t - 2$, we infer that $\frac{3st-3s-t}{3t-2} > \frac{(t-1)(3s+1)}{3t}$. Since $y = s - \lceil\frac{s-2t+3}{3t-2}\rceil \geq s - \frac{s-2t+3+3t-3}{3t-2} = \frac{3st-3s-t}{3t-2} > \frac{(t-1)(3s+1)}{3t}$, we have $\frac{t-1}{y} < \frac{3t}{3s+1}$, and we are done.

Now assume that $b_0 = 3t+2$. Let $x = t$ and $y = s$. We infer that $\frac{3t}{3s+2} < \frac{t}{s} < \frac{3t+2}{3s+1}$, and we are done.

Case 3. $b \equiv 2 \pmod 3$. Since p is prime, we infer that $b_0 \not\equiv 2 \pmod 3$. Suppose $b = 3s+2$.

Subcase 3.1. $b_0 \equiv 0 \pmod 3$. Suppose $b_0 = 3t$. Recall that $b_0 = 3t \equiv 1 \pmod 2$. Since $c = b+1 < \frac{p-1}{2} = b + \frac{b_0-1}{2}$, we infer that $b_0 > 3$ and thus $t \geq 3$. If $s < 3t-3$, then let $x = t-1$ and $y = s$. We infer that $\frac{3t-2}{3s+3} < \frac{t-1}{s} < \frac{3t}{3s+2}$, and we are done. Next assume that $s \geq 3t-3$. Choose $y = s - \lceil\frac{s-3t+4}{3t-2}\rceil$ and $x = t-1$. We will show that $\frac{3t-2}{3s+3} < \frac{t-1}{y} < \frac{3t}{3s+2}$. Since $y = s - \lceil\frac{s-3t+4}{3t-2}\rceil \leq s - \frac{s-3t+4}{3t-2} = \frac{3st-3s+3t-4}{3t-2} < \frac{(t-1)(3s+3)}{3t-2}$, we have $\frac{3t-2}{3s+3} < \frac{t-1}{y}$. Since $t \geq 3$ and $s \geq 3t-3$, we infer that $\frac{3st-3s-1}{3t-2} > \frac{(t-1)(3s+2)}{3t}$. Since $y = s - \lceil\frac{s-3t+4}{3t-2}\rceil \geq s - \frac{s-3t+4+3t-3}{3t-2} = \frac{3st-3s-1}{3t-2} > \frac{(t-1)(3s+2)}{3t}$, we have $\frac{t-1}{y} < \frac{3t}{3s+2}$, and we are done.

Subcase 3.2. $b_0 \equiv 1 \pmod 3$. Suppose $b_0 = 3t+1$. Recall that $b_0 = 3t+1 \equiv 1 \pmod 2$. Hence $t \equiv 0 \pmod 2$.

If $s > 2t$, then let $x = t$ and $y = s$. We infer that $\frac{3t-1}{3s+3} < \frac{t}{s} < \frac{3t+1}{3s+2}$, and we are done.

If $s < \frac{3t-3}{2}$, then let $x = t-1$ and $y = s$. We infer that $\frac{3t-1}{3s+3} < \frac{t-1}{s} < \frac{3t+1}{3s+2}$, and we are done.

Next assume that $\frac{3t-3}{2} \leq s \leq 2t$.

If $t > 5$, then let $x = t-1$ and $y = s-1$. We infer that $\frac{3t-1}{3s+3} < \frac{t-1}{s-1} < \frac{3t+1}{3s+2}$, and we are done. If $t \leq 5$, we have $t = 2$ or 4.

If $t = 2$, then $b_0 = 7$. Since $\frac{3}{2} \leq s \leq 4$, we have $2 \leq s \leq 4$. If $s \leq 3$, then $b \leq 11$ and $p \leq 29$, yielding a contradiction to $p \geq 31$. If $s = 4$, then $b = 14$ and $p = 35$, yielding a contradiction to that p is prime.

If $t = 4$, then $b_0 = 13$. Since $\frac{9}{2} \leq s \leq 8$, we have $5 \leq s \leq 8$. If $s = 5$, then $b = 17$ and $p = 47$, so the results follows from Lemma 2.3. If $s = 6$, then $b = 20$ and $p = 53$, so the results follows from Lemma 2.3. If $s = 7$, then $b = 23$ and $p = 59$, so the results follows from Lemma 2.3. If $s = 8$, then $b = 26$ and $p = 65$, yielding a contradiction to that p is prime. □

We are now in the position to prove the main theorem.

Proof of Theorem 1.2

We divide the proof according to the following three cases.

Case 1. $\lceil\frac{p}{c}\rceil < \lceil\frac{p}{b}\rceil$. Suppose that $\lceil\frac{p}{c}\rceil = m < \frac{p}{b}$. Let $k = 1$. Then $ma \leq mb < p$, and we are done.

Case 2. $\lceil \frac{p}{c} \rceil = \lceil \frac{p}{b} \rceil$ and $k_1 \le \frac{b}{a}$. Suppose $\lceil \frac{k_1 p}{c} \rceil = m < \frac{k_1 p}{b}$. Let $k = k_1$. Then $ma \le m\frac{b}{k_1} < p$, and we are done.

Case 3. $\lceil \frac{p}{c} \rceil = \lceil \frac{p}{b} \rceil$ and $k_1 > \frac{b}{a}$. Then $k_1 \ge 2$.

If $a - 2 \ge \frac{b}{k_1}$, then $\frac{(k_1-1)p(a-2)}{bc} > \frac{2(k_1-1)}{k_1} \ge 1$, a contradiction to (3.3). Hence we may assume that $a - 2 < \frac{b}{k_1} < a$.

Now assume that $b = k_1\ell + k_0$, where $0 \le k_0 < k_1$. Then $a - 2 \le \ell < \ell + 1 \le a$.

Subcase 3.1. $a = \ell + 1$. Then $c = a + b - 2 = (k_1 + 1)\ell + k_0 - 1$.

Suppose $\frac{p}{c} > 3$. By (3.3) we have $1 > \frac{3(\ell-1)(k_1-1)}{k_1\ell+k_0} \ge \frac{3\ell k_1 - 3k_1 - 3\ell + 3}{k_1\ell + k_1 - 1}$. Hence $2\ell k_1 - 3\ell - 4k_1 + 4 < 0$. This implies that $\ell = 2$ or $\ell = 3,\ k_1 = 2$.

If $\ell = 2$, then $a = 3$. If $\frac{p}{c} > 4$, then by (3.3) we have $1 > \frac{4(\ell-1)(k_1-1)}{k_1\ell+k_0} = \frac{4k_1-4}{2k_1+k_0}$. Hence $2k_1 - k_0 - 4 < 0$ and thus $k_1 = 2$. Hence $b = 4$ or 5. If $b = 4$, then $c = 5$, so the result follows from Lemma 2.4 (3). If $b = 5$, then $c = 6$, so the result follows from Lemma 2.4 (4). Next assume that $3 < \frac{p}{c} < 4$. Since $\lceil \frac{p}{c} \rceil = \lceil \frac{p}{b} \rceil$ we have $3 < \frac{p}{c} < \frac{p}{b} < 4$. By (3.3) we have $1 > \frac{3(\ell-1)(k_1-1)}{k_1\ell+k_0} = \frac{3k_1-3}{2k_1+k_0}$. Hence $k_1 - k_0 - 3 < 0$ and thus $k_0 = k_1 - 1$ or $k_1 - 2$. If $k_0 = k_1 - 1$, then $b = 3k_1 - 1$ and $c = 3k_1$, so the result follows from Lemma 3.1. If $k_0 = k_1 - 2$, then $b = 3k_1 - 2$ and $c = 3k_1 - 1$, so it follows from Lemma 3.2 that this case is impossible.

If $\ell = 3,\ k_1 = 2$, then $a = 4$ and $b = 6$ or 7. If $b = 6$, then $c = 8$, so the result follows from Lemma 2.4 (1). If $b = 7$, then $c = 9$, so the result follows from Lemma 2.4 (2).

Suppose that $3 > \frac{p}{c} > 2$. Since $\lceil \frac{p}{c} \rceil = \lceil \frac{p}{b} \rceil$ we have $2 < \frac{p}{c} < \frac{p}{b} < 3$. By (3.3) we have $1 > \frac{2(\ell-1)(k_1-1)}{k_1\ell+k_0} \ge \frac{2\ell k_1 - 2k_1 - 2\ell + 2}{k_1\ell + k_1 - 1}$. Hence $\ell k_1 - 2\ell - 3k_1 + 3 < 0$. This implies that $k_1 = 2$ or $k_1 = 3,\ \ell \le 5$ or $k_1 = 4,\ \ell \le 4$ or $k_1 \ge 5,\ \ell \le 3$.

If $k_1 = 2$, then $k_0 = 0$ or 1. Since $\frac{2p}{c} \le m_1 < \frac{2p}{b}$, we infer that $m_1 = 5$. If $5a < p$, we are done. Hence we may assume that $p < 5a = 5\ell + 5$. Since $p > 2c = 6\ell + 2k_0 - 2$, we have $5\ell + 5 > 6\ell + 2k_0 - 2$ and thus $\ell < 7$. Since $p \ge 31$, we infer that $\ell \ge 6$. Hence $a \ge 7$. Since $p < 5\ell + 5 < 42$, by Lemma 2.3 we have $\mathrm{ind}(S) = 1$.

If $k_1 = 3$ and $\ell \le 5$, then $b = k_1\ell + k_0 \le 17$. Hence $p < 3b \le 51$, so the result follows from Lemma 2.3.

If $k_1 = 4$ and $\ell \le 4$, then $b = k_1\ell + k_0 \le 19$. Hence $p < 3b \le 57$, so the result follows from Lemma 2.3.

If $k_1 \ge 5$ and $\ell = 3$, then $a = 4$. By (3.3) we have $1 > \frac{2\times 2\times(k_1-1)}{3k_1+k_0}$. Hence $k_1 - k_0 - 4 < 0$ and thus $k_0 = k_1 - 1$ or $k_1 - 2$ or $k_1 - 3$. If $k_0 = k_1 - 1$, then $b = 4k_1 - 1$ and $c = 4k_1 + 1$, so the result follows from Lemma 3.3. If $k_0 = k_1 - 2$, then $b = 4k_1 - 2$ and $c = 4k_1$, yielding a contradiction (by Lemma 3.4). If $k_0 = k_1 - 3$, then $b = 4k_1 - 3$ and $c = 4k_1 - 1$, yielding a contradiction (by Lemma 3.5).

If $k_1 \geq 5$ and $\ell = 2$, then $a = 3$. Therefore, the result follows from Lemma 3.6.

Subcase 3.2. $a = \ell + 2$. Then $c = a + b - 2 = (k_1 + 1)\ell + k_0$.

Suppose $\frac{p}{c} > 3$. By (3.3) we have $1 > \frac{3\ell(k_1-1)}{k_1\ell+k_0} \geq \frac{3\ell k_1 - 3\ell}{k_1\ell+k_1-1}$. Hence $2\ell k_1 - 3\ell - k_1 + 1 < 0$, which is impossible since $k_1 \geq 2$ and $\ell \geq 1$.

Next assume that $3 > \frac{p}{c} > 2$, by (3.3) we have $1 > \frac{2\ell(k_1-1)}{k_1\ell+k_0} \geq \frac{2\ell k_1 - 2\ell}{k_1\ell+k_1-1}$. Hence $\ell k_1 - 2\ell - k_1 + 1 < 0$. This implies that $k_1 = 2$ or $\ell = 1$.

If $k_1 = 2$, then $k_0 = 0$ or 1. Since $\lceil \frac{p}{c} \rceil = \lceil \frac{p}{b} \rceil$ we have $2 < \frac{p}{c} < \frac{p}{b} < 3$. Since $\frac{2p}{c} \leq m_1 < \frac{2p}{b}$, we infer that $m_1 = 5$. If $5a < p$, we are done. Hence we may assume that $p < 5a = 5\ell + 10$. Since $p > 2c = 6\ell + 2k_0$, we have $5\ell + 10 > 6\ell + 2k_0$ and thus $\ell < 10$. Since $p \geq 31$, we infer that $\ell \geq 5$. Hence $a \geq 7$. Since $p < 5\ell + 10 < 60$, by Lemma 2.3 we have $\mathrm{ind}(S) = 1$.

If $\ell = 1$, then $a = 3$, $b = k_1 + k_0$, $c = k_1 + k_0 + 1$. By (3.3) we have $1 > \frac{2\ell(k_1-1)}{k_1\ell+k_0} = \frac{2k_1-2}{k_1+k_0}$. Hence $k_1 - k_0 - 2 < 0$ and thus $k_0 = k_1 - 1$. Then $b = 2k_1 - 1$ and $c = 2k_1$. Suppose $p = 2b + b_0 = 4k_1 - 2 + b_0$. Then b_0 is odd. Since $c < \frac{p-1}{2}$, we infer that $3 < b_0 < 2k_1 - 1$. By (3.3) we have $1 > \frac{(k_1-1)(4k_1-2+b_0)}{(2k_1-1)(2k_1)}$. Hence $b_0k_1 - 4k_1 + 2 - b_0 < 0$. If $b_0 \geq 6$, then $0 > b_0(k_1-1) - 4k_1 + 2 \geq 6k_1 - 6 - 4k_1 + 2 \geq 0$, a contradiction. Hence we must have $b_0 = 5$. Then $0 > b_0(k_1-1) - 4k_1 + 2 = 5k_1 - 5 - 4k_1 + 2 = k_1 - 3$ and thus $k_1 = 2$. Then $p = 11$, yielding a contradiction.

This completes the proof. □

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