Distance Fibonacci numbers, distance Lucas numbers and their applications

Dorota Bród[1], *Krzysztof Piejko, Iwona Włoch*
Rzeszów University of Technology
Faculty of Mathematics and Applied Physics
al. Powstańców Warszawy 12, 35-959 Rzeszów, Poland
dorotab@prz.edu.pl, piejko@prz.edu.pl, iwloch@prz.edu.pl

Abstract

In this paper we define new generalizations of the Lucas numbers, which also generalize the Perrin numbers. This generalization is based on the concept of k-distance Fibonacci numbers. We give interpretations of these numbers with respect to special decompositions and coverings, also in graphs. Moreover, we show some identities for these numbers, which often generalize known classical relations for the Lucas numbers and the Perrin numbers. We give an application of the distance Fibonacci numbers for building the Pascal's triangle.

AMS Subject Classification: 11B37, 11C20, 15B36, 05C69
Key Words: Fibonacci numbers, Lucas numbers, matching, covering

1 Introduction and preliminary results

In general we use the standard notation, see [3, 4]. The Fibonacci sequence is defined by $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$ with $F_0 = F_1 = 1$. There are some versions of the Fibonacci sequence, the most popular is the Lucas sequence defined as follows $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$ with $L_0 = 2$, $L_1 = 1$. In the literature have existed many interesting generalizations of the Fibonacci sequence and the like, see for example [5]-[10],[12]. A very natural is the concept of distance Fibonacci numbers introduced recently in [1], which generalize the Fibonacci numbers in the distance sense. Let $k \geq 2$, $n \geq 0$ be integers. The distance Fibonacci numbers $Fd(k,n)$ are defined recursively in the following way

$$Fd(k,n) = Fd(k,n-k+1) + Fd(k,n-k) \text{ for } n \geq k \tag{1}$$

[1] corresponding author

with initial conditions $Fd(k,n) = 1$ for $n = 0, 1, \ldots, k-1$. Note that $Fd(2,n) = F_n$. Moreover, $Fd(3,n) = Pv(n)$, where $Pv(n)$ is the n-th Padovan number defined by the recurrence relation $Pv(n) = Pv(n-2) + Pv(n-3)$ for $n \geq 3$ with $Pv(0) = Pv(1) = Pv(2) = 1$. The Padovan numbers have many interesting applications and generalizations, also in graphs. It is worth mentioning that $Pv(n-3)$ is the maximum value of the number of all maximal independent sets including the set of pendant vertices among all n-vertex trees, see [11]. Some generalization of the Padovan numbers with respect to distance independent sets in graphs was given in [13].

The Table 1 includes initial words of the distance Fibonacci numbers $Fd(k,n)$ for special k and n.

Tab.1. The distance Fibonacci numbers $Fd(k,n)$

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
F_n	1	1	2	3	5	8	13	21	34	55	89	144	233	377
$Fd(3,n)$	1	1	1	2	2	3	4	5	7	9	12	16	21	28
$Fd(4,n)$	1	1	1	1	2	2	2	3	4	4	5	7	8	9
$Fd(5,n)$	1	1	1	1	1	2	2	2	2	3	4	4	4	5
$Fd(6,n)$	1	1	1	1	1	1	2	2	2	2	2	3	4	4

Many interesting interpretations and properties of $Fd(k,n)$ can be found in [1], [2]. Among others, they proved that the number $Fd(k,n)$ has the following combinatorial interpretation, see [1].

Let $k \geq 2$, $n \geq k-1$ be integers. Let $X = \{1, 2, \ldots, n\}$ and $\mathcal{Y} = \{\mathcal{Y}_t; t \in T\}$ be the family of subsets of the set X, such that $\mathcal{Y}_t$ contains consecutive integers and

(i). $|\mathcal{Y}_t| \in \{k, k-1\}$ for all $t \in T$,

(ii). $\mathcal{Y}_t \cap \mathcal{Y}_s = \emptyset$ for $t \neq s$,

(iii). $0 \leq |X \setminus \bigcup_{t \in T} \mathcal{Y}_t| \leq k-2$,

(iv). for each $m \in (X \setminus \bigcup_{t \in T} \mathcal{Y}_t)$ either $m = n$ or $m + 1 \in (X \setminus \bigcup_{t \in T} \mathcal{Y}_t)$.

The family $\mathcal{Y}$ is called as a *quasi-k-decomposition of the set X*. It was proved.

Theorem 1 [1] *Let $k \geq 2$, $n \geq k-1$ be integers. Then the number of all quasi-k-decompositions of the set X is equal to $Fd(k,n)$.*

In this paper we shall show another application of the number $Fd(k,n)$, namely if k is sufficiently large then numbers $Fd(k,n)$ may be used for building the Pascal's triangle. We introduce two cyclic versions of the distance Fibonacci numbers $Fd(k,n)$, which generalize the classical Lucas numbers L_n and the Perrin numbers $Pr(n)$. In particular, we define distance Lucas numbers $Ld^{(1)}(k,n)$ and $Ld^{(2)}(k,n)$ of the first and the second kind, respectively. We give some identities between $Fd(k,n)$, $Ld^{(1)}(k,n)$

and $Ld^{(2)}(k,n)$ and we shall show that in some cases they generalize classical relations for the Fibonacci numbers F_n, the Lucas numbers L_n, the Padovan numbers $Pv(n)$ and the Perrin numbers $Pr(n)$.

Theorem 2 *Let $m \geq 1$, $k \geq 2$, $k > m$. Then for $n = mk, mk+1, \ldots, (m+1)(k-1)$*
(i). $Fd(k,n) = 2^m$,
(ii). $Fd(k,(m+1)(k-1)+l) = 2^m + \sum_{t=0}^{l-1} \binom{m}{t}$ for $l = 1, 2, \ldots, m$.

Proof. In the proof of formula (i) we use the induction on m. For $m = 1$, by the definition of $Fd(k,n)$, we have $Fd(k,n) = 2$ for $n = k, k+1, \ldots, 2k-2$. Assume that $Fd(k, mk+l) = 2^m$ for $l = 0, 1, \ldots, k-m-1$. We will prove that $Fd(k,(m+1)k+l) = 2^{m+1}$ for $l = 0, 1, \ldots, k-m-2$. By (1) and the induction hypothesis, we obtain

$$Fd(k,(m+1)k+l) = Fd(k,mk+l) + Fd(k,mk+l+1) = 2^m + 2^m = 2^{m+1},$$

which ends the proof of (i).

To prove (ii) we also use the induction on m. If $m = 1$, then we obtain $Fd(n,2k-1) = 2 + \binom{1}{0} = 3 = Fd(n,2k-1)$. Assume that (ii) is true for an arbitrary $m \geq 1$ and $k \geq 2$, $k > m$. We will prove that $Fd(k,(m+2)(k-1)+l) = 2^{m+1} + \sum_{t=0}^{l-1} \binom{m+1}{t}$ for $l = 1, 2, \ldots, m+1$.

By (1) and the induction hypothesis, we have

$$\begin{aligned}
&Fd(k,(m+2)(k-1)+l) = Fd(k,(m+1)(k-1)+l-1+k) = \\
&= Fd(k,(m+1)(k-1)+l-1) + Fd(k,(m+1)(k-1)+l) = \\
&= 2^m + \sum_{t=0}^{l-1} \binom{m}{t} + 2^m + \sum_{t=0}^{l-2} \binom{m}{t} = \\
&= 2^{m+1} + \binom{m}{0} + \sum_{t=1}^{l-1} \binom{m}{t} + \sum_{t=1}^{l-1} \binom{m}{t-1} = \\
&= 2^{m+1} + \binom{m+1}{0} + \sum_{t=1}^{l-1} \binom{m+1}{t} = 2^{m+1} + \sum_{t=0}^{l-1} \binom{m+1}{t},
\end{aligned}$$

which completes the proof. □

Corollary 3 *Let k, l, m be integers, $m \geq 1$, $k \geq 2$, $k > m$, $l = 0, \ldots, m-1$. Then*

$$Fd(k, m(k-1)+l+1) - Fd(k, m(k-1)+l) = \binom{m-1}{l}.$$

Proof. By Theorem 2, we obtain

$$Fd(k, m(k-1)+l+1) - Fd(k, m(k-1)+l) =$$
$$= 2^{m+1} + \sum_{t=0}^{l} \binom{m-1}{t} - 2^{m+1} - \sum_{t=0}^{l-1} \binom{m-1}{t} = \binom{m-1}{l},$$

which ends the proof. □

It is interesting that for $m \geq 1$ and sufficiently large $k \geq m+1$, by Theorem 2 (and Corollary 3) we have that in the sequence of the distance Fibonacci numbers a word 2^m appears exactly $k-m$ times. Let m be a fixed integer. Then the distance Fibonacci sequence has the subsequence of consecutive words of the form:

$$\ldots \underbrace{2^m, 2^m, \ldots, 2^m}_{(k-m)-\text{times}}, a_1, a_2, a_3, \ldots, a_{m-1}, a_m, \underbrace{2^{m+1}, 2^{m+1}, \ldots, 2^{m+1}}_{(k-m-1)-\text{times}} \ldots,$$

where the numbers $a_1 - 2^m, a_2 - a_1, a_3 - a_2, \ldots, a_{m-1} - a_m, 2^{m+1} - a_m$ form $m-1$ row of Pascal's triangle. It can be ilustrated in the following way

$$\begin{array}{ccccccccccccccccccccc}
\overbrace{1\ldots1}^{k-\text{times}} & & \overbrace{2\ldots2}^{(k-1)-\text{times}} & & 3 & & \overbrace{4\ldots4}^{(k-2)-\text{times}} & & 5 & 7 & & \overbrace{8\ldots8}^{(k-3)-\text{times}} & & 9 & 12 & 15 & & \overbrace{16\ldots16}^{(k-4)-\text{times}} & 17 & 21 & 27 \; 31 \; 32 \ldots \\
 & 1 & & 1 & & 1 & & 1 & & 2 & 1 & & 1 & & 3 & & 3 \;\; 1 & & 1 & 4 & 6 \;\; 4 \;\; 1
\end{array}$$

2 Distance Lucas numbers $Ld^{(1)}(k,n)$

In this section we introduce the first generalization of the Lucas numbers L_n. Let $k \geq 2$, $n \geq 0$ be integers. The distance Lucas numbers of the first kind $Ld^{(1)}(k,n)$ are defined by the k-th order linear recurrence relation

$$Ld^{(1)}(k,n) = Ld^{(1)}(k, n-k+1) + Ld^{(1)}(k, n-k) \text{ for } n \geq k$$

and for $0 \leq n \leq k-1$ we have the following initial conditions

$$Ld^{(1)}(k,n) = \begin{cases} 1 & \text{if } k+n \text{ is odd,} \\ 2 & \text{if } k+n \text{ is even.} \end{cases}$$

We can observe that if $k=2$, then $Ld^{(1)}(2,n) = L_n$.

The Table 2 includes a few first words of the distance Lucas sequence of the first kind $Ld^{(1)}(k,n)$ for special values of k and n.

Tab.2. The distance Lucas numbers $Ld^{(1)}(k,n)$ of the first kind

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
L_n	2	1	3	4	7	11	18	29	47	76	123	199	322	521
$Ld^{(1)}(3,n)$	1	2	1	3	3	4	6	7	10	13	17	23	30	40
$Ld^{(1)}(4,n)$	2	1	2	1	3	3	3	4	6	6	7	10	12	13
$Ld^{(1)}(5,n)$	1	2	1	2	1	3	3	3	3	4	6	6	6	7
$Ld^{(1)}(6,n)$	2	1	2	1	2	1	3	3	3	3	3	4	6	6

Now we give a combinatorial representation of the distance Lucas numbers $Ld^{(1)}(k,n)$ for fixed $k \geq 2$, $n \geq 3$ and $n \geq k-1$. Let $X = \{1,2,\ldots,n\}$. For $i,j \in X$ we define $i \oplus j$ as follows

$$i \oplus j = \begin{cases} i+j & \text{for } i+j \leq n, \\ i+j-n & \text{for } i+j > n. \end{cases}$$

In other words we say that X contains n cyclically consecutive integers. For fixed $2 \leq k \leq n+1$ let $\mathcal{C}^{(1)}(k,n) = \{C_i^{(1)}; i = 1,2,\ldots,p\}$ such that $C_1^{(1)} = \{t_0 \oplus 1, t_0 \oplus 2, \ldots, t_1\}$, $C_2^{(1)} = \{t_1 \oplus 1, t_1 \oplus 2, \ldots, t_2\}$, ..., $C_p^{(1)} = \{t_{p-1} \oplus 1, t_{p-1} \oplus 2, \ldots, t_p\}$ and the following conditions hold

(i). $(t_p = 1$ and $|C_p^{(1)}| = k)$ or $t_0 = n$,

(ii). $|C_i^{(1)}| \in \{k-1, k\}$ for $i = 1,2,\ldots,p$,

(iii). $n-k+2 \leq \sum\limits_{i=1}^{p} |C_i^{(1)}| \leq n$.

The family $\mathcal{C}^{(1)}(k,n)$ is called as *cyclic quasi-k-decomposition of the first kind* of the set X.

Theorem 4 *Let $n \geq 3$, $2 \leq k \leq n+1$ be integers. The number of all families $\mathcal{C}^{(1)}(k,n)$ is equal to the number $Ld^{(1)}(k,n)$.*

Proof. For $n = k-1$ the Theorem is obvious. Let $n \geq k$. By the definition of the family $\mathcal{C}^{(1)}(k,n)$ we deduce that the following possibilities have to be considered:

(1) $t_p = 1$ and $|C_p^{(1)}| = k$.

Since the subset $\{n-k+2,\ldots,n,1\} \in \mathcal{C}^{(1)}(k,n)$, we have that $\mathcal{C}^{(1)}(k,n) = \mathcal{F}(k,n) \cup \{n-k+2,\ldots,n,1\}$, where $\mathcal{F}(k,n)$ is an arbitrary quasi-k-decomposition of the set $X \setminus \{n-k+2,\ldots,n,1\}$, which is isomorphic to the set $X' = \{1,2,\ldots,n-k\}$. Using the combinatorial representation of the number $Fd(k,n)$, it is immediately follows that there are $Fd(k,n-k)$ families $\mathcal{C}^{(1)}(k,n)$ including the subset $\{n-k+2,\ldots,n,1\}$.

(2) $t_0 = n$ and $|C_1^{(1)}| = k$.

Proving analogously to the case (1), we obtain $Fd(k,n-k)$ families $\mathcal{C}^{(1)}(k,n)$ including the subset $\{1,2,\ldots,k\}$.

(3) $t_0 = n$ and $|C_1^{(1)}| = k - 1$.
Proving analogously to the case (1), we obtain $Fd(k, n-k+1)$ families $\mathcal{C}^{(1)}(k,n)$ including the subset $\{1, 2, \ldots, k-1\}$.

Altogether, this gives that the number of all families $\mathcal{C}^{(1)}(k,n)$ is equal to $Fd(k, n-k+1) + 2Fd(k, n-k)$.

Claim.

$$Ld^{(1)}(k,n) = Fd(k, n-k+1) + 2Fd(k, n-k) \quad \text{for} \quad n \geq k. \tag{2}$$

Proof (by induction on n). If $n = k$ then the result is obvious. Let $n > k$. Assuming (2) to hold for an arbitrary n, we will prove it for $n+1$. Using definitions of the numbers $Ld^{(1)}(k,n)$ and $Fd(k,n)$ and the induction hypothesis, we obtain

$$\begin{aligned} & Ld^{(1)}(k,n+1) = Ld^{(1)}(k, n-k+1) + Ld^{(1)}(k, n-k+2) = \\ = \ & Fd(k, n-2k+2) + Fd(k, n-2k+3) + 2F(k, n-2k+1) + \\ + \ & 2Fd(k, n-2k+2) = Fd(k, n-k+2) + 2Fd(k, n-k+1), \end{aligned}$$

which ends the proof. □

Corollary 5 *Let $k \geq 2$, $n \geq k$ be integers. Then*

$$Ld^{(1)}(k,n) = Fd(k, n-k+1) + 2Fd(k, n-k).$$

If $k = 2$, then we obtain the well-known identity $L_n = F_{n-1} + 2F_{n-2}$. In particular, if $k = 2$ then the number $Ld^{(1)}(k,n)$ gives the total number of decompositions of the set $X = \{1, 2, \ldots, n\}$, $n \geq 3$, on one-element and two-elements subsets with the assumption that elements from X are cyclically consecutive. Note that for $n < 3$ or $n < k-1$ the numbers $Ld^{(1)}(k,n)$ do not have such combinatorial representations.

Now we give the graph representation of the number $Ld^{(1)}(k,n)$. We use the standard definitions and notation of the graph theory, see [4].

Let $\mathcal{H} = \{H_1, H_2, \ldots, H_m\}$, $m \geq 1$, be a collection of m given graphs. A subgraph $M \subseteq G$ is a *$\mathcal{H}$-matching* of G if each component of M is isomorphic to H_i, $1 \leq i \leq m$. If $H_i = H$ for all $i = 1, 2, \ldots, m$, then the definition of $\mathcal{H}$-matching reduces to the definition of H-matching. Moreover, if $H = K_2$ then we obtain the definition of a matching in the classical sense.

Some results concerning H-matching covering problems and H-matching counting problems can be found in [14] and [10], respectively. We recall that a *perfect matching* is a subset of independent edges that meet every vertex of a graph. A *perfect $\mathcal{H}$-matching* of G is a collection M of vertex disjoint graphs such that every $H \in M$ is isomorphic to some $H_i \in \mathcal{H}$, $1 \leq i \leq m$ and M meets every vertex of a graph G. If the collection M is a perfect $\mathcal{H}$-matching of a subgraph $R \subseteq G$ and a subgraph induced by $V(G \setminus R)$ is either empty or it is connected and each subgraph of $G \setminus R$ is nonisomorphic to any H_i, $1 \leq i \leq m$, then we will say that M is a *quasi-perfect-$\mathcal{H}$-matching* of a graph G. Clearly, if $V(G \setminus R)$ is empty then quasi-perfect-$\mathcal{H}$-matching is a perfect matching of G.

Using this terminology we can observe that the set X corresponds to the vertex set of the graph C_n, $n \geq k-1$ and $n \geq 3$ with $V(C_n) = \{x_1, \dots, x_n\}$ and with the numbering of the vertices in the natural fashion. Then each $C_i^{(1)}$, $1 \leq i \leq p$, corresponds to P_t, where $t \in \{k-1, k\}$. Consequently, the family $\mathcal{C}^{(1)}$ corresponds to a special quasi-perfect-$\{P_k, P_{k-1}\}$-matching α of a graph C_n such that x_1 is the pendant vertex of a subgraph $P_t \in \alpha$, $t \in \{k, k-1\}$, $\{x_{n-k+3}, \dots, x_1\} \notin \alpha$. Moreover, the definition of the family $\mathcal{C}^{(1)}$ gives that all subgraphs $P_t \in \alpha$ are consecutive similarly to subsets $C_i^{(1)} \in \mathcal{C}^{(1)}$, $1 \leq i \leq p$. Finally, $Ld^{(1)}(k, n)$ is equal to the number of all such quasi-perfect-$\{P_k, P_{k-1}\}$-matchings α of the graph C_n.

3 Distance Lucas numbers $Ld^{(2)}(k, n)$

In this section we introduce the second generalization of the Lucas numbers L_n. Let $k \geq 2$, $n \geq 0$ be integers. The generalized Lucas numbers of the second kind $Ld^{(2)}(k, n)$ are defined by the k-th order linear recurrence relation

$$Ld^{(2)}(k, n) = Ld^{(2)}(k, n-k+1) + Ld^{(2)}(k, n-k) \text{ for } n \geq k$$

and for $0 \leq n \leq k-1$ we have the following initial conditions

$$Ld^{(2)}(k, n) = \begin{cases} k & \text{if } k+n \text{ is even,} \\ k-1 & \text{if } k+n \text{ is odd.} \end{cases}$$

We can observe that if $k = 2$, then $Ld^{(2)}(2, n) = L_n$. Moreover, $Ld^{(2)}(3, n) = Pr(n+2)$, where $Pr(n)$ is the n-th Perrin number defined recursively by $Pr(n) = Pr(n-2) + Pr(n-3)$ for $n \geq 3$ with $Pr(0) = 3$, $Pr(1) = 0$, $Pr(2) = 2$. It is interesting and worth mentioning that the Perrin numbers $Pr(n)$ are a cyclic version of the Padovan numbers $Pv(n)$, similarly as the Lucas numbers L_n are a cyclic version of the Fibonacci numbers F_n. The Perrin numbers have different combinatorial interpretations, also in graphs with respect to the number of maximal independent sets in C_n. Recently a generalization of the Perrin sequence with respect to distance independent sets was given in [13].

The Table 3 includes a few first words of the distance Lucas sequence of the second kind $Ld^{(2)}(k, n)$ for special values of k and n.

Tab.3. The distance Lucas numbers $Ld^{(2)}(k,n)$ of the second kind

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
L_n	2	1	3	4	7	11	18	29	47	76	123	199	322	521
$Ld^{(2)}(3,n)$	2	3	2	5	5	7	10	12	17	22	29	39	51	68
$Ld^{(2)}(4,n)$	4	3	4	3	7	7	7	10	14	14	17	24	28	31
$Ld^{(2)}(5,n)$	4	5	4	5	4	9	9	9	9	13	18	18	18	22
$Ld^{(2)}(6,n)$	6	5	6	5	6	5	11	11	11	11	11	16	22	22

Analogously as for the numbers $Ld^{(1)}(k,n)$ firstly we give a combinatorial representations of the distance Lucas numbers $Ld^{(2)}(k,n)$ for fixed $k \geq 2$ and $n \geq k-1$.

Let $X = \{1,2,\ldots,n\}$, $n \geq k-1$, $n \geq 3$, contains n cyclically consecutive integers. For fixed $2 \leq k \leq n+1$ let $\mathcal{C}^{(2)}(k,n) = \{C_i^{(2)}; i = 1,2,\ldots,p\}$ such that $C_i^{(2)} = \{t_{i-1} \oplus 1, t_{i-1} \oplus 2, \ldots, t_i\}$ for $i = 1,2,3,\ldots,p$ and
(i). $1 \in C_1^{(2)}$,
(ii). $|C_i^{(2)}| \in \{k-1,k\}$ for $i = 1,2,\ldots,p$,
(iii). $n-k+2 \leq \sum_{i=1}^{p} |C_i^{(2)}| \leq n$.
The family $\mathcal{C}^{(2)}(k,n)$ is called as a *cyclic quasi-k-decomposition of the second kind* of the set X.

Theorem 6 *Let $k \geq 2$, $n \geq k-1$, $n \geq 3$ be integers. The number of all families $\mathcal{C}^{(2)}(k,n)$ is equal to the number $Ld^{(2)}(k,n)$.*

Proof. It is easily seen that for $n = k-1$ the Theorem holds. Assume that $n \geq k$. The definition of the family $\mathcal{C}^{(2)}(k,n)$ immediately gives two possibilities:
(1) $|C_1^{(2)}| = k-1$.
Since $1 \in C_1^{(2)}$,we have that there are exactly $k-1$ subsets $C_1^{(2)}$ of the form $\{n-k+3,\ldots,1\}$, $\{n-k+4,\ldots,1,2\},\ldots,\{1,2,\ldots,k-1\}$. This means that $\mathcal{C}^{(2)}(k,n) = \mathcal{F}(k,n) \cup C_1^{(2)}$, where $C_1^{(2)}$ is defined above and $\mathcal{F}(k,n)$ is an arbitrary quasi-k-decomposition of the set $X \setminus C_1^{(2)}$, which is isomorphic to the set $X' = \{1,2,\ldots,n-k+1\}$. Since the set $C_1^{(2)}$ can be chosen on $k-1$ ways, by Theorem 1, we obtain $(k-1)Fd(k,n-k+1)$ families $\mathcal{C}^{(2)}(k,n)$ including the subset $C_1^{(2)}$ of the cardinality $k-1$.
(2) $|C_1^{(2)}| = k$.
Proving analogously to the case (1), we obtain $kFd(k,n-k)$ families $\mathcal{C}^{(2)}(k,n)$ including the subset $C_1^{(2)}$ of the cardinality k.

Finally, from the above cases we obtain that the number of all families $\mathcal{C}^{(2)}(k,n)$ is equal to $(k-1)Fd(k,n-k+1) + kFd(k,n-k)$.

Claim.

$$Ld^{(2)}(k,n) = (k-1)Fd(k,n-k+1) + kFd(k,n-k) \quad \text{for} \quad n \geq k. \qquad (3)$$

Proof (by induction on n). If $n = k$ then the result is obvious. Assume that $n > k$ and the formula (3) holds for an arbitrary n. We will prove it for $n+1$. Using definitions of the numbers $Ld^{(2)}(k,n)$ and $Fd(k,n)$ and the induction hypothesis, we obtain

$$\begin{aligned}
&Ld^{(2)}(k,n+1) = Ld^{(2)}(k,n-k+2) + Ld^{(2)}(k,n-k+1) = \\
&= (k-1)Fd(k,n-2k+3) + kFd(k,n-2k+2) + \\
&+(k-1)Fd(k,n-2k+2) + kFd(k,n-2k+1)) = \\
&= (k-1)(Fd(k,n-2k+3) + Fd(k,n-2k+2)) + \\
&+k(Fd(k,n-2k+2) + kFd(k,n-2k+1)) = \\
&= (k-1)Fd(k,n-k+2) + kFd(k,n-k+1),
\end{aligned}$$

which ends the proof. □

Corollary 7 *Let $k \geq 2$, $n \geq k$, $n \geq 3$ be integers. Then*

$$Ld^{(2)}(k,n) = (k-1)Fd(k,n-k+1) + kFd(k,n-k).$$

If $k = 2$, then we obtain the well-known identity $L_n = F_{n-1} + 2F_{n-2}$. Analogously as for $Ld^{(1)}(k,n)$ if $n < k-1$ or $n < 3$, the numbers $Ld^{(2)}(k,n)$ do not have the above combinatorial representations.

The numbers $Ld^{(2)}(k,n)$ have also the graph representations with respect to the number of special quasi-perfect-matchings. Similarly to $Ld^{(1)}(k,n)$ the set X corresponds to the vertex set of the graph C_n, $n \geq k-1$ and $n \geq 3$. Then each $C_i^{(2)} \in \mathcal{C}_i^{(2)}$, $1 \leq i \leq p$ corresponds to P_t, where $t \in \{k-1, k\}$ and consequently $\mathcal{C}^{(2)}$ corresponds to a special quasi-perfect-$\{P_k, P_{k-1}\}$-matching β of a graph C_n such that there is $P_t \in \beta$, where $t \in \{k-1, k\}$. Moreover, the definition of the family $\mathcal{C}^{(2)}$ gives that there is $P_t \in \beta$, $t \in \{k-1, k\}$ such that $x_1 \in V(P_t)$ and all subgraphs $P_t \in \beta$ are consecutive in the same way as subsets $C_i^{(2)} \in \mathcal{C}^{(2)}$, $1 \leq i \leq p$. Finally, $Ld^{(2)}(k,n)$ is equal to the number of all quasi-perfect-$\{P_k, P_{k-1}\}$-matchings of the graph C_n.

4 Identities for $Fd(k,n)$, $Ld^{(1)}(k,n)$ and $Ld^{(2)}(k,n)$

In this section we present the list of identities for the distance Fibonacci numbers $Fd(k,n)$ and the distance Lucas numbers of the first kind $Ld^{(1)}(k,n)$ and the second kind $Ld^{(2)}(k,n)$, which generalize known identities for the Fibonacci, the Lucas, the Padovan and the Perrin numbers.

Theorem 8 *Let k, m, n be integers, $k \geq 2$, $m \geq 1$, $n \geq 0$. Then*

$$Fd(k, n+k-1) + \sum_{i=1}^{m} Fd(k, n+ki) = Fd(k, n+(m+1)k-1). \qquad (4)$$

Proof (by induction on m). For $m = 1$ we have the obvious equation $Fd(k, n+k-1) + Fd(k, n+k) = Fd(k, n+2k-1)$. Assume that formula (4) is true for an arbitrary $m \geq 1$. We will prove that $Fd(k, n+k-1) + \sum_{i=1}^{m+1} Fd(k, n+ik) = Fd(k, n+(m+2)k-1)$. By the induction hypothesis and the definition of $Fd(k,n)$, we have

$$\begin{aligned} & Fd(k, n+k-1) + \sum_{i=1}^{m+1} Fd(k, n+ik) = \\ = \; & Fd(k, n+k-1) + \sum_{i=1}^{m} Fd(k, n+ik) + Fd(k, n+(m+1)k) = \\ = \; & Fd(k, n+(m+1)k-1) + Fd(k, n+(m+1)k) = \\ = \; & Fd(k, n+(m+2)k-1), \end{aligned}$$

which ends the proof. □

Putting $n = 0$ or $n = 1$, respectively in Theorem 8 we obtain the following

Corollary 9 *Let $k \geq 2$ and $m \geq 1$. Then*

(*i*). $\sum_{i=0}^{m} Fd(k, ki) = Fd(k, (m+1)k-1)$,

(*ii*). $\sum_{i=0}^{m} Fd(k, ki+1) = Fd(k, (m+1)k) - 1$.

For $k = 2$ we obtain the well-known identities for the Fibonacci numbers:

$$\sum_{i=0}^{m} F_{2i} = F_{2m+1}, \quad \sum_{i=0}^{m} F_{2i+1} = F_{2m+2} - 1.$$

For $k = 3$ Theorem 8 gives the following formula for the Padovan numbers

Corollary 10 *For $m \geq 1$ and $n \geq 0$*

$$\sum_{i=1}^{m} Pv(n+3i) = Pv(n+3m+2) - Pv(n+2).$$

Note that for $n = 0, 1, 2$ we obtain the well-known identities for the Padovan numbers, respectively:

$\sum_{i=1}^{m} Pv(3i) = Pv(3m+2) - 1, \quad \sum_{i=1}^{m} Pv(3i+1) = Pv(3m+3) - 2,$
$\sum_{i=1}^{m} Pv(3i+2) = Pv(3m+4) - 2.$

The following identity is true for both kinds of distance Lucas numbers $Ld^{(1)}(k,n)$ and $Ld^{(2)}(k,n)$. The proof of this theorem is analogous as in Theorem 8, so we omit it.

Theorem 11 *Let $k \geq 2$, $k > m$, $m \geq 1$, $n \geq 0$. Then for $j = 1, 2$*

$$\sum_{i=1}^{m} Ld^{(j)}(k, n+ki) = Ld^{(j)}(k, n+(m+1)k-1) - Ld^{(j)}(k, n+k-1).$$

Since $Ld^{(2)}(3,n) = Pr(n+2)$, we obtain, by Theorem 11, the following identity for the Perrin numbers

Corollary 12 *For $m \geq 1$ and $n \geq -2$*

$$\sum_{i=1}^{m} Pr(n+2+3i) = Pr(n+3m+4) - Pr(n+4).$$

Note that for $n = -2, -1, 0$ we obtain the well-known identities for the Perrin numbers:

$\sum_{i=1}^{m} Pr(3i) = Pr(3m+2) - 2, \quad \sum_{i=1}^{m} Pr(3i+1) = Pr(3m+3) - 3,$
$\sum_{i=1}^{m} Pr(3i+2) = Pr(3m+4) - 2.$

Putting $n = 0$ or $n = 1$ in Theorem 11, we obtain, respectively

Corollary 13 *Let $k \geq 2$. Then*

(*i*). $\sum_{i=1}^{m} Ld^{(1)}(k, ki) = Ld^{(1)}(k, (m+1)k-1) - 1,$

(*ii*). $\sum_{i=1}^{m} Ld^{(1)}(k, ki+1) = Ld^{(1)}(k, (m+1)k) - 3,$

(*iii*). $\sum_{i=1}^{m} Ld^{(2)}(k, ki) = Ld^{(2)}(k, (m+1)k-1) - k + 1,$

(*iv*). $\sum_{i=1}^{m} Ld^{(2)}(k, ki+1) = Ld^{(2)}(k, (m+1)k) - 2k + 1.$

For $k = 2$ formulas (*i*), (*iii*) and (*ii*), (*iv*) give the well-known identities for the Lucas numbers.
$\sum_{i=1}^{m} L_{2i} = L_{2m+1} - 1, \quad \sum_{i=1}^{m} L_{2i+1} = L_{2m+2} - 3.$

Theorem 14 *Let $k \geq 2$ and $n \geq k+1$ be integers. Then*

(*i*). $Fd(k,n) + Fd(k,n-k+2) = Fd(k,n+1) + Fd(k,n-k)$,

(*ii*). $Ld^{(1)}(k,n+k-2) = Fd(k,n-2) + Fd(k,n+k-2)$,

(*iii*). $Ld^{(2)}(k,n+k-2) = Fd(k,n-2) + (k-1)Fd(k,n+k-2)$,

(*iv*). $Ld^{(j)}(k,n+1) = Ld^{(j)}(k,n) + Ld^{(j)}(k,n-k+2) - Ld^{(j)}(k,n-k)$ *for* $j = 1,2$.

Proof. (*i*). By the definition of $Fd(k,n)$, we have

$$\begin{aligned}&Fd(k,n+1) + Fd(k,n-k) - Fd(k,n-k+2) =\\ &= Fd(k,n-k+1) + Fd(k,n-k+2) +\\ &+ Fd(k,n-k) - Fd(k,n-k+2) =\\ &= Fd(k,n-k+1) + Fd(k,n-k) = Fd(k,n).\end{aligned}$$

(*ii*). By Corollary 5 and by the definition of $L^{(1)}(k,n)$, we obtain

$$\begin{aligned}&L^{(1)}(k,n+k-2) = Fd(k,n-1) + 2Fd(k,n-2) =\\ &= Fd(k,n-2) + Fd(k,n-1) + Fd(k,n-2) =\\ &= Fd(k,n+k-2) + Fd(k,n-2).\end{aligned}$$

(*iii*). analogously as in (*ii*).

(*iv*). analogously as in (*i*). □

For $k = 2$ and $n \geq 2$ we obtain well-known identities, note that (*ii*) and (*iii*) give the same relations.

$$F_n = \tfrac{1}{2}(F_{n-2} + F_{n+1}), \quad L_n = F_{n-2} + F_n, \quad L_n = \tfrac{1}{2}(L_{n+1} + L_{n-2}).$$

The following result may be proved in much the same way as Theorem 2.

Theorem 15 *Let $m \geq 1$, $k \geq 2$, $k > m$. Then*

(*i*). $Ld^{(1)}(k,n) = 3 \cdot 2^{m-1}$ *for* $n = mk, mk+1, \dots, (m+1)(k-1)$,

(*ii*). $Ld^{(1)}(k,(m+1)(k-1)+l) = 3 \cdot 2^{m-1} + 3\sum_{t=0}^{l-1}\binom{m-1}{t} - 2\binom{m-1}{l-1}$ *for* $l = 1,2,\dots,m$,

(*iii*). $Ld^{(2)}(k,n) = (2k-1)2^{m-1}$ *for* $n = mk, mk+1, \dots, (m+1)(k-1)$,

(*iv*). $Ld^{(2)}(k,(m+1)(k-1)+l) = (2k-1)2^{m-1} + (2k-1)\sum_{t=0}^{l-1}\binom{m-1}{t} - k\binom{m-1}{l-1}$ *for* $l = 1,2,\dots,m$.

References

[1] U. Bednarz, A. Włoch, M. Wołowiec-Musiał, *Distance Fibonacci numbers, their interpretations and matrix generators*, Commentationes Mathematicae, (2012), in print.

[2] U. Bednarz, A. Włoch, M. Wołowiec-Musiał, communications.

[3] C. Berge, Principles of combinatorics, Academic Press New York and London (1971).

[4] R. Diestel, Graph theory, Springer-Verlag, Heidelberg, New York, Inc., 2005.

[5] S. Falcon, *On the k-Lucas numbers*, Int. J. Contemp. Math. Sciences 6 (21) (2011), 1039–1050.

[6] S. Falcon, *On the powers of k-Fibonacci numbers*, Ars Combinatoria, in print.

[7] S. Falcon, A.Plaza, *The k-Fibonacci sequence and the Pascal 2-triangle*, Chaos, Solitions & Fractals 33.1 (2007), 38–49.

[8] S. Falcon, A.Plaza, *On k-Fibonacci numbers of arithmetic indexes*, Applied Mathematics and Computation 208 (2009), 180–185.

[9] M. Kwaśnik, I. Włoch, *The total number of generalized stable sets and kernels in graphs*, Ars Combinatoria 55 (2000) 139–146.

[10] A. Włoch, *On generalized Fibonacci numbers and k-distance K_p-matchings in graphs*, Discrete Applied Mathematics 160 (2012), 1399–1405.

[11] I. Włoch, *Trees with extremal numbers of maximal independent sets including the set of leaves*, Discrete Mathematics 308 (2008), 4768–4772.

[12] A. Włoch, I. Włoch, *Generalized sequences and k-independent sets in graphs*, Discrete Applied Mathematics 158 (2010), 1966–1970.

[13] A. Włoch, I. Włoch, *Generalized Padovan numbers, Perrin numbers and maximal k-independent sets in graphs*, Ars Combinatoria 99 (2011), 359–364.

[14] R. Yuster, *Combinatorial and computational aspect of graph packing and graph decomposition*, Computer Science Review 1 (2007), 12–26.