On the Laplacian spectral characterization of the generalized T-shape trees*

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Abstract A T-shape tree $T(l_1, l_2, l_3)$ is obtained from three paths P_{l_1+1}, P_{l_2+1} and P_{l_3+1} by identifying one of their pendent vertices. A generalized T-shape tree $T_s(l_1, l_2, l_3)$ is obtained from $T(l_1, l_2, l_3)$ by appending two pendent vertices to exactly s pendent vertices of $T(l_1, l_2, l_3)$, where $1 \leq s \leq 3$ is a positive integer. In this paper, we firstly show that the generalized T-shape tree $T_2(l_1, l_2, l_3)$ is determined by its Laplacian spectrum. Applying similar arguments for the trees $T_1(l_1, l_2, l_3)$ and $T_3(l_1, l_2, l_3)$ one can obtain that any the generalized T-shape tree on n vertices is determined by its Laplacian spectrum.

Keywords: T-shape tree; Laplacian spectra; cospectral graphs

1 Introduction

Throughout this paper, we concern only with simple undirected graphs (loops and multiple edges are not allowed). Let $G = (V, E)$ be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$. Let $A(G)$ be the $(0,1)$-adjacent matrix of G, the *Laplacian matrix* of G is defined to be $L(G) = D(G) - A(G)$, where D is the diagonal matrix of vertex degrees. The polynomials $\phi(G;\lambda) = \det(\lambda I - A(G))$ and $\psi(G;\mu) = \det(\mu I - L(G))$, where I is the identity matrix, are the *characteristic polynomials* of G w.r.t. $A(G)$ and $L(G)$. The adjacent spectrum (Laplacian spectrum) of G consists of all the eigenvalues (together with their multiplicities) of matrix $A(G)$ (or $L(G)$). Since $A(G)$ and $L(G)$ are real symmetric matrices, their eigenvalues are real numbers. So we can assume that $\lambda_1 \geq \lambda_2 \geq \cdots \geq$

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λ_n and $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n = 0$ are the adjacent and the Laplacian eigenvalues of G, respectively. Two graphs are said to be *cospectral* w.r.t. adjacent (Laplacian) matrix if they have the same adjacent (Laplacian) spectrum. A graph G is said to be determined by its adjacent (Laplacian) spectrum (DAS/DLS for short) if for any graph H, $\phi(G;\lambda) = \phi(H;\lambda)$ (or $\psi(G;\mu) = \psi(H;\mu)$) implies that H is isomorphic to G.

Determining which graphs are determined by their spectrum is a difficult problem in the theory of graph spectra. In [3], van Dam and Haemers proposed the following more modest problem: which trees are determined by their spectrum? It is proved by Shen et al. [6] that Z_n and some graphs related to Z_n are determined by their adjacent spectrum as well as Laplacian spectrum. W.Wang et al. [7] proved that the T-shape tree is determined by Laplacian spectrum. G. R. Omidi et al. [8] showed that starlike trees determined by their Laplacian spectrum. Subsequently, Slobodan K. Simiċ et al. [9] characterized that some forests determined by their Laplacian or signless Laplacian spectrum. In [10], Fan et al. given a Laplacian spectral characterization of $T_3(l_1, l_2, l_3)$. For a recent survey of the subject, one can consult [11].

A *T-shape tree* $T(l_1, l_2, l_3)$ is obtained from three paths P_{l_1+1}, P_{l_2+1} and P_{l_3+1} by identifying one of their pendent vertices (see [7]). A *generalized T-shape tree* $T_s(l_1, l_2, l_3)$ is obtained from $T(l_1, l_2, l_3)$ by appending two pendent vertices to s ($1 \leq s \leq 3$) pendent vertices of $T(l_1, l_2, l_3)$, where $n = l_1+l_2+l_3+2s+1$. In Figure 1, $T_2(l_1, l_2, l_3)$ is showed and $\mathbb{T}_2(k, l, m)$ can be viewed as its line graph; in Figure 3, the tree $T_1(l_1, l_2, l_3)$ and $T_3(l_1, l_2, l_3)$ are showed. An *open quipu* is a tree G of maximum degree 3 such that all the vertices of degree 3 lied on a path. In Figure 1, we describe an open quipu $OQ_3(e_1, e_2, e_3, e_4, e_5, e_6, e_7)$ and $\mathbb{Q}_3(b_1, b_2, b_3, b_4, b_5, b_6, b_7)$ can be viewed as its line graph. Similarly, we can describe $OQ_2(e_1, e_2, e_3, e_4, e_5)$ and $\mathbb{Q}_2(b_1, b_2, b_3, b_4, b_5)$. The degree sequence of a graph G is written as $d(G) = (0^{x_0}, 1^{x_1}, \ldots, k^{x_k}, \ldots, \Delta^{x_\Delta})$ where k^{x_k} means that G has x_k vertices of degree k and $x_0+x_1+\cdots+x_\Delta = n$. The notions and symbols not defined here are standard, see [1] for any undefined terms.

In this paper, we complete the Laplacian spectral characterization of the generalized T-shape tree $T_2(l_1, l_2, l_3)$. Using a similar argument as in the proof of Theorem 3.1, one can prove that the generalized T-shape tree $T_1(l_1, l_2, l_3)$ is also determined by its Laplacian spectrum. Thus, combining the result on the tree $T_3(l_1, l_2, l_3)$ in literature [10], we obtain that any the generalized T-shape tree on n vertices is determined by its Laplacian spectrum.

The paper is organized as follows. In Section 2, some useful lemmas are cited. In Section 3, we first proved that no two non-isomorphic graphs $\mathbb{T}_2(k, l, m)$ are cospectral with respect to the adjacency matrix, and the degree sequence of graph which cospectral to the generalized T-shape tree

$T_2(l_1, l_2, l_3)$ with respect to Laplacian spectrum is determined. At last, we prove that the generalized T-shape tree $T_2(l_1, l_2, l_3)$ is determined by its Laplacian spectrum. Further, we show that any the generalized T-shape tree on n vertices is determined by its Laplacian spectrum.

2 Preliminaries

In this section we give some useful lemmas that are needed in the next section.

Lemma 2.1 ([1] pp.37). *For any vertex u of a graph G, $C(u)$ be the sets of all cycles Z containing u. Then*

$$P_G(\lambda) = \lambda P_{G-u}(\lambda) - \textstyle\sum_{uv \in E(G)} P_{G-u-v}(\lambda) - 2\sum_{Z \in C(u)} P_{G-V(Z)}(\lambda).$$

We summarize some results of [3] and [4] in the following lemma.

Lemma 2.2 ([3][4]). *Let G be a graph. For the adjacency matrix and the Laplacian matrix, the following can be obtained from the spectrum.*
(i) The number of vertices.
(ii) The number of edges.
For the adjacency matrix the following follows from the spectrum.
(iii) The number of closed walks of any length.
(iv) Whether G is bipartite.
For the Laplacian matrix the following follows from the spectrum.
(v) The number of spanning trees.
(vi) The number of components.
(vii) The sum of the square of degrees of vertices.

The *line graph* $\mathcal{L}(G)$ of graph G is a graph whose vertices corresponding the edges of G, and where two vertices are adjacent iff the corresponding edges of G are adjacent.

Lemma 2.3 ([2]). *For two graphs G and H, if $\mathcal{L}(G) \cong \mathcal{L}(H)$ with $\{G, H\} \neq \{K_3, K_{1,3}\}$, then $G \cong H$.*

Lemma 2.4 ([1]). *Let P_n denote the path on n vertices. Then*

$$\phi(P_n; \lambda) = \prod_{j=1}^{n} (\lambda - 2\cos\frac{\pi j}{n+1}) = \frac{\sin\left((n+1)\arccos\lambda/2\right)}{\sin\left(\arccos\lambda/2\right)}.$$

Let $\lambda = 2\cos\theta$, set $t^{1/2} = e^{i\theta}$, it is useful to write the characteristic polynomial of P_n in the following form (see [7]):

$$\phi(P_n; t^{1/2} + t^{-1/2}) = t^{-n/2}(t^{n+1} - 1)/(t - 1). \tag{1}$$

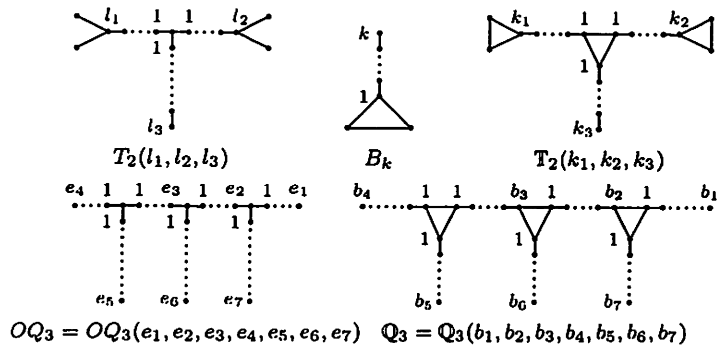


Figure 1: Graphs $T_2(l_1, l_2, l_3)$, B_k, $\mathbb{T}_2(k_1, k_2, k_3)$, OQ_3 and $\mathbb{Q}_3$

Lemma 2.5 ([5]). *Let T be a tree with n vertices and $\mathcal{L}(T)$ be its line graph. Then for $i = 1, 2, \cdots, n-1$, $\mu_i(T) = \lambda_i(\mathcal{L}(T)) + 2$.*

Lemma 2.6 ([8]). *If two trees T and T^* are cospectral w.r.t. the Laplacian matrix, then $\mathcal{L}(T)$ and $\mathcal{L}(T^*)$ are cospectral w.r.t. the adjacency matrix.*

Let $N_G(H)$ be the number of subgraphs of graph G which are isomorphic to H and let $N_G(i)$ be the number of closed walks of length i in G. Let $N'_H(i)$ be the number of closed walks of H of length i which contain all the edges of H and $S_i(G)$ be the set of all the connected subgraphs H of G such that $N'_H(i) \neq 0$. Then

$$N_G(i) = \sum_{H \in S_i(G)} N_G(H) N'_H(i). \tag{2}$$

Based on above Eq.(2) it provides some formulae for calculating the number of closed walks of length 2, 3, 4, 5 for any graphs and of length 6, 7 for graphs without cycles $C_i (i = 4, 5, 6, 7, 8)$.

Lemma 2.7 ([12]). *The number of closed walks of length $k (k = 2, 3, 4, 5, 6, 7)$ of a graph G without cycles $C_i (i = 4, 5, 6, 7, 8)$ are given in the following, where m is the number of edges of graph G.*

(*i*) $N_G(2) = 2m, \quad N_G(3) = 6N_G(C_3)$.
(*ii*) $N_G(4) = 2m + 4N_G(P_3)$.
(*iii*) $N_G(5) = 30N_G(C_3) + 10N_G(G_1)$.
(*iv*) $N_G(6) = 2m + 12N_G(P_3) + 6N_G(P_4) + 12N_G(K_{1,3}) + 24N_G(C_3) + 24N_G(G_5)$.
(*v*) $N_G(7) = 126N_G(C_3) + 84N_G(G_1) + 14N_G(G_2) + 14N_G(G_3) + 28N_G(G_4)$ (*see Figure* 2).

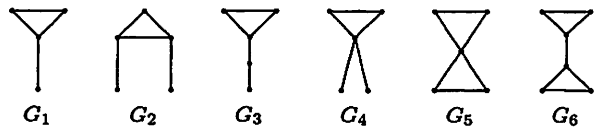


Figure 2: Some related graphs

Lemma 2.8 ([15]). *Let G be a graph with $V(G) \neq 0$ and $E(G) \neq 0$. Then*

$$\Delta(G) + 1 \leq \mu_1 \leq \max\{\frac{d_u(d_u + m_u) + d_v(d_v + m_v)}{d_u + d_v}, uv \in V(G)\}$$

where $\Delta(G)$ denote the maximum vertex degree of G, and m_v the average of degrees of the vertices adjacent to the vertex v in G.

Lemma 2.9 ([3]). *For bipartite graphs, the sum of cubes of degrees is determined by the Laplacian spectrum.*

Lemma 2.10 ([16]). *For a graph G, let H be a subgraph of G. Then $\mu_1(H) \leq \mu_1(G)$.*

Lemma 2.11 ([16]). *Let G be a connected bipartite graph. If G_{uv} is the graph obtained from G by subdividing the edge uv of G, then we have:*
(i) If $G = C_{2k}$, then $\mu_1(G_{uv}) < \mu_1(G) = 4$.
(ii) If uv is not in an internal path of $G \neq C_{2k}$, then $\mu_1(G_{uv}) > \mu_1(G)$.
(iii) If uv belongs to an internal path of G, then $\mu_1(G_{uv}) < \mu_1(G)$.

3 Main Results

Let B_k be the graph consisting of K_3 and P_{k-1} by identifying one vertex of K_3 and one end vertex of P_{k-1}, and the graph $\mathbb{T}_2(k, l, m)$ be the graph consisting of B_{k-1}, B_{l-1} and P_{m-1} which is showed in Figure 1.

Lemma 3.1. *No two non-isomorphic graphs $\mathbb{T}_2(k, l, m)$ are cospectral with respect to the adjacency matrix.*

Proof. Suppose that $G = \mathbb{T}_2(k_1, k_2, k_3)$ and $H = \mathbb{T}_2(l_1, l_2, l_3)$ are cospectral with respect to the adjacency matrix. Then

$$k_1 + k_2 + k_3 = l_1 + l_2 + l_3. \tag{3}$$

Without loss of generality we assume that $k_1 \leq k_2$ and $l_1 \leq l_2$, and will show that G and H are isomorphic. Let $\phi(B_k) = \phi(B_k; \lambda)$, $\phi(P_k) = \phi(P_k; \lambda)$ and

$\phi(G)=\phi(G;\lambda)$ be the characteristic polynomials of graphs B_k, P_k and G, respectively. By Lemma 2.1,

$$\phi(B_k)=(\lambda-2)(\lambda+1)^2\phi(P_{k-1})-(\lambda^2-1)\phi(P_{k-2}); \tag{4}$$

$$\begin{aligned}\phi(G)=&\ \phi(B_{k_1-1})\phi(B_{k_2-1})\phi(P_{k_3-1})\phi(K_3)-\phi(K_2)(\phi(B_{k_1-2})\phi(B_{k_2-1})\phi(P_{k_3-1})\\ &+\phi(B_{k_1-1})\phi(B_{k_2-2})\phi(P_{k_3-1})+\phi(B_{k_1-1})\phi(B_{k_2-1})\phi(P_{k_3-2}))\\ &+\lambda(\phi(B_{k_1-1})\phi(B_{k_2-2})\phi(P_{k_3-2})+\phi(B_{k_1-2})\phi(B_{k_2-1})\phi(P_{k_3-2})\\ &+\phi(B_{k_1-2})\phi(B_{k_2-2})\phi(P_{k_3-1}))-\phi(B_{k_1-2})\phi(B_{k_2-2})\phi(P_{k_3-2}).\end{aligned} \tag{5}$$

From Eqs.(1), (4) and (5), it can be computed by using Maple 9.5 that

$$\phi(G;t^{1/2}+t^{-1/2})(t-1)^3t^{\frac{k_1+k_2+k_3+4}{2}}=\eta_1(t)+\eta_2(t)+\eta_3(t)+\eta_4(t), \tag{6}$$

where

$$\begin{aligned}\eta_1(t)=&-1+5t+6t^{\frac{3}{2}}-5t^2-20t^{\frac{5}{2}}-17t^3+6t^{\frac{7}{2}}+25t^4+24t^{\frac{9}{2}}+11t^5+2t^{\frac{11}{2}};\\ \eta_2(t)=&\ (t^{k_1}+t^{k_2})(t+4t^{\frac{3}{2}}+5t^2-2t^{\frac{5}{2}}-15t^3-22t^{\frac{7}{2}}-15t^4-2t^{\frac{9}{2}}+5t^5\\ &+4t^{\frac{11}{2}}+t^6)+t^{k_3}(-t-2t^{\frac{3}{2}}+t^2+8t^{\frac{5}{2}}+11t^3+2t^{\frac{7}{2}}-13t^4-20t^{\frac{9}{2}}\\ &-15t^5-6t^{\frac{11}{2}}-t^6);\\ \eta_3(t)=&\ t^{k_1+k_2}(t+6t^{\frac{3}{2}}+15t^2+20t^{\frac{5}{2}}+13t^3-2t^{\frac{7}{2}}-11t^4-8t^{\frac{9}{2}}-t^5+2t^{\frac{11}{2}}\\ &+t^6)+(t^{k_1+k_3}+t^{k_2+k_3})(-t-4t^{\frac{3}{2}}-5t^2+2t^{\frac{5}{2}}+15t^3+22t^{\frac{7}{2}}+15t^4\\ &+2t^{\frac{9}{2}}-5t^5-4t^{\frac{11}{2}}-t^6);\\ \eta_4(t)=&\ t^{k_1+k_2+k_3}(-2t^{\frac{3}{2}}-11t^2-24t^{\frac{5}{2}}-25t^3-6t^{\frac{7}{2}}+17t^4+20t^{\frac{9}{2}}+5t^5\\ &-6t^{\frac{11}{2}}-5t^6+t^7).\end{aligned}$$

Analogously to above discussion we have

$$\phi(H;t^{1/2}+t^{-1/2})(t-1)^3t^{\frac{l_1+l_2+l_3+4}{2}}=\xi_1(t)+\xi_2(t)+\xi_3(t)+\xi_4(t), \tag{7}$$

where $\xi_1(t)$, $\xi_2(t),\xi_3(t)$ and $\xi_4(t)$ are obtained from $\xi_1(t)$, $\eta_2(t),\eta_3(t)$ and $\eta_4(t)$ by replacing the parameters k_1,k_2 and k_3 with l_1,l_2 and l_3, respectively. From (3) we have $\xi_1(t)=\eta_1(t)$ and $\xi_4(t)=\eta_4(t)$. By comparing Eqs.(6) and (7), we have

$$\eta_2(t)+\eta_3(t)=\xi_2(t)+\xi_3(t). \tag{8}$$

From Eq.(8) we have

$$\begin{aligned}&(t^{k_1}+t^{k_2})g_1(t)+t^{k_3}g_2(t)+t^{k_1+k_2}h_1(t)+(t^{k_1+k_3}+t^{k_2+k_3})h_2(t)\\ &=(t^{l_1}+t^{l_2})g_1(t)+t^{l_3}g_2(t)+t^{l_1+l_2}h_1(t)+(t^{l_1+l_3}+t^{l_2+l_3})h_2(t).\end{aligned} \tag{9}$$

where

$$\begin{aligned}
g_1(t) &= t + 4t^{\frac{3}{2}} + 5t^2 - 2t^{\frac{5}{2}} - 15t^3 - 22t^{\frac{7}{2}} - 15t^4 - 2t^{\frac{9}{2}} + 5t^5 + 4t^{\frac{11}{2}} + t^6;\\
g_2(t) &= -t - 2t^{\frac{3}{2}} + t^2 + 8t^{\frac{5}{2}} + 11t^3 + 2t^{\frac{7}{2}} - 13t^4 - 20t^{\frac{9}{2}} - 15t^5 - 6t^{\frac{11}{2}} - t^6;\\
h_1(t) &= t + 6t^{\frac{3}{2}} + 15t^2 + 20t^{\frac{5}{2}} + 13t^3 - 2t^{\frac{7}{2}} - 11t^4 - 8t^{\frac{9}{2}} - t^5 + 2t^{\frac{11}{2}} + t^6;\\
h_2(t) &= -t - 4t^{\frac{3}{2}} - 5t^2 + 2t^{\frac{5}{2}} + 15t^3 + 22t^{\frac{7}{2}} + 15t^4 + 2t^{\frac{9}{2}} - 5t^5 - 4t^{\frac{11}{2}} - t^6.
\end{aligned}$$

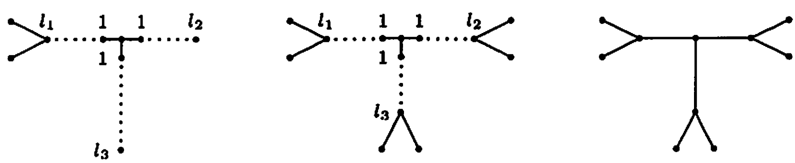
Taking the derivative for both sides of Eq.(9) with respect to t, and set $t = 1$, one can get that

$$2(k_3 - l_3) + 3(k_1 + k_2 - l_1 - l_2) = 0. \tag{10}$$

From Eqs.(3) and (10) we have $k_3 = l_3$. Removing the identical terms from both sides of Eq.(9) we obtain that

$$\begin{aligned}
&(t^{k_1} + t^{k_2})g_1(t) + (t^{k_1+k_3} + t^{k_2+k_3})h_2(t)\\
&\quad = (t^{l_1} + t^{l_2})g_1(t) + (t^{l_1+l_3} + t^{l_2+l_3})h_2(t).
\end{aligned} \tag{11}$$

Now, the leading term of the left side and that of the right side of Eq.(11) are $t^{k_2+k_3+6}$ and $t^{l_2+l_3+6}$, respectively. So we have $k_2 = l_2$. Obviously, $k_1 = l_1$ by Eq.(3). Thus, G and H are isomorphic. The proof is completed. □



$T_1(l_1, l_2, l_3)$ $T_3(l_1, l_2, l_3)$ $T_3(1, 1, 1)$

Figure 3: Graphs $T_1(l_1, l_2, l_3)$, $T_3(l_1, l_2, l_3)$ and $T_3(1, 1, 1)$

Here, we consider the Laplacian spectral radii of tree $T_2(l_1, l_2, l_3)$.

Lemma 3.2. *Let G be the generalized T-shape tree $T_2(l_1, l_2, l_3)$. Then*

$$\mu_1(T_2(l_1, l_2, l_3)) < 5.$$

Proof. Obviously, $T_2(l_1, l_2, l_3)$ is a subgraph of $T_3(l_1, l_2, l_3)$, and $T_3(l_1, l_2, l_3)$ can be obtained from $T_3(1, 1, 1)$ by subdividing each internal path. In the light of Lemmas 2.10 and 2.11 (iii) we get

$$\begin{aligned}
\mu_1(G) &= \mu_1(T_2(l_1, l_2, l_3)) \le \mu_1(T_3(l_1, l_2, l_3)) < \mu_1(T_3(1, l_2, l_3))\\
&< \mu_1(T_3(1, 1, l_3)) < \mu_1(T_3(1, 1, 1)) = 5.
\end{aligned}$$

Thus, the lemma holds. □

Lemma 3.3. *Let G be a connected bipartite graph, and H be a graph cospectral to G with respect to Laplacian spectrum. If $\lfloor \mu_1(G) \rfloor \leq 5$, then the degree sequence of H is determined by the shared spectrum, where $\lfloor x \rfloor$ is taken the greatest integer less than or equal to x.*

Proof. Let x_i and y_i be the number of vertices of degree i in G and H, respectively. By Lemmas 2.2 and 2.10, we have $\sum_{1\leq i\leq \Delta} x_i = \sum_{1\leq i\leq \Delta} y_i$, $\sum_{1\leq i\leq \Delta} ix_i = \sum_{1\leq i\leq \Delta} iy_i$, $\sum_{1\leq i\leq \Delta} i^2x_i = \sum_{1\leq i\leq \Delta} i^2y_i$ and $\sum_{1\leq i\leq \Delta} i^3x_i = \sum_{1\leq i\leq \Delta} i^3y_i$. Because of $\mu_1(G) = \mu_1(H)$, by Lemma 2.8 we obtain that $\Delta(H) \leq 4$. So

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = y_1 + y_2 + y_3 + y_4, \\ x_1 + 2x_2 + 3x_3 + 4x_4 = y_1 + 2y_2 + 3y_3 + 4y_4, \\ x_1 + 4x_2 + 9x_3 + 16x_4 = y_1 + 4y_2 + 9y_3 + 16y_4, \\ x_1 + 8x_2 + 27x_3 + 64x_4 = y_1 + 8y_2 + 27y_3 + 64y_4. \end{cases}$$

From the nonsingularity of vandermonde determinant one can easily obtain that $x_i = y_i$ for $i = 1, 2, 3, 4$. Thus, the degree sequence of H is determined by the shared spectrum. □

According to Lemma 2.2 we known that if a graph H Lalacian cospectral to the generalized T-shape tree $T_2(l_1, l_2, l_3)$, then H is also a tree. Moreover, by Lemmas 3.2 and 3.3 graph H has the same degree sequence as $T_2(l_1, l_2, l_3)$. By the degree sequence one can easily gets the following result.

Lemma 3.4. *For the generalized T-shape tree $T_2(l_1, l_2, l_3)$, let H be a graph Laplacian cospectral with $T_2(l_1, l_2, l_3)$. Then H may be a T_2-shape tree or OQ_3.*

Note that the line graphs of $T_2(l_1, l_2, l_3)$ and OQ_3, respectively, like graphs $\mathbb{T}_2(a_1, a_2, a_3)$ and $\mathbb{Q}_3(b_1, \ldots, b_7)$. For the convenience, we firstly prove that the following lemma holds.

Lemma 3.5. *For graphs $\mathbb{T}_2(a_1, a_2, a_3)$ and $\mathbb{Q}_3(b_1, \ldots, b_7)$, there don't exist adjacency cospectral non-isomorphic pairs for the distinct parameters a_i and b_j, where $i = 1, 2, 3$ and $j = 1, 2, \ldots, 7$.*

Proof. Let $\mathbb{G} = \mathbb{T}_2(a_1, a_2, a_3)$, without loss of generality assume that $a_1 \leq a_2$ in the following, we distinguish three cases to discuss it bellow.

Case 1. $a_1 \geq 2$.

Subcase 1.1. If $a_3 \geq 2$, let $\mathbb{H}$ be a $\mathbb{Q}_3(b_1, \ldots, b_7)$ that is cospectral with $\mathbb{G}$ w.r.t. adjacent matrix. Obviously, $\Delta(\mathbb{G}) = 3$ and $\Delta(\mathbb{H}) \leq 4$. Let y_i be the number of vertices of degree i in $\mathbb{H}$, by Lemma 2.2 graphs $\mathbb{G}$ and

$\mathbb{H}$ have the same number of vertices, edges and triangles. So we have

$$\begin{cases} y_1+y_2+y_3+y_4=n, \\ y_1+2y_2+3y_3+4y_4=2(n+2), \\ y_2+\binom{3}{2}y_3+\binom{4}{2}y_4=5\binom{3}{2}+n-6. \end{cases} \quad (12)$$

From the system of linear equation one has $(y_1,y_2,y_3,y_4)=(1-y_4,n-6+3y_4,5-3y_4,y_4)$. Moreover, $y_4=0,1$ since $y_i\geq 0$ for $i=1,2,3,4$. Thus, the degree sequence of $\mathbb{H}$ may be $(1^1,2^{n-6},3^5)$ and $(2^{n-3},3^2,4^1)$.

Obviously, $(2^{n-3},3^2,4^1)$ is a degree sequence of $\mathbb{Q}_3(b_1,\dots,b_7)$ if and only if one of parameters b_2,b_3 is no less than 2, the others equal to 1. Let $\mathbb{H}_1$ be the graph $\mathbb{Q}_3(b_1,\dots,b_7)$ with the degree sequence $(2^{n-3},3^2,4^1)$. Since $N_{\mathbb{H}_1}(C_3)=N_{\mathbb{G}}(C_3)=3$, but $N_{\mathbb{H}_1}(G_1)=6$ and $N_{\mathbb{G}}(G_1)=5$, by (iii) of Lemma 2.7 we get $N_{\mathbb{H}_1}(5)\neq N_{\mathbb{G}}(5)$. Thus, it is impossible.

$(1^1,2^{n-6},3^5)$ is a degree sequence of $\mathbb{Q}_3(b_1,\dots,b_7)$ if and only if there at most exists one parameter in $\{b_1,b_4,b_5,b_7\}$ that is no less than 2. Meanwhile, $b_2,b_3\geq 2$. Suppose that $b_4\geq 2$ or $b_6\geq 2$ by the symmetry. For graph $\mathbb{Q}_3(1,b_2,b_3,1,1,b_6,1)$, by Lemma 3.1 $\mathbb{G}\cong\mathbb{Q}_3(1,b_2,b_3,1,1,b_6,1)$ if $a_3\geq 2$ and $b_6\geq 2$.

Let $\mathbb{H}_2=\mathbb{Q}_3(1,b_2,b_3,b_4,1,1,1,1)$, we prove that $\mathbb{G}$ ($a_3\geq 2$) and $\mathbb{H}_2$ ($b_2,b_3,b_4\geq 2$) are not cospectral w.r.t. adjacency matrix.

Assume that $\mathbb{G}$ and $\mathbb{H}_2$ have the same adjacent spectrum, then they have the same number of vertices and the same characteristic polynomials. Let $\phi(\mathbb{G};\lambda)$ and $\phi(\mathbb{H}_2;\lambda)$ be the characteristic polynomials of $\mathbb{G}$ and $\mathbb{H}_2$, respectively. Let v be the degree 3 vertex linked the path P_{b_4} in $\mathbb{H}_2$, by Lemma 2.1 we have

$$\begin{aligned}\phi(\mathbb{H}_2,\lambda) &= (\lambda\phi(P_{b_4-1})-\phi(P_{b_4-2}))\phi(D_{b_3+1})-\phi(P_{b_4-1})\phi(D_{b_3}) \\ &\quad -(\lambda-2)\phi(P_{b_4-1})\phi(D_{b_3-1}),\end{aligned} \quad (13)$$

where
$\phi(D_k)=\lambda\phi(B_{b_2+k})-\phi(P_{k-1})\phi(B_{b_2})-\phi(P_k)\phi(B_{b_2-1})-2\phi(P_{k-1})\phi(B_{b_2-1})$.
From Eqs.(1), (4) and (13), it can be computed by using Maple 9.5 that

$$\phi(\mathbb{H}_2;t^{1/2}+t^{-1/2})(t-1)^3t^{\frac{b_2+b_3+b_4+4}{2}}=\varphi_1(t)+\varphi_2(t)+\varphi_3(t)+\varphi_4(t), \quad (14)$$

where

$$\begin{aligned}
\varphi_1(t) &= -1+5t+2t^{\frac{3}{2}}-5t^2-8t^{\frac{5}{2}}-2t^3+4t^{\frac{7}{2}}+t^4+t^5+2t^{\frac{11}{2}}+t^6; \\
\varphi_2(t) &= t^{b_2}(t+4t^{\frac{3}{2}}+4t^2-2t^{\frac{5}{2}}-5t^3-2t^{\frac{7}{2}}-t^4+t^5+t^6) \\
&\quad +t^{b_3}(t-3t^2-2t^{\frac{5}{2}}+2t^3+4t^{\frac{7}{2}}+t^4-2t^{\frac{9}{2}}-5t^5) \\
&\quad +t^{b_4}(-t+2t^{\frac{3}{2}}+2t^2-2t^{\frac{5}{2}}+5t^3-2t^{\frac{7}{2}}+5t^4+4t^{\frac{9}{2}}-2t^{\frac{11}{2}}-t^6); \\
\varphi_3(t) &= t^{b_2+b_4}(t^2+2t^{\frac{5}{2}}-3t^3-4t^{\frac{7}{2}}-2t^4+2t^{\frac{9}{2}}+3t^5-t^6) \\
&\quad +t^{b_3+b_4}(-t-2t^2-t^3+2t^{\frac{7}{2}}+5t^4+2t^{\frac{9}{2}}-4t^5-4t^{\frac{11}{2}}-t^6) \\
&\quad +t^{b_2+b_3}(-t+2t^{\frac{3}{2}}-4t^{\frac{5}{2}}-5t^3+2t^{\frac{7}{2}}+5t^4+2t^{\frac{9}{2}}-2t^5-2t^{\frac{11}{2}}+t^6); \\
\varphi_4(t) &= t^{b_2+b_3+b_4}(-t-2t^{\frac{3}{2}}-t^2-t^3-4t^{\frac{7}{2}}+2t^4+8t^{\frac{9}{2}}+5t^5-2t^{\frac{11}{2}}-5t^6+t^7).
\end{aligned}$$

From Eq.(6) we have $\phi(\mathbb{G}; t^{1/2}+t^{-1/2})(t-1)^3 t^{\frac{a_1+a_2+a_3+4}{2}} = \eta_1(t)+\eta_2(t)+\eta_3(t)+\eta_4(t)$. Note that $a_1+a_2+a_3 = b_2+b_3+b_4$, it is easy to check that $\phi(\mathbb{G}; t^{1/2}+t^{-1/2}) \neq \phi(\mathbb{H}_2; t^{1/2}+t^{-1/2})$. Further, $\phi(\mathbb{G}; \lambda) \neq \phi(\mathbb{H}_2; \lambda)$, a contradiction.

Subcase 1.2. If $a_3 = 1$, let $\mathbb{H}$ be a graph that is cospectral with $\mathbb{G}$ w.r.t. adjacenct matrix. Similar to Eq.(12), we have the degree sequence of $\mathbb{H}$ is $(1^0, 2^{n-4}, 3^4, 4^0)$. From the Lemma 3.1 we know that $\mathbb{G} \cong \mathbb{Q}_3(1, b_2, b_3, 1, 1, 1, 1)$.

Case 2. $a_1 = 1$, $a_2 \geq 2$.

Subcase 2.1. If $a_3 \geq 2$, let $\mathbb{H}$ be a $\mathbb{Q}_3(b_1, \ldots, b_7)$ that is cospectral with $\mathbb{G} = \mathbb{T}_2(1, a_2, a_3)$ w.r.t. adjacent matrix. By Lemma 2.2 we know that graphs $\mathbb{G}$ and $\mathbb{H}$ have the same number of vertices, edges and triangles, one can easily obtain that the degree sequence of Q is either $(1^2, 2^{n-8}, 3^6)$, $(1^1, 2^{n-5}, 3^3, 4^1)$ or $(2^{n-2}, 4^2)$.

Let $\mathbb{H}_1$ and $\mathbb{H}_2$ be the graph $\mathbb{Q}_3(b_1, \ldots, b_7)$ with the degree sequence $(1^2, 2^{n-8}, 3^6)$ and $(2^{n-2}, 4^2)$, respectively. By Figure 1 we get that $N_{\mathbb{G}}(C_3) = N_{\mathbb{H}_1}(C_3) = N_{\mathbb{H}_2}(C_3) = 3$, but, $N_{\mathbb{G}}(G_1) = 7$, $N_{\mathbb{H}_1}(G_1) = 6$, $N_{\mathbb{H}_2}(G_1) = 8$. It follows from (iii) of Lemma 2.7 that $N_{\mathbb{G}}(5) \neq N_{\mathbb{H}_1}(5)$, and $N_{\mathbb{G}}(5) \neq N_{\mathbb{H}_2}(5)$. Thus, $\mathbb{G}$ is not adjacency cospectral with $\mathbb{H}_1$ and $\mathbb{H}_2$, respectively.

$(1^1, 2^{n-5}, 3^3, 4^1)$ is a degree sequence of $\mathbb{Q}_3(b_1, \ldots, b_7)$ if and only if there is a parameter in $\{b_2, b_3\}$ equals to 1, the other is no less than 2. Meanwhile, one of $\{b_1, b_4, b_5, b_6\}$ is greater than 1, the others are equal to 1. Without loss of generality, we assume that $b_2 \geq 2$, $b_3 = 1$ and $b_6 \geq 2$, or $b_5 \geq 2$, or $b_7 \geq 2$.

For the graph $\mathbb{Q}_3(1, b_2, 1, 1, 1, b_6, 1)$, by Lemma 3.1 $\mathbb{Q}_3(1, b_2, 1, 1, 1, b_6, 1) \cong \mathbb{G}$.

For graphs $\mathbb{H}_3 = \mathbb{Q}_3(1, b_2, 1, 1, b_5, 1, 1)$, let $\phi(\mathbb{H}_3; \lambda)$ be the characteristic polynomial of $\mathbb{H}_3$. Applying Lemma 2.1 at vertex of degree 4 in $\mathbb{H}_3$ we have

$$\begin{aligned}\phi(\mathbb{H}_3; \lambda) = {} & \lambda\phi(P_{b_5+1})\phi(B_{b_3+1}) - (\phi(P_{b_5})\phi(B_{b_3+1}) + \lambda\phi(P_{b_5-1})\phi(B_{b_3+1}) \\ & +\phi(P_{b_5+1})\phi(B_{b_3}) + \lambda\phi(P_{b_5+1})\phi(B_{b_3-1})) - 2(\phi(P_{b_5+1})\phi(B_{b_3+1}) \\ & +\phi(P_{b_5+1})\phi(B_{b_3-1})).\end{aligned} \tag{15}$$

From Eqs.(1), (4) and (15), it can be computed by using Maple 9.5 that

$$\phi(\mathbb{H}_3; t^{1/2}+t^{-1/2})(t-1)^3 t^{\frac{b_3+b_5+5}{2}} = \tau_1(t)+\tau_2(t)+\tau_3(t), \tag{16}$$

where

$$\begin{aligned}\tau_1(t) &= -1 + 2t^{\frac{1}{2}} + 5t + 8t^2 - 12t^{\frac{5}{2}} - 5t^3 + 4t^{\frac{7}{2}} + 7t^4 + 6t^{\frac{9}{2}} + 2t^5; \\ \tau_2(t) &= t^{b_3}(t + 4t^{\frac{3}{2}} + 5t^2 + 2t^{\frac{5}{2}} - t^3 - 4t^{\frac{7}{2}} - 4t^4 - 2t^{\frac{9}{2}} - t^5 - t^6 + t^7) \\ &\quad +t^{b_5}(-t + t^2 + t^3 + t^{\frac{7}{2}} + 4t^4 + 4t^{\frac{9}{2}} + t^5 - 2t^{\frac{11}{2}} - 5t^6 - 4t^{\frac{13}{2}} - t^7); \\ \tau_3(t) &= t^{b_3+b_5+1}(t^2 - 6t^{\frac{5}{2}} - 7t^3 - 4t^{\frac{7}{2}} + 5t^4 + 12t^{\frac{9}{2}} + 8t^5 - 5t^6 - 2t^{\frac{13}{2}} + t^7).\end{aligned}$$

Obviously, $b_2 + b_5 = a_2 + a_3$. From Eq.(16) we get $\phi(\mathbb{G}; t^{1/2} + t^{-1/2})(t - 1)^3 t^{\frac{a_2+a_3+5}{2}} = \eta_1(t)+\eta_2(t)+\eta_3(t)+\eta_4(t)$. It is easy to check that $\phi(\mathbb{G}; t^{1/2} + t^{-1/2}) \neq \phi(\mathbb{H}_3; t^{1/2} + t^{-1/2})$. Further, $\phi(\mathbb{G}; \lambda) \neq \phi(\mathbb{H}_3; \lambda)$. Thus $\mathbb{G}$ and $\mathbb{H}_3$ have distinct adjacent spectrum.

For graph $\mathbb{H}_4 = \mathbb{Q}_3(1, b_2, 1, 1, 1, 1, b_7)$, let $\phi(\mathbb{H}_4; \lambda)$ be the characteristic polynomial of $\mathbb{H}_4$, we can prove that $\phi(\mathbb{G}; \lambda) \neq \phi(\mathbb{H}_4; \lambda)$ by the same method. So $\mathbb{G}$ and $\mathbb{H}_4$ are not cospectral w.r.t. adjacent matrix.

Subcase 2.2. If $a_3 = 1$, i.e., $\mathbb{G} = \mathbb{T}_2(1, a_2, 1)$. It is easy to obtain that the degree sequence of $\mathbb{Q}_3(b_1, \ldots, b_7)$ is $(1^1, 2^{n-6}, 3^5)$ or $(2^{n-3}, 3^2, 4^1)$.

Let $\mathbb{H}_x$ and $\mathbb{H}_y$ be the graph with the degree sequence of $(1^1, 2^{n-6}, 3^5, 4^0)$ and $(1^0, 2^{n-3}, 3^2, 4^1)$, respectively. From Figure 1 one can easily obtain that $N_{\mathbb{H}_x}(C_3) = N_{\mathbb{G}}(C_3) = 3$, but $N_{\mathbb{H}_x}(G_1) = 5$, $N_{\mathbb{G}}(G_1) = 6$. So, $N_{\mathbb{H}_x}(5) \neq N_{\mathbb{G}}(5)$ by (iii) of Lemma 2.7. Thus, $\mathbb{H}_x$ is not adjacency cospectral with $\mathbb{G}$. For the graphs $\mathbb{H}_y$, it likes graph $\mathbb{G}$, by Lemma 3.1 we known that G and $\mathbb{H}_y$ are isomorphic.

Case 3. $a_1 = a_2 = 1$.

Subcase 3.1. If $a_3 \geq 2$, let $\mathbb{H}$ be a $\mathbb{Q}_3(b_1, \ldots, b_7)$ that is cospectral with $\mathbb{T}_2(1, a_2, a_3)$ w.r.t. adjacent matrix. By Lemma 2.2 we know that graphs $\mathbb{G}$ and $\mathbb{H}$ have the same number of vertices, edges and triangles, one can easily obtain that the degree sequence of $\mathbb{H}$ may be $(1^3, 2^{n-10}, 3^7)$, $(1^2, 2^{n-7}, 3^4, 4^1)$ or $(1^1, 2^{n-4}, 3^1, 4^2)$.

Let $\mathbb{H}_1$, $\mathbb{H}_2$ and $\mathbb{H}_3$ be the graphs with the degree sequence of $(1^3, 2^{n-10}, 3^7)$, $(1^2, 2^{n-7}, 3^4, 4^1)$ and $(1^1, 2^{n-4}, 3^1, 4^2)$, respectively. Since $N_{\mathbb{G}}(C_3) = N_{\mathbb{H}_i}(C_3) = 3$ for $i = 1, 2$, and $N_{\mathbb{G}}(G_1) = 9$. However, $N_{\mathbb{H}_1}(G_1) = 5$, $N_{\mathbb{H}_2}(G_1) = 8$, by (iii) of Lemma 2.7 we have $N_{\mathbb{G}}(5) \neq N_{\mathbb{H}_i}(5)$ for $i = 1, 2$. Hence, $\mathbb{H}_i (i = 1, 2)$ are not cospectral to $\mathbb{G}$ w.r.t. adjacent matrix.

For the graph $\mathbb{H}_3$, if $b_6 \geq 2$ then, by Lemma 3.1 we know that $\mathbb{G}$ and $\mathbb{H}_3$ are isomorphic; otherwise, $b_4 \geq 2$ (see Figure 1). Then

$$N_{\mathbb{G}}(C_3) = 3, N_{\mathbb{G}}(G_1) = 3, N_{\mathbb{G}}(G_2) = 8, N_{\mathbb{G}}(G_3) = 14 \textit{ or } 15, N_{\mathbb{G}}(G_4) = 4,$$

$$N_{\mathbb{H}_3}(C_3) = 3, N_{\mathbb{H}_3}(G_1) = 3, N_{\mathbb{H}_3}(G_2) = 6, N_{\mathbb{H}_3}(G_3) = 13 \textit{ or } 14, N_{\mathbb{H}_3}(G_4)=4.$$

By (v) of Lemma 2.7 we get $N_{\mathbb{H}_3}(7) \neq N_{\mathbb{G}}(7)$. Thus, $\mathbb{H}_3$ is not cospectral to $\mathbb{G}$ w.r.t. adjacency matrix.

Subcase 3.2. If $a_3 = 1$, i.e., $\mathbb{G} = \mathbb{T}_2(1, 1, 1)$. It is easy to obtain that $\mathbb{G}$ does not exists adjacency cospectral non-isomorphic pairs with $\mathbb{Q}_3(b_1, \ldots, b_7)$.

Summarize all situations above, there is no cospectral pairs w.r.t. adjacent spectrum among $T_2(a_1, a_2, a_3)$ and $Q_3(b_1, \ldots, b_7)$, respectively.

□

Using the previous facts, we show that the generalized T-shape tree $T_2(l_1, l_2, l_3)$ is determined by its Laplacian spectrum.

Theorem 3.1. *Let G be the generalized T-shape tree $T_2(l_1, l_2, l_3)$ on n vertices. Then G is determined by its Laplacian spectrum.*

Proof. Let H be any graph that is cospectral with G w.r.t. the Laplacian spectrum. From Lemmas 3.4 and 3.5 one can claim that H is just a T_2-shape tree. According to Lemma 2.6, $\mathcal{L}(H)$ and $\mathcal{L}(G)$ are cospectral w.r.t. adjacency spectrum. Note that $\mathcal{L}(G)$ and $\mathcal{L}(H)$ are graph $\mathbb{T}_2(k_1, k_2, k_3)$. It follows that $\mathcal{L}(G) \cong \mathcal{L}(H)$ by Lemma 3.1. Thus, by Lemma 2.3 graph H is isomorphic to G. So, the proof is completed. □

Theorem 3.2. *Let G be any the generalized T-shape tree with n vertices. Then G is determined by its Laplacian spectrum.*

Proof. For the generalized T-shape tree $T_s(l_1, l_2, l_3)$ $(s = 1, 2, 3)$, we distinguish three cases as follows.

If $s = 1$, the graph $T_1(l_1, l_2, l_3)$ which is shown in Figure 3. Let $\mathbb{T}_1(k_1, k_2, k_3)$ and $\mathbb{Q}_2(b_1, b_2, b_3, b_4, b_5)$ be graphs isomorphic to the line graphs of $T_1(l_1, l_2, l_3)$ and $OQ_2(e_1, e_2, e_3, e_4, e_5)$, respectively. Then the characteristic polynomial of $\mathbb{T}_1(k_1, k_2, k_3)$ is obtained from Eq.(5) by replacing $\phi(B_{k_2-1})$ and $\phi(B_{k_2-2})$, with $\phi(P_{k_2-1})$ and $\phi(P_{k_2-2})$ respectively. Applying a similar argument with the proof of Lemma 3.1 we get that no two non-isomorphic graphs $\mathbb{T}_1(k_1, k_2, k_3)$ are cospectral w.r.t. the adjacent matrix. Let H be a graph cospectral to $T_1(l_1, l_2, l_3)$ w.r.t. Laplacian spectrum. Since $\mu(T_1(l_1, l_2, l_3)) < \mu(T_3(l_1, l_2, l_3)) \leq \mu(T_3(1, 1, 1)) = 5$, by Lemma 3.3 we know that the degree sequence of H is the same as graph $T_1(l_1, l_2, l_3)$. Then H is either graph OQ_2 or T_1-shape tree. Similar to the proof of Lemma 3.5 one can easily prove that $\mathbb{T}_1(a_1, a_2, a_3)$ and $\mathbb{Q}_2(b_1, b_2, b_3, b_4, b_5)$ don't exist adjacency cospectral non-isomorphic pairs for distinct parameters a_i and b_j, where $i = 1, 2, 3$ and $j = 1, 2, \ldots, 5$. Thus, H is just a T_1-shape tree. Further, it is easy to show that $T_1(l_1, l_2, l_3)$ is determined by its Laplacian spectrum.

If $s = 2$, the Laplacian characterization of $T_2(l_1, l_2, l_3)$ see Theorem 3.1.

If $s = 3$, the graph $T_3(l_1, l_2, l_3)$ which is shown in Figure 3, Fan et al. has proved that $T_3(l_1, l_2, l_3)$ is determined by its Laplacian spectrum in [10]. In addition, let $\mathbb{T}_3(k_1, k_2, k_3)$ be the line graph of $T_3(l_1, l_2, l_3)$. Then the characteristic polynomial of $\mathbb{T}_3(k_1, k_2, k_3)$ is obtained from Eq.(5) by replacing $\phi(P_{k_3-1})$ with $\phi(B_{k_3-1})$, and replacing $\phi(P_{k_3-2})$ with $\phi(B_{k_3-2})$, respectively. Applying the same discussion of Lemma 3.1 one can easily show that no two non-isomorphic graphs $\mathbb{T}_3(k_1, k_2, k_3)$ are cospectral with respect to the adjacent matrix, it will simplify the proof of Fan's result in literature [10].

So the proof is completed. □

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