A TRANSFORMATION FOR THE AL-SALAM-CARLITZ POLYNOMIALS

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ABSTRACT. In this paper, we use the q-difference operator and the Andrews-Askey integral to give a transformation for the Al-Salam-Carlitz polynomials. As applications, we obtain an expansion of the Carlitz identity and some other identities for Al-Salam-Carlitz polynomials .

1. INTRODUCTION AND MAIN RESULT

The following is the well-known Rogers-Szegö polynomials:

$$h_n(x|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^k. \tag{1.1}$$

The Rogers-Szegö polynomials play an important role in the theory of orthogonal polynomials, particularly in the study of the Askey-Wilson integral [3, 6, 7] . The Rogers-Szegö polynomials is a special case of a more general polynomials, the Al-Salam-Carlitz polynomials. The Al-Salam-Carlitz polynomials $\varphi_n^{(a)}(x|q)$ is defined as [11]

$$\varphi_n^{(a)}(x|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^k (a;q)_k. \tag{1.2}$$

The Rogers-Szegö polynomials and the Al-Salam-Carlitz polynomials are related by

$$h_n(x|q) = \varphi_n^{(0)}(x|q).$$

In this paper, we give a transformation for the Al-Salam-Carlitz polynomials. The main result is the following theorem:

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Theorem 1.1. *We have*

$$\sum_{k=0}^{n} \frac{(a,b,bx;q)_k}{(q;q)_k}(-ab)^{n-k}q^{(n-k)(n+k-1)/2}\varphi_{n-k}^{(bq^k)}(x|q) = \frac{(a;q)_{n+1}}{(q;q)_n}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}\frac{(-b)^k q^{\binom{k}{2}}}{1-aq^{n-k}}\varphi_k^{(b)}(x|q). \tag{1.3}$$

Applications of the transformation formula are also given, which include an expansion of the Carlitz identity and some other identities for the Al-Salam-Carlitz polynomials.

2. NOTATIONS AND KNOWN RESULTS

We first recall some definitions, notations and known results in [5] which will be used for the proof of Theorem 1.1. Throughout the whole paper, it is supposed that $0 < q < 1$. The q-shifted factorials are defined as

$$(a;q)_0 = 1, \quad (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k), \quad (a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k). \tag{2.1}$$

We also adopt the following compact notation for multiple q-shifted factorial:

$$(a_1, a_2, \ldots, a_m; q)_n = (a_1;q)_n(a_2;q)_n \ldots (a_m;q)_n, \tag{2.2}$$

where n is an integer or ∞. The q-binomial coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}. \tag{2.3}$$

The following is the special case of the q-binomial theorem [2]

$$\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}(-1)^k q^{\binom{k}{2}}x^k = (x;q)_n. \tag{2.4}$$

Carlitz [4] discovered the following transformation formula

$$\sum_{k=0}^{n} \frac{(a,b;q)_k}{(q;q)_k}(-ab)^{n-k}q^{(n-k)(n+k-1)/2} = \frac{(a;q)_{n+1}}{(q;q)_n}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}\frac{(-b)^k q^{\binom{k}{2}}}{1-aq^{n-k}}. \tag{2.5}$$

The q-difference operator is defined by [9]

$$D_q\{f(a)\} = \frac{1}{a}[f(a) - f(aq)]. \tag{2.6}$$

In this paper, D_q acts on the variable a. The following property of D_q is straightforward:

$$D_q^n\{\frac{(at;q)_\infty}{(as;q)_\infty}\} = s^n(t/s;q)_n\frac{(atq^n;q)_\infty}{(as;q)_\infty}. \tag{2.7}$$

F.H Jackson defined the q-integral by [8]

$$\int_0^d f(t)d_qt = d(1-q)\sum_{n=0}^{\infty} f(dq^n)q^n, \tag{2.8}$$

and

$$\int_c^d f(t)d_qt = \int_0^d f(t)d_qt - \int_0^c f(t)d_qt. \tag{2.9}$$

The following is the Andrews-Askey integral [1], which can be derived from Ramanujan's ${}_1\psi_1$:

$$\int_c^d \frac{(qt/c,qt/d;q)_\infty}{(at,bt;q)_\infty}d_qt = \frac{d(1-q)(q,dq/c,c/d,abcd;q)_\infty}{(ac,ad,bc,bd;q)_\infty}, \tag{2.10}$$

provided that no zero factors in the denominator of the integrals.

3. THE PROOF OF THEOREM 1.1

In this section, we use the Andrews-Askey integral, the q-difference operator and the Al-salam and Carlitz polynomials to prove the theorem 1.1. Before the proof, we give the following q-integral formula:

Lemma 3.1. *We have*

$$\int_x^1 \frac{(qt/x,qt;q)_\infty t^n}{(at;q)_\infty}d_qt = \frac{(1-q)(q,q/x,x;q)_\infty}{(ax,a;q)_\infty}\varphi_n^{(a)}(x|q), \tag{3.1}$$

provided that no zero factors in the denominator.

Proof. By the definition of D_q and the Andrews-Askey integral (2.10),we can easily have

$$\int_c^d D_q\{\frac{(qt/c,qt/d;q)_\infty}{(at,bt;q)_\infty}\}d_qt = D_q\{\int_c^d \frac{(qt/c,qt/d;q)_\infty}{(at,bt;q)_\infty}d_qt\}. \tag{3.2}$$

Iterating this n times gives

$$\int_c^d D_q^n\{\frac{(qt/c,qt/d;q)_\infty}{(at,bt;q)_\infty}\}d_qt = D_q^n\{\int_c^d \frac{(qt/c,qt/d;q)_\infty}{(at,bt;q)_\infty}d_qt\}. \tag{3.3}$$

By direct calculation, we know

$$D_q^n\{\frac{(qt/c,qt/d;q)_\infty}{(at,bt;q)_\infty}\} = \frac{(qt/c,qt/d;q)_\infty t^n}{(at,bt;q)_\infty}. \tag{3.4}$$

Using the Andrews-Askey integral (2.10) and the following Leibniz rule for D_q [10]:

$$D_q^n\{f(a)g(a)\} = \sum_{k=0}^{n} q^{k(k-n)} \left[\begin{array}{c} n \\ k \end{array} \right] D_q^k\{f(a)\} D_q^{n-k}\{g(q^k a)\}$$

gets

$$\begin{aligned}
(3.5) \qquad & D_q^n\{\int_c^d \frac{(qt/c, qt/d; q)_\infty}{(at, bt; q)_\infty} d_q t\} \\
= \;& D_q^n\{\frac{d(1-q)(q, dq/c, c/d, abcd; q)_\infty}{(ac, ad, bc, bd; q)_\infty}\} \\
= \;& \frac{d(1-q)(q, dq/c, c/d; q)_\infty}{(bc, bd; q)_\infty} D_q^n\{\frac{(abcd; q)_\infty}{(ac; q)_\infty} \cdot \frac{1}{(ad; q)_\infty}\} \\
= \;& \frac{d(1-q)(q, dq/c, c/d; q)_\infty}{(bc, bd; q)_\infty} \\
\times \;& \sum_{k=0}^{n} q^{k(k-n)} \left[\begin{array}{c} n \\ k \end{array} \right] D_q^k\{\frac{(abcd; q)_\infty}{(ac; q)_\infty}\} D_q^{n-k}\{\frac{1}{(adq^k; q)_\infty}\}.
\end{aligned}$$

Employing (2.7), we have

$$\begin{aligned}
(3.6) \qquad & D_q^n\{\int_c^d \frac{(qt/c, qt/d; q)_\infty}{(at, bt; q)_\infty} d_q t\} \\
= \;& \frac{d(1-q)(q, dq/c, c/d, abcd; q)_\infty}{(ac, ad, bc, bd; q)_\infty} \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right] \frac{(ad, bd; q)_k}{(abcd; q)_k} c^k d^{n-k}.
\end{aligned}$$

Combining (3.3), (3.4) and (3.6) gives

$$\begin{aligned}
(3.7) \qquad & \int_c^d \frac{(qt/c, qt/d; q)_\infty t^n}{(at, bt; q)_\infty} d_q t \\
= \;& \frac{d(1-q)(q, dq/c, c/d, abcd; q)_\infty}{(ac, ad, bc, bd; q)_\infty} \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right] \frac{(ad, bd; q)_k}{(abcd; q)_k} c^k d^{n-k}.
\end{aligned}$$

Letting $b = 0$, $c = x$ and $d = 1$ in (3.7) gets (3.1). □

Now, we give the proof of theorem 1.1

Proof. Rewriting (2.5) as

$$\sum_{k=0}^{n}\frac{(a;q)_k}{(q;q)_k}(-a)^{n-k}q^{(n-k)(n+k-1)/2}\frac{b^{n-k}}{(bq^k;q)_\infty} \tag{3.8}$$
$$= \frac{(a;q)_{n+1}}{(q;q)_n}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}\frac{(-1)^k q^{\binom{k}{2}}}{1-aq^{n-k}}\cdot\frac{b^k}{(b;q)_\infty}.$$

Letting $b=bt$ in (3.8) and then multiplying the equation (3.8) by $(qt/x,qt;q)_\infty$ gets

$$\sum_{k=0}^{n}\frac{(a;q)_k}{(q;q)_k}(-ab)^{n-k}q^{(n-k)(n+k-1)/2}\frac{(qt/x,qt;q)_\infty t^{n-k}}{(btq^k;q)_\infty} \tag{3.9}$$
$$= \frac{(a;q)_{n+1}}{(q;q)_n}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}\frac{(-1)^k q^{\binom{k}{2}}b^k}{1-aq^{n-k}}\cdot\frac{(qt/x,qt;q)_\infty t^k}{(bt;q)_\infty}.$$

Taking the q-integral on both sides of the above identity with respect to variable t, we have

$$\sum_{k=0}^{n}\frac{(a;q)_k}{(q;q)_k}(-ab)^{n-k}q^{(n-k)(n+k-1)/2}\int_x^1\frac{(qt/x,qt;q)_\infty t^{n-k}}{(btq^k;q)_\infty}\,d_qt \tag{3.10}$$
$$= \frac{(a;q)_{n+1}}{(q;q)_n}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}\frac{(-1)^k q^{\binom{k}{2}}b^k}{1-aq^{n-k}}\int_x^1\frac{(qt/x,qt;q)_\infty t^k}{(bt;q)_\infty}\,d_qt.$$

Using (3.1), we get

$$\int_x^1\frac{(qt/x,qt;q)_\infty t^{n-k}}{(btq^k;q)_\infty}d_qt = \frac{(1-q)(q,q/x,x;q)_\infty}{(bq^kx,bq^k;q)_\infty}\,\varphi_{n-k}^{(bq^k)}(x|q), \tag{3.11}$$

and

$$\int_x^1\frac{(qt/x,qt;q)_\infty t^k}{(bt;q)_\infty}d_qt = \frac{(1-q)(q,q/x,x;q)_\infty}{(bx,a;q)_\infty}\,\varphi_k^{(b)}(x|q), \tag{3.12}$$

Substituting (3.11) and (3.12) into (3.10) gives (1.3).

□

4. Some applications of the transformation formula

In this section, we give some applications of the transformation formula. First, we obtain the following identity, which can be thought as an expansion of the Carlitz identity:

Theorem 4.1. *Let t, n are integers such that $0 \le t \le n$, then we have*

$$(4.1)\qquad \sum_{k=0}^{n} \sum_{i=\max\{0,t+k-n\}}^{\min\{k,t\}} \begin{bmatrix} k \\ i \end{bmatrix} \begin{bmatrix} n-k \\ t-i \end{bmatrix} \frac{(a,b;q)_k (bq^k;q)_{t-i}}{(q;q)_k}$$
$$\times\ a^{n-k}(-b)^{n-k+i} q^{(n-k)(n+k-1)/2+\binom{i}{2}}$$
$$= \frac{(a;q)_{n+1}(b;q)_t}{(q;q)_n} \sum_{k=t}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} k \\ t \end{bmatrix} \frac{(-b)^k q^{\binom{k}{2}}}{1-aq^{n-k}}.$$

Proof. Using (2.4) gives

$$(4.2)\qquad (bx;q)_k = \sum_{i=0}^{k} \begin{bmatrix} k \\ i \end{bmatrix}_q (-1)^i q^{\binom{i}{2}} b^i x^i.$$

Substituting the above expansion into (1.3) and then comparing the coefficients of x^t in (1.3) gives (4.1). Thus, we finish the proof.

□

It is obvious that the case $t = 0$ of (4.1) results in the Carlitz identity (2.5). Then we have the following identities for the Al-Salam-Carlitz polynomials.

Theorem 4.2. *We have*

$$(4.3)\sum_{k=0}^{n} \frac{(a,ax;q)_k}{(q;q)_k}[a^{n-k}\varphi_{n-k}^{(aq^k)}(x|q) - a^{n+1-k}\varphi_{n+1-k}^{(aq^k)}(x|q)] = \frac{(a,ax;q)_{n+1}}{(q;q)_n}.$$

Proof. Let $t = n$ in (4.1), we get

$$(4.4)\qquad \sum_{k=0}^{n} \frac{(a;q)_k a^{n-k}}{(q;q)_k} = \frac{(aq;q)_n}{(q;q)_n}.$$

Rewriting (4.4) as

$$(4.5)\qquad \sum_{k=0}^{n} \frac{1}{(q;q)_k} \cdot \frac{a^{n-k} - a^{n+1-k}}{(aq^k;q)_\infty} = \frac{1}{(q;q)_n} \cdot \frac{1}{(aq^{n+1};q)_\infty}.$$

Letting $a = at$ in (4.5) and then multiplying the equation (4.5) by $(qt/x, qt;q)_\infty$ gets

$$(4.6)\qquad \sum_{k=0}^{n} \frac{1}{(q;q)_k} \cdot \frac{(a^{n-k}t^{n-k} - a^{n+1-k}t^{n+1-k})(qt/x,qt;q)_\infty}{(aq^k t;q)_\infty}$$
$$= \frac{1}{(q;q)_n} \cdot \frac{(qt/x,qt;q)_\infty}{(aq^{n+1}t;q)_\infty}.$$

Taking the q-integral on both sides of the above identity with respect to variable t, we have

$$(4.7)\qquad \sum_{k=0}^{n}\frac{1}{(q;q)_k}\cdot\int_x^1\frac{(a^{n-k}t^{n-k}-a^{n+1-k}t^{n+1-k})(qt/x,qt;q)_\infty}{(aq^kt;q)_\infty}\,d_qt$$
$$= \frac{1}{(q;q)_n}\cdot\int_x^1\frac{(qt/x,qt;q)_\infty}{(aq^{n+1}t;q)_\infty}\,d_qt.$$

Using (3.1), we get

$$(4.8)\int_x^1\frac{(a^{n-k}t^{n-k}-a^{n+1-k}t^{n+1-k})(qt/x,qt;q)_\infty}{(aq^kt;q)_\infty}\,d_qt$$
$$= a^{n-k}\int_x^1\frac{(qt/x,qt;q)_\infty t^{n-k}}{(aq^kt;q)_\infty}\,d_qt-a^{n+1-k}\int_x^1\frac{(qt/x,qt;q)_\infty t^{n+1-k}}{(aq^kt;q)_\infty}\,d_qt$$
$$= \frac{(1-q)(q,q/x,x;q)_\infty}{(aq^k,axq^k;q)_\infty}\,[a^{n-k}\varphi_{n-k}^{(aq^k)}(x|q)-a^{n+1-k}\varphi_{n+1-k}^{(aq^k)}(x|q)],$$

and

$$(4.9)\qquad \int_x^1\frac{(qt/x,qt;q)_\infty}{(aq^{n+1}t;q)_\infty}\,d_qt=\frac{(1-q)(q,q/x,x;q)_\infty}{(aq^{n+1},axq^{n+1};q)_\infty}.$$

Substituting (4.8) and (4.10) into (4.7) gives (4.3). □

Corollary 4.3. When $n\geq 0$ and $a\neq 0$, we have

$$(4.10)\qquad \varphi_n^{(a)}(1/a\,|q)=1/a^n.$$

Proof. Letting $x=1/a$ in (4.3) gives

$$(4.11)\qquad a^n\varphi_n^{(a)}(1/a\,|q)-a^{n+1}\varphi_{n+1}^{(a)}(1/a\,|q)=0.$$

Combining (4.11) and $\varphi_0^{(a)}(1/a\,|q)=1$ gives (4.10). □

Corollary 4.4. When $n\geq 1$ and $a\neq 0$, we have

$$(4.12)\qquad \varphi_n^{(a)}(1/aq\,|q)-a\varphi_{n+1}^{(a)}(1/aq\,|q)$$
$$= (\frac{1}{a^n}-\frac{1}{a^{n-1}})(\frac{1}{q^n}-\frac{1}{q^{n+1}}).$$

Proof. Letting $x=1/aq$ in (4.3) gives

$$(4.13)\qquad a^n\varphi_n^{(a)}(1/aq\,|q)-a^{n+1}\varphi_{n+1}^{(a)}(1/aq\,|q)$$
$$+\ \frac{(1-a)(1-1/q)}{1-q}[a^{n-1}\varphi_{n-1}^{(aq)}(1/aq\,|q)-a^n\varphi_n^{(aq)}(1/aq\,|q)]=0.$$

Using (4.10) gets

$$(4.14)\qquad \varphi_{n-1}^{(aq)}(1/aq\,|q)=\frac{1}{(aq)^{n-1}},\ \varphi_n^{(aq)}(1/aq\,|q)=\frac{1}{(aq)^n}.$$

Substituting (4.14) into (4.13) gives (4.12). □

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