On the $\delta - \alpha-$open sets and the $\delta - \alpha-$continuous functions

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Abstract

This paper introduces the new notions of $\delta - \alpha-$open sets and the $\delta - \alpha-$continuous functions in the topological spaces and investigates some of their properties.
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1 Preliminaries

Throughout this paper, $Cl(A)$ and $Int(A)$ denote the closure and interior of A, respectively. A point $x \in X$ is called a δ-cluster point of A if $A \cap int(cl(B)) \neq \emptyset$ for each open set B containing x.The set of all δ-cluster points of A is called the δ-closure of A and is denoted by $Cl_\delta(A)$.If $Cl_\delta(A) = A$,then A is called δ-closed.The set $\{x \in X : x \in G \subset A$ for some regular open set G of $X\}$ is called the δ-interior of A and denoted by $Int_\delta(A)$.

First we recall some definitions used in the sequel.

Definition 1.1. A subset A of a topological space(X, τ) is said to be

(1) pre-open [1] if $A \subset Int(Cl(A))$.
(2) semi-open[2] if $A \subset Cl(Int(A))$.
(3)α-open[3] if $A \subset Int(Cl(Int(A)))$.
(4)β-open[4] if $A \subset Cl(Int(Cl(A)))$.
(5)δ-preopen [5] if $A \subset Int(Cl_\delta(A))$.
(6)δ-semi-open[6] if $A \subset Cl(Int_\delta(A))$.

(7)$\delta-\beta$-open[7]if $A \subset Cl(Int(Cl_\delta(A)))$

(8)δ-open[7] if $A = Int_\delta(A)$

Lemma 1.1.[8] For a subset A of a topological space(X,τ) ,the following properties hold:

(1)If A is open in (X,τ),then $Cl_\delta(A) = Cl(A)$.

(2)If A is closed in (X,τ),then $Int_\delta(A) = Int(A)$.

2 $\delta-\alpha-$open sets

Definition 2.1. A subset A of a topological space (X,τ) is said to be $\delta-\alpha-$open set , if $A \subset Int(Cl(Int_\delta(A)))$.

The complement of a $\delta-\alpha-$open set is said to be $\delta-\alpha-$closed. The family of all $\delta-\alpha-$open(resp.$\delta-\alpha-$closed) sets in a topological space (X,τ) is denoted by $\delta\alpha O(X,\tau)$ (resp.$\delta\alpha C(X,\tau)$)

Definition 2.2. A point $x \in X$ is called the $\delta-\alpha-$cluster point of A, if $A \cap U \neq \emptyset$ for every $\delta-\alpha-$open set U of X containing x.

The set of all $\delta-\alpha-$cluster points of A is called $\delta-\alpha-$closure of A,denoted by $\alpha Cl_\delta(A)$.

From the definition above we obtain that $x \in \alpha Cl_\delta(A)$ if and only if $A \cap V \neq \emptyset$ for every $V \in \delta\alpha O(X,\tau)$ containing x.And A is $\delta-\alpha-$closed if and only if $A = \alpha Cl_\delta(A)$.

Proposition 2.1 Let A be a subset of a topological space (X,τ), the following properties hold:

(1) If A is $\delta-\alpha-$open in (X,τ),then it is $\alpha-$open in (X,τ)

(2) If A is closed in (X,τ),then $\delta-\alpha-$open and $\alpha-$open equivalent.

Proof: (1)This is obvious since $Int_\delta(A) \subset Int(A)$.

(2)It is obvious from lemma 1.1.

Remark 2.1: If we have an $\delta-\alpha-$ set in a subspace of a space it is not an $\delta-\alpha-$ set in the space. And also when $\delta-\alpha-$ set in a space it is not an $\delta-\alpha-$ set in a subspace.

For example Let $X = \{a,b,c,d\}$,$\tau = \{\emptyset, X, \{a\}, \{b\}, \{d\}, \{a,b\}, \{a,d\}, \{b,d\}\{a,b,c\}\{a,b,d\}\}$. And $\delta\alpha O(X,\tau) = \{\emptyset, X, \{a\}, \{b\}, \{d\}, \{a,b\}, \{a,d\}, \{b,d\}\{a,b,c\}\{a,b,d\}\}$. $A = \{a,c,d\}$, $\tau_A = \{\emptyset, A, \{a\}, \{d\}, \{a,d\}, \{a,c\}$. So (A,τ_A) is a subspace of X.

let $B_1 = \{a\}$. B_1 is an $\delta - \alpha-$ set in the (X, τ). $Int_\delta(B_1) = \emptyset$ in the subspace (A, τ_A). And so B_1 is not an $\delta - \alpha-$ set in the subspace (A, τ_A).

Let $B_2 = \{a, c\}$, $Int(Cl(Int_\delta(B_2))) = \{a, c\}$.$B_2$ is an $\delta - \alpha-$ set in the subspace (A, τ_A).B_2 is not an $\delta - \alpha-$ set in the space (X, τ).

Remark 2.2: The converse of Proposition 2.1 (1) is not true. For example Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{a\}\{c\}\{a, c\}, \{a, b\}\{a, b, c\}$ $\{a, c, d\}\}$. $A = \{a\}$, $Int(A) = \{a\}$, $Cl(Int(A)) = \{a, b, d\}$, $Int(Cl(Int(A))) = \{a, b\}$. So A is a $\alpha-$open. But $Int_\delta(A) = \emptyset$, A is not a $\delta - \alpha-$open.

From the Definition 1.1 and the proportion 2.1. we have

$$
\begin{array}{lll}
 & \nearrow \delta - \text{semiopen} \Rightarrow \text{semi} - \text{open} \Rightarrow & \beta - \text{open} \\
 & & \Downarrow \\
\delta - \alpha - \text{open} & \rightarrow \alpha - \text{open} \Rightarrow \alpha - \text{open} \Rightarrow & \delta - \beta - \text{open} \\
 & & \Uparrow \\
 & \searrow \delta - \text{open} \Rightarrow \text{open} \Rightarrow \text{preopen} \Rightarrow & \delta - \text{preopen}
\end{array}
$$

Proposition 2.2 $A \subset X$ is a $\delta - \alpha-$closed if and only if $Cl(Int(Cl_\delta)) \subset A$

Proof: A subset A is a $\delta - \alpha-$closed if and only if $X - A$ is $\delta - \alpha-$open. Then $X - A \subset Int(Cl(Int_\delta(X - A))) = Int(Cl(X - Cl_\delta(A))) = Int(X - Int(Cl_\delta(A))) = X - Cl(Int(Cl_\delta(A)))$.

Proposition 2.3 Let A be a subset of a topological space (X, τ), the following properties hold:

(1)$A \subset \alpha Cl_\delta(A)$.

(2)If $A \subset B$,then $\alpha Cl_\delta(A) \subset \alpha Cl_\delta(B)$

(3)$\alpha Cl_\delta(A) = \cap\{F \in \delta\alpha C(X, \tau) | A \subset F\}$

(4)If A_α is a $\delta - \alpha-$closed set of X for each $\alpha \in \Delta$,then $\cap\{A_\alpha | \alpha \in \Delta\}$ is $\delta - \alpha-$closed.

(5)$\alpha Cl_\delta(A)$ is $\delta - \alpha-$closed,that is $\alpha Cl_\delta(\alpha Cl_\delta(A)) = \alpha Cl_\delta(A)$

(6)(a)$\alpha Cl_\delta(\cap\{A_{\alpha:\alpha\in\Delta}\}) \subset \cap\{\alpha Cl_\delta(A_\alpha) : \alpha \in \Delta\}$

(b)$\alpha Cl_\delta(\cup\{A_{\alpha:}\alpha \in \Delta\}) = \cup\{\alpha Cl_\delta(A_\alpha) : \alpha \in \Delta\}$

Proof:(1) Suppose $x \notin \alpha Cl_\delta(A)$.There exists $V \in \delta\alpha O(x, \tau)$ containing x such that $A \cap V = \emptyset$,hence $x \notin A$.

(2)Similar with (1).

(3)Suppose $x \in \alpha Cl_\delta(A)$.For any $V \in \delta\alpha O(x,\tau)$ containing x and any $\delta-\alpha-$closed set F containing A.We have $\emptyset \neq A \cap V \subset F \cap V$ and hence $x \in \alpha Cl_\delta(F) = F$.This shows that $x \in \cap\{F \subset X | A \subset F and F is \delta-\alpha-closed\}$.So $\alpha Cl_\delta(A) \subset \cap\{F \in \delta\alpha C(X,\tau) | A \subset F\}$.Conversly, suppose that $x \notin \alpha Cl_\delta(A)$. There exists $V \in \delta\alpha O(x,\tau)$ containing x such that $A \cap V = \emptyset$,$X - V$ is a $\delta-\alpha-$closed set which contains A and does not contain x.Therefore we obtain $x \notin \cap\{F \in \delta\alpha C(X,\tau) | A \subset F\}$.So this completes the proof.

(4)It is obviously from (1)and(2).

(5)It is obviously from (3)and(4).

(6)(a)It is obviously from (2)

(b)It is obviously $\alpha Cl_\delta(\cup\{A_{\alpha:\alpha\in\Delta}\}) \subset \cup\{\alpha Cl_\delta(A_\alpha) : \alpha \in \Delta\}$ from (2).Conversly,Suppose $x \in \alpha Cl_\delta(\cup\{A_{\alpha:\alpha\in\Delta}\})$, There exists $U \in \delta\alpha O(x,\tau)$ containing x such that $(\cup A_{\alpha:\alpha\in\Delta}) \cap U = \cup(A_\alpha \cap U : \alpha \in \Delta) \neq \emptyset$.So There is at least a $\alpha_0 \in \Delta$ such that $A_{\alpha_0} \cap U \neq \emptyset$,$x \in \alpha Cl_\delta(A_0)$.So $x \in \cup\{\alpha Cl_\delta(A_\alpha) : \alpha \in \Delta\}$ and $\alpha Cl_\delta(\cup\{A_{\alpha}:\alpha \in \Delta\}) \subset \cup\{\alpha Cl_\delta(A_\alpha) : \alpha \in \Delta\}$

3 $\delta-\alpha-$continuous functions

Definition 3.1. A function $f : (X,\tau) \to (Y,\sigma)$ is said to be $\delta-\alpha-$continuous function, if for each $x \in X$ and each $\delta-\alpha-$open set V containing $f(x)$, there is a $\delta-\alpha-$open set U in X containing x such that $f(U) \subset V$.

Proposition 3.1. A function $f : (X,\tau) \to (Y,\sigma)$ is said to be $\delta-\alpha-$continuous if and only if the inverse image of each $\delta-\alpha-$open set is $\delta-\alpha-$open set.

Definition 3.2. Let (X,τ) be a topological space,$x \in X$ and $\{x_s, s \in S\}$ be a net of X .We say that the net $\{x_s, s \in S\}$ $\delta-\alpha-$converges to x and write $x_s \xrightarrow{(\delta-\alpha)} x$, if for each $\delta-\alpha-$open set U containing x there exists an element $s_0 \in S$ such that $s \geq s_0$ implies $x_s \in U$.

Definition 3.3. A topological space (X,τ) is called $\delta-\alpha-$connected. if X can not be expressed by the disjoint union of two nonempty

$\delta-\alpha-$open sets.

Definition 3.4. A net $\{f_\mu.\mu \in M\}$ in $\delta\alpha(X,Y)$,$\delta-\alpha-$continuously converges to $f \in \delta\alpha(X,Y)$ if for every net $\{x_\lambda, \lambda \in \Lambda\}$ in X which $\delta-\alpha-$ converges to $x \in X$,we have the net $\{f_\mu(x_\lambda), (\lambda,\mu) \in \Lambda \times M\}$ converges to $f(x)$ in Y(here $\delta\alpha(X,Y)$ denotes all $\delta-\alpha-$continuous function X into Y).

Definition 3.5. The $\delta-\alpha-$frontier of a subset A of a space X is given by $\alpha Fr_\delta(A) = \alpha Cl_\delta(A) \cap \alpha Cl_\delta(X-A)$.

Theorem 3.1. For a function $f:(X,\tau) \to (Y,\sigma)$, the followings are equivalent:

(1) f is $\delta-\alpha-$continuous.

(2)The inverse image of each $\delta-\alpha-$closed set is $\delta-\alpha-$closed.

(3) For any set $A \subset X$,$f(\alpha Cl_\delta(A)) \subset \alpha Cl_\delta(f(A))$

(4) For any set $B \subset Y$,$\alpha Cl_\delta(f^{-1}(B)) \subset f^{-1}(\alpha Cl_\delta(B))$

Proof: It is obvious from Proposition 2.3 .

Theorem 3.2. If $f:(X,\tau) \to (Y,\sigma)$ is a $\delta-\alpha-$continuous surjection and (X,τ) is $\delta-\alpha-$connected,then (Y,σ) is $\delta-\alpha-$connected.

Proof:Suppose that Y is not a $\delta-\alpha-$connected.There exist nonempty $\delta-\alpha-$open sets A and B such that $Y = A \cup B$.Since f is $\delta-\alpha-$continuous ,$f^{-1}(A)$ and $f^{-1}(B)$ are $\delta-\alpha-$open in X.On the other hand, $f^{-1}(A)$ and $f^{-1}(B)$ are nonempty disjoint sets and $X = f^{-1}(A) \cup f^{-1}(B)$.This shows that X is not a $\delta-\alpha-$connected which is a contradiction.

Theorem 3.3. $f:(X,\tau) \to (Y,\sigma)$ is $\delta-\alpha-$continuous at $x \in X$ if and only if for every net $\{x_\lambda : \lambda \in \Lambda\}$ in X which $\delta-\alpha-$ converges to a point x,we have that the net $\{f(x_\lambda) : \lambda \in \Lambda\}$in Y $\delta-\alpha-$converges to a point $f(x)$.

Proof:Let us suppose that f is $\delta-\alpha-$continuous at $x \in X$ and Let $\{x_\lambda : \lambda \in \Lambda\}$ be a net in X such that $\delta-\alpha-$ converges to a point x.Then for every $\delta-\alpha-$open set V containing $f(x)$ in Y,there exists $\delta-\alpha-$open set U containing x in X such that $f(U) \subset V$.Since $\{x_\lambda : \lambda \in \Lambda\}$,there exists an element $\lambda_0 \in \Lambda$ such that $x_\lambda \in U$,for every $\lambda \in \Lambda$,$\lambda \geq \lambda_0$.Thus $f(x_\lambda) \in V$,for every $\lambda \geq \lambda_0$,$\lambda \in \Lambda$ and therefore the net $\{f(x_\lambda) : \lambda \in \Lambda\}$in Y $\delta-\alpha-$ converges to a point $f(x)$.

Conversely,if the function f is not $\delta-\alpha-$continuous at $x \in X$,then for some $\delta-\alpha-$open set V containing $f(x)$.We have $f(U) \not\subset V$ for every $\delta-\alpha-$open set U containing x in X. Thus for every $\delta-\alpha-$open set U containing x we can find $x_\mu \in U$ such that $f(x_\mu) \notin V$.Let $N_{(x)}$ be the set of all $\delta-\alpha-$open set containing x in X.The set $N_{(x)}$ with the relation of inverse inclusion,that is $U_1 \leq U_2$ if and only if $U_2 \subset U_1$,form a directed set.Therefore the net $\{x_U, U \in N_{(x)}\}$ $\delta-\alpha-$ converges to a point x in X,but the net $\{f(x_U), U \in N_{(x)}\}$ does not $\delta-\alpha-$ converges to a point $f(x)$ in Y.Hence the function f is $\delta-\alpha-$continuous at $x \in X$.

Theorem 3.4. A net $\{f_\mu.\mu \in M\}$ in $\delta\alpha(X,Y)$,$\delta-\alpha-$continuously converges to $f \in \delta\alpha(X,Y)$ if and only if for every $x \in X$ and every $\delta-\alpha-$ open V containing $f(x)$ in Y,there exist an element $\mu_0 \in M$ and $\delta-\alpha-$ open U containing x in X such that $f_\mu(U) \subset V$ for every $\mu \geq \mu_0, \mu \in M$.

Proof:Let $x \in X$ and V be a $\delta-\alpha-$ open set containing $f(x)$ in Y such that for every $\mu \in M$ and every $\delta-\alpha-$ open set containing U containing $x \in X$,there exists $\mu' \geq \mu, \mu' \in M$ such that $f_{\mu'} \not\subset V$.Then for every $\delta-\alpha-$ open set containing U containing $x \in X$ we can choose a point $x_\mu \in U$ such that $f_{\mu'}(x_\mu) \notin V$.Therefore the net $\{x_U, U \in \delta\alpha O(X,x)\}$ $\delta-\alpha-$ converges to x, but the $f_\mu(x_U)$, $(U,\mu) \in \delta\alpha O(X,x) \times M$ does not converge to $f((x)$ in Y.

Conversely.Let $\{x_\lambda, \lambda \in \Lambda\}$ be net in $\delta\alpha(X,Y)$ which is $\delta-\alpha-$converge to x in X and V be an arbitrary $\delta-\alpha-$ open set containing containing $f(x)$ in Y.By assumption,there exists a $\delta-\alpha-$ open set containing U containing x in X and an element $\mu_0 \in M$ such that $f_\mu(U) \subset V$,for every $\mu \geq \mu_0, \mu \in M$.Since the net $\{x_\lambda, \lambda \in \Lambda\}$ $\delta-\alpha-$converge to x in X.There exists $\lambda_0 \in \Lambda$ such that $x_\lambda \in U$,for every $\lambda \in \Lambda, \lambda \geq \lambda_0$.Let $(\lambda_0, \mu_0) \in \Lambda \times M$.Then for every $(\lambda,\mu) \in \Lambda \times M$,$(\lambda,\mu) \geq (\lambda_0,\mu_0)$,we have $f_\mu(x_\lambda) \in f_\mu(U) \subset V$.Thus the net $\{f_\mu(x_\lambda), (\lambda,\mu) \in \Lambda \times M\}$ converge to $f(x)$ in Y.

Theorem 3.5. A function $f:(X,\tau) \to (Y,\sigma)$ is not $\delta-\alpha-$continuous at x if and only if $x \in \alpha Fr_\delta(f^{-1}(S))$ for some $\delta-\alpha-$open set S in Y containing $f(x)$.

Proof:Suppose that f is not $\delta-\alpha-$continuous at x.There exists a

$\delta-\alpha-$ open set S containing $f(x)$ for which $f(A) \not\subset S$ for every $A \in \delta\alpha O(X,x)$.We have $f(A)\cap(Y-S) \neq \emptyset$ and $A\cap(X-f^{-1}(S)) \neq \emptyset$ for every $A \in \delta\alpha O(X,x)$.Hence $x \in \alpha Cl_\delta(X-f^{-1}(S))$.Since $x \in f^{-1}(S)$ we obtain $x \in \alpha Cl_\delta(f^{-1}(S))$ and hence $x \in \alpha Fr_\delta(f^{-1}(S))$.

Sufficiency:Suppose that there exists a $\delta-\alpha-$ open set S in Y containing $f(x)$ such that $x \in \alpha Fr_\delta(f^{-1}(S))$ for $x \in X$.Let f be $\delta-\alpha-$continuous at x.Thre exists a $\delta-\alpha-$ open set A such that $x \in A$ and $A \subset f^{-1}(S)$.Thus $x \notin \alpha Cl_\delta(X - f^{-1}(S))$.This is a contradiction.

4 $\delta-\alpha R_0$ and $\delta-\alpha R_1$ spaces

Definition 4.1. A topological space (X,τ) is said to be $\delta-\alpha R_0$ if every $\delta-\alpha-$ open set contains the $\delta-\alpha-$closure of each of its singletons.

Definition 4.2. A topological space (X,τ) is said to be $\delta-\alpha R_1$ if for x,y in X with $\alpha Cl_\delta(\{x\}) \neq \alpha Cl_\delta(\{y\})$,there exist disjoint $\delta-\alpha-$ open sets U and V such that $\alpha Cl_\delta(\{x\}$ is a subset of U and $\alpha Cl_\delta(\{y\}$ is a subset of V.

Definition 4.3. Let A be a subset of a space X.The $\delta-\alpha$kernel of A denoted by $\alpha Ker_\delta(A) = \cap\{O \in \delta\alpha O(X,\tau) : A \subset O\}$.

Proposition 4.1 . Let (X,τ) be a topological space and $x \in X$.Then $y \in \alpha Ker_\delta(\{x\})$ of and only if $x \in \alpha Cl_\delta(\{y\}$.

Proof: Suppose $y \notin \alpha Ker_\delta(\{x\})$.Then there exists a $\delta-\alpha-$ open set V containing x such that $y \notin V$.Therefore we have $x \notin \alpha Cl_\delta(\{y\}$. The converse is similarly shown.

Proposition 4.2 .The following statement are equivalent for any points x and y in a topological space (X,τ):

(1)$\alpha Ker_\delta(\{x\}) \neq \alpha Ker_\delta(\{x\})$

(2)$\alpha Cl_\delta(\{x\}) \neq \alpha Cl_\delta(\{y\})$

proof: (1) $\rightarrow$ (2) Suppose $\alpha Ker_\delta(\{x\}) \neq \alpha Ker_\delta(\{x\})$,then there exists a point z in X such that $z \in \alpha Ker_\delta(\{x\})$ and $z \notin \alpha Ker_\delta(\{y\})$.From $z \in \alpha Ker_\delta(\{x\})$ it follows that $\{x\}\cap\alpha Cl_\delta(\{z\}) \neq \emptyset$ which implies $x \in \alpha Cl_\delta(\{z\})$.By $z \notin \alpha Ker_\delta(\{y\})$,we have $\{y\} \cap$

$\alpha Cl_\delta(\{z\}) = \emptyset$.Therefore it follows that $\alpha Cl_\delta(\{x\}) \neq \alpha Cl_\delta(\{y\})$.Now $\alpha Ker_\delta(\{x\}) \neq \alpha Ker_\delta(\{x\})$ implies $\alpha Cl_\delta(\{x\}) \neq \alpha Cl_\delta(\{y\})$.

(2) $\rightarrow$ (1) Suppose $\alpha Cl_\delta(\{x\}) \neq \alpha Cl_\delta(\{y\})$.Then there exists a point z in X such that $z \in \alpha Cl_\delta(\{x\})$ and $z \notin \alpha Cl_\delta(\{y\})$ .It follows that exists a $\delta - \alpha-$ open containing z therefore x but not y,namely,$y \notin \alpha Ker_\delta(\{x\})$ and thus $\alpha Ker_\delta(\{x\}) \neq \alpha Ker_\delta(\{x\})$.

Theorem 4.1. If (X,τ) is $\delta - \alpha R_1$,then (X,τ) is $\delta - \alpha R_0$.

proof: Let U be $\delta - \alpha-$ open and $x \in U$.If $y \notin U$,then since $x \notin \alpha Cl_\delta(\{y\})$, $\alpha Cl_\delta(\{x\}) \neq \alpha Cl_\delta(\{y\})$. Hence there exists a $\delta - \alpha-$ open V_y such that $\alpha Cl_\delta(\{y\}) \subset V_y$ and $x \notin V_y$,which implies $y \notin \alpha Cl_\delta(\{x\})$.Thus $\alpha Cl_\delta(\{x\}) \subset U$. Therefore (X,τ) is $\delta - \alpha R_0$.

Question. Does there exist a space which is $\delta - \alpha R_0$ is not $\delta - \alpha R_1$.

Remark 4.1 The $\delta - \alpha R_1$ spaces and the $\delta - \alpha R_0$ spaces are not kept under the $\delta - \alpha-$ continuous function.

For example Let $X = \{a, c, d\}$, $\tau = \{\emptyset, X, \{a\}, \{d\}, \{a, d\}, \{a, c\}\}$. And $\delta\alpha O(X,\tau) = \{\emptyset, X, \{d\}, \{a, c\}\}$. $\delta\alpha C(X,\tau) = \{\emptyset, X, \{d\}, \{a, c\}\}$. $\alpha Cl_\delta\{a\} = \alpha Cl_\delta\{c\} = \{a, c\}$, $\alpha Cl_\delta\{d\} = \{d\}$,$\alpha Cl_\delta\{a\} \neq \alpha Cl_\delta\{d\}$. Let $U = \{a, c\}$,$V = \{d\}$ and $U \cap V = \emptyset$.So U and V is $\delta - \alpha-$open and $\alpha Cl_\delta\{a\} \subset U$,$\alpha Cl_\delta\{d\} \subset V$.So X is a $\delta - \alpha R_1$ space. And it is also a $\delta - \alpha R_0$ space from Theorem 4.1.

Let $Y = \{a_1, b_1, c_1\}$,$\sigma = \{\emptyset, Y, \{a_1\}, \{b_1\}, \{a_1, b_1\}\}$. And $\delta\alpha O(Y,\tau_1) = \delta\alpha C(Y,\tau_1) = \{\emptyset, Y, \{a_1\}, \{b_1\}, \{a_1, b_1\}\}$. $\alpha Cl_\delta\{a_1\} = \{a_1, c_1\}$, $\alpha Cl_\delta\{b_1\} = \{b_1, c_1\}$,$\alpha Cl_\delta\{c_1\} = \{c_1\}$, $\alpha Cl_\delta\{a_1\} \neq \alpha Cl_\delta\{b_1\}$. Obviously Y is not a $\delta - \alpha R_1$ space, and Y is not a $\delta - \alpha R_0$ space

Let $f : (X,\tau) \rightarrow (Y,\sigma)$ be a function such that $f(a) = f(c) = b_1$ and $f(d) = a_1$ Clearly the map f is $\delta - \alpha-$continuous.

From Proposition 4.2 it is obvious that

Theorem 4.2. A topological space (X,τ) is $\delta - \alpha R_1$ if and only if for x,$y \in X$,$\alpha Ker_\delta(\{x\}) \neq \alpha Ker_\delta(\{x\})$,there exist disjoint $\delta - \alpha-$ open sets U and V such that $\alpha Cl_\delta(\{x\}) \subset U$ and $\alpha Cl_\delta(\{y\}) \subset V$.

Theorem 4.3. A topological space (X,τ) is $\delta - \alpha R_0$ if and only if for x,$y \in X$,$\alpha Cl_\delta(\{x\}) \neq \alpha Cl_\delta(\{y\})$ implies $\alpha Cl_\delta(\{x\}) \cap \alpha Cl_\delta(\{y\}) = \emptyset$.

proof: Necessity.Assume (X,τ) is $\delta - \alpha R_0$ and x,$y \in X$ such that

$\alpha Cl_\delta(\{x\}) \neq \alpha Cl_\delta(\{y\})$.Then there exist $z \in \alpha Cl_\delta(\{x\})$ such that $z \notin \alpha Cl_\delta(\{y\})$ (or $z \in \alpha Cl_\delta(\{y\})$ such that $z \notin \alpha Cl_\delta(\{x\})$).There exists $V \in \delta\alpha O(X,\tau)$ such that $y \notin V$ and $z \in V$;hence $x \in V$.Therefore we have $x \notin \alpha Cl_\delta(\{y\})$.Thus $x \in X - \alpha Cl_\delta(\{y\}) \in \delta\alpha O(X,\tau)$ which implies $\alpha Cl_\delta(\{x\}) \subset X - \alpha Cl_\delta(\{y\}) \in \delta\alpha O(X,\tau)$ and $\alpha Cl_\delta(\{x\}) \cap \alpha Cl_\delta(\{y\}) = \emptyset$.The proof for otherwise is similar.

Sufficiency. Let $V \in \delta\alpha O(X,\tau)$ and $x \in V$.We will show that $\alpha Cl_\delta(\{x\}) \subset V$.Really let $y \notin V$,i.e.,$y \in X - V$.Then $x \neq y$ and $x \notin \alpha Cl_\delta(\{y\})$.This show that $\alpha Cl_\delta(\{x\}) \neq \alpha Cl_\delta(\{y\})$ .By assumption $\alpha Cl_\delta(\{x\}) \cap \alpha Cl_\delta(\{y\}) = \emptyset$.Hence $y \notin \alpha Cl_\delta(\{x\})$.Therefore $\alpha Cl_\delta(\{x\}) \subset V$.

Theorem 4.4.A topological space (X,τ) is $\delta - \alpha R_0$ if and only if for x,$y \in X$,$\alpha Ker_\delta(\{x\}) \neq \alpha Ker_\delta(\{y\})$ implies $\alpha Ker_\delta(\{x\}) \cap \alpha Ker_\delta(\{y\}) = \emptyset$.

proof: Assume (X,τ) is $\delta - \alpha R_0$ space .Thus by Proposition 4.2 ,for any points x and y in X if $\alpha Ker_\delta(\{x\}) \neq \alpha Ker_\delta(\{y\})$ then $\alpha Cl_\delta(\{x\}) \neq \alpha Cl_\delta(\{y\})$. Now we prove that $\alpha Ker_\delta(\{x\}) \cap \alpha Ker_\delta(\{y\}) = \emptyset$. Assume that $z \in \alpha Ker_\delta(\{x\}) \cap \alpha Ker_\delta(\{y\})$.By $z \in \alpha Ker_\delta(\{x\})$ and Proposition 4.1,it follows that $x \in \alpha Cl_\delta(\{z\})$.Since $x \in \alpha Cl_\delta(\{z\})$, by Theorem 4.3 $\alpha Cl_\delta(\{x\}) = \alpha Cl_\delta(\{z\})$. Similarly, we have $\alpha Cl_\delta(\{y\}) = \alpha Cl_\delta(\{z\}) = \alpha Cl_\delta(\{x\})$. This is a contradiction. Therefore we have $\alpha Ker_\delta(\{x\}) \cap \alpha Ker_\delta(\{x\}) = \emptyset$.

Conversely,let (X,τ) be a topological space such that for any points x and y in X, $\alpha Ker_\delta(\{x\}) \neq \alpha Ker_\delta(\{y\})$ implies $\alpha Ker_\delta(\{x\}) \cap \alpha Ker_\delta(\{x\}) = \emptyset$.If $\alpha Cl_\delta(\{x\}) \neq \alpha Cl_\delta(\{y\})$, then by Proposition 4.2 $\alpha Ker_\delta(\{x\}) \neq \alpha Ker_\delta(\{y\})$. Therefore $\alpha Ker_\delta(\{x\}) \cap \alpha Ker_\delta(\{y\}) = \emptyset$ which implies $\alpha Cl_\delta(\{x\}) \neq \alpha Cl_\delta(\{y\})$. Because $z \in \alpha Cl_\delta(\{x\})$ implies $x \in \alpha Ker_\delta(\{z\})$ and therefore $\alpha Ker_\delta(\{x\}) \cap \alpha Ker_\delta(\{z\}) \neq \emptyset$. By hypothesis, we have $\alpha Ker_\delta(\{x\}) = \alpha Ker_\delta(\{z\})$. Then $z \in \alpha Cl_\delta(\{x\}) \neq \alpha Cl_\delta(\{y\})$ implies that $\alpha Ker_\delta(\{x\}) = \alpha Ker_\delta(\{z\} = \alpha Ker_\delta(\{y\}))$.This is a contradiction. Therefore we have $\alpha Cl_\delta(\{x\}) \cap \alpha Cl_\delta(\{y\}) = \emptyset$ and by Theorem 4.3 (X,τ) is $\delta - \alpha R_0$ space.

Theorem 4.5. For a topological space (X,τ),the following properties are equivalent:

(1)(X,τ) is $\delta - \alpha R_0$ space .

(2)For any nonempty set A and $G \in \delta\alpha O(X,\tau)$ such that $A \cap G \neq \emptyset$,there exists $F \in \delta\alpha C(X,\tau)$ such that $A \cap F \neq \emptyset$ and $F \subset G$.

(3) Any $G \in \delta\alpha O(X,\tau)$,$G = \cup\{F \in \delta\alpha C(X,\tau) | F \subset G\}$.

(4) Any $F \in \delta\alpha C(X,\tau)$,$F = \cap\{F \in \delta\alpha C(X,\tau) | F \subset G\}$.

(5)For any $x \in X$, $\alpha Cl_\delta(\{x\}) \subset \alpha Ker_\delta(\{x\})$.

proof: (1) $\rightarrow$ (2) Let A be a nonempty set of X and $G \in \delta\alpha O(X,\tau)$ such that $A \cap G \neq \emptyset$. There exists $x \in A \cap G$.Since $x \in G \in \delta\alpha O(X,\tau)$,$\alpha Cl_\delta(\{x\}) \subset G$. Set $F = \alpha Cl_\delta(\{x\})$,then $F \in \delta\alpha C(X,\tau)$,$F \subset G$ and $A \cap F \neq \emptyset$.

(2) $\rightarrow$ (3) Let $G \in \delta\alpha O(X,\tau)$,then $G \supset \cup\{F \in \delta\alpha C(X,\tau) | F \subset G\}$.Let x be any point of G.Therefore we have $x \in F \subset \cup\{F \in \delta\alpha C(X,\tau) | F \subset G\}$ and hence $G = \cup\{F \in \delta\alpha C(X,\tau) | F \subset G\}$.

(3) $\rightarrow$ (4) This is obvious.

(4) $\rightarrow$ (5) Let x be any point of X and $y \notin \alpha Ker_\delta(\{x\})$.There exists $V \in \delta\alpha O(X,\tau)$ such that $x \in V$ and $y \notin V$,hence $\alpha Cl_\delta(\{y\}) \cap V = \emptyset$.By (4)$(\cap\{F \in \delta\alpha C(X,\tau) | alphaCl_\delta(\{y\}) \subset G\}) \cap V = \emptyset$ and there exists $G \in \delta\alpha O(X,\tau)$ such that $x \notin G$ and $alphaCl_\delta(\{y\}) \subset G$.Therefore $\alpha Cl_\delta(\{x\}) \cap G = \emptyset$ and $y \notin \alpha Cl_\delta(\{x\})$. Consequently we obtain $\alpha Cl_\delta(\{x\}) \subset \alpha Ker_\delta(\{x\})$.

(5) $\rightarrow$ (1) Let $G \in \delta\alpha O(X,\tau)$ and $x \in G$. Let $y \in \alpha Ker_\delta(\{x\})$,then $x \in \alpha Cl_\delta(\{y\})$ and $y \in G$. This implies that $\alpha Ker_\delta(\{x\}) \subset G$. Therefore, we obtain $x \in \alpha Cl_\delta(\{x\}) \subset \alpha Ker_\delta(\{x\}) \subset G$.This shows that (X,τ) is $\delta - \alpha R_0$ space.

Corollary 4.3 . For a topological space (X,τ),the following properties are equivalent:

(1)(X,τ) is a $\delta - \alpha R_0$ space.

(2)$\alpha Cl_\delta(\{x\}) = \alpha Ker_\delta(\{x\})$ for all $x \in X$.

proof: (1) $\rightarrow$ (2) Suppose that (X,τ) is $\delta - \alpha R_0$ space. By Theorem 4.5 $\alpha Cl_\delta(\{x\}) \subset \alpha Ker_\delta(\{x\})$ for each $x \in X$. Let $y \in \alpha Ker_\delta(\{x\})$, By Corollary 6.1,$x \in \alpha Cl_\delta(\{y\})$ and by Theorem 4.3 $\alpha Cl_\delta(\{x\}) = \alpha Cl_\delta(\{y\})$.Therefore $y \in \alpha Cl_\delta(\{x\})$ hence $\alpha Ker_\delta(\{x\}) \subset \alpha Cl_\delta(\{x\})$ for all $x \in X$.This shows that $\alpha Cl_\delta(\{x\}) = \alpha Ker_\delta(\{x\})$.

(2) $\rightarrow$ (1) This is obvious by Theorem 4.5.

Theorem 4.6. For a topological space (X,τ),the following properties are equivalent:

(1)(X,τ) is a $\delta-\alpha R_0$ space.

(2)$x \in \alpha Cl_\delta(\{y\})$ if only if $y \in \alpha Cl_\delta(\{x\})$.

proof: (1) $\to$ (2) Suppose that (X,τ) is $\delta-\alpha R_0$ space. Let $x \in \alpha Cl_\delta(\{y\})$ and D be any $\delta-\alpha-$ open set such that $y \in D$. Therefore every $\delta-\alpha-$ open set which contains y contains x.Hence $y \in \alpha Cl_\delta(\{x\})$.

(2) $\to$ (1) Let U be a $\delta-\alpha-$ open set and $x \in U$.If $y \notin U$,then $x \notin \alpha Cl_\delta(\{y\})$ and hence $y \notin \alpha Cl_\delta(\{x\})$.This implies that $\alpha Cl_\delta(\{x\}) \subset U$.Hence (X,τ) is $\delta-\alpha R_0$ space.

Theorem 4.7. For a topological space (X,τ),the following properties are equivalent:

(1)(X,τ) is a $\delta-\alpha R_0$ space.

(2) If F is $\delta-\alpha-$closed,then $F = \alpha Ker_\delta(\{F\}))$.

(3) If F is $\delta-\alpha-$closed and $x \in F$,then $\alpha Ker_\delta(\{x\})) \subset F$.

(4) If $x \in X$,then $\alpha Ker_\delta(\{x\})) \subset \alpha Cl_\delta(\{x\})$.

proof: (1) $\to$ (2) This is obviously by Theorem 4.5.

(2) $\to$ (3) In general $A \subset B$ implies $\alpha Ker_\delta(\{A\})) \subset \alpha Ker_\delta(\{B\}))$. Therefore it follows from (2) that $\alpha Ker_\delta(\{x\})) \subset \alpha Cl_\delta(\{x\})$.

(3) $\to$ (4) Since $x \in \alpha Cl_\delta(\{x\})$ and $\alpha Cl_\delta(\{x\})$ is $\delta-\alpha-$closed,by(3) $\alpha Ker_\delta(\{x\})) \subset \alpha Cl_\delta(\{x\})$.

(4) $\to$ (1) We show the implication. by using By Theorem 4.6.Let $x \in \alpha Cl_\delta(\{y\})$.Then by Proposition 2.1 $y \in \alpha Ker_\delta(\{x\}))$.Since $x \in \alpha Cl_\delta(\{x\})$ and $\alpha Cl_\delta(\{x\})$ is $\delta-\alpha-$closed, by (4) we obtain $y \in \alpha Ker_\delta(\{x\})) \subset \alpha Cl_\delta(\{x\})$.Therefore $x \in \alpha Cl_\delta(\{y\})$ implies $y \in \alpha Cl_\delta(\{x\})$

Proposition 4.4 . Let (X,τ) be a topological space and let x,y be any two points in X such that every net in X $\delta-\alpha-$converges to x.Then $x \in \alpha Cl_\delta(\{y\})$.

proof: Suppose that $x_n = y$ for each $n \in N$.Then $\{x_n\}_{n\in N}$ is a net in $\alpha Cl_\delta(\{y\})$.By the fact that $\{x_n\}_{n\in N}$ $\delta-\alpha-$converges to y,then $\{x_n\}_{n\in N}$ $\delta-\alpha-$converges to x and this means that $x \in \alpha Cl_\delta(\{y\})$.

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