On Component Order Edge Connectivity Of a Complete Bipartite Graph

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Abstract: The traditional parameter used as a measure of vulnerability of a network modeled by a graph with perfect nodes and edges that may fail is edge connectivity λ. For the complete bipartite graph $K_{p,q}$ where $1 \leq p \leq q$, $\lambda(K_{p,q}) = p$. In this case, failure of the network means that the surviving subgraph becomes disconnected upon the failure of individual edges. If, instead, failure of the network is defined to mean that the surviving subgraph has no component of order greater than or equal to some preassigned number k, then the associated vulnerability parameter, the k-component order edge connectivity $\lambda_c^{(k)}$, is the minimum number of edges required to fail so that the surviving subgraph is in a failure state. We determine the value of $\lambda_c^{(k)}\left(K_{p,q}\right)$ for arbitrary $1 \leq p \leq q$ and $4 \leq k \leq p+q$. As it happens, the situation is relatively simple when p is small and more involved when p is large.

1. THE MODEL

Networks are represented by graphs with nodes corresponding to the stations and edges corresponding to the links. We assume $G = (V, E)$ is a simple graph, where V is the set of nodes and E is the set of edges. We use the notation $n(G) = |V|$ for the order of the graph G and $e(G) = |E|$ for the size of the graph G. Unless specifically stated, we follow the standard graph theory notation found in [4]. In addition any numerical value is assumed to be an integer.

In the traditional edge-failure model it is assumed that nodes are perfectly reliable but edges may fail. When a set F of edges fail we refer to F as an **edge-failure set** and the surviving subgraph G-F as an **edge-failure state** if G-F is disconnected.

Definition 1.1: The **edge-connectivity** of G, denoted by $\lambda(G)$ or simply λ, is given by $\lambda(G) = \min\{|F| : F \subseteq E,\ F \text{ is an edge-failure set}\}$.

For example, consider the complete bipartite graph $K_{p,q}$, with $1 \le p \le q$. We will refer to the two maximal sets of independent nodes as the parts of the $K_{p,q}$. It is easily seen that $\lambda(K_{p,q}) = p$, i.e., the order of the smaller part.

One drawback of the traditional edge-failure model is that the graph G-F is an edge-failure state if it is disconnected, and no consideration is given to whether or not there exists a relatively "large" surviving component, which in itself may be capable of performing the desired network function. Therefore in 2006 Boesch et al. [1] introduced a new edge-failure model, the k-component order edge-failure model. In this model, when a set F of edges fail we refer to F as a **k-component order edge-failure set** and the surviving subgraph G-F as a **k-component order edge-failure state** or simply **failure state** if G-F contains no component of order at least k, where k is a predetermined threshold value.

Definition 1.2: Let $2 \le k \le n$ be a predetermined threshold value. The **k-component order edge-connectivity** of G, denoted by $\lambda_c^{(k)}(G)$ or simply $\lambda_c^{(k)}$, is given by $\lambda_c^{(k)}(G) = \min\{|F| : F \subseteq E,\ F \text{ is a } k\text{-component edge-failure set}\}$. We refer to the set of edges F as a **minimum k-component order edge-failure set** and to the resulting graph G-F as a **maximum k-component order edge-failure state (maximum failure state).**

Remarks: 1) The parameter k-component order edge-connectivity can be considered a special case of Harary's conditional edge-connectivity, which is the minimum number of edges whose removal results in a graph whose components satisfy a specified graph theoretic property P. Properties considered by Harary included number of cycles, bounds on the degree, diameter, and Hamiltonicity. He did not consider bounds on the order of the components.

2) Since every 2-component order edge-failure state must consist of isolated nodes and therefore is edgeless, $\lambda_c^{(2)}(G) = e(G)$. The components of any 3-component order edge-failure state must be either independent edges or isolated nodes. It follows that $\lambda_c^{(3)}(G) = e(G) - |M|$, where M is a maximum matching of G. **Thus we will assume that the threshold value k is at least 4.**

Figure 1.3 depicts a maximum 5-component order edge-failure state for $K_{3,7}$. Maximum failure states may not be unique. In fact there exist two other non-isomorphic maximum failure states for $K_{3,7}$ (see Section 6).

Formulas for $\lambda_c^{(k)}(G)$ have been found for specific classes of graphs [1,2,5]. For example, $\lambda_c^{(k)}(P_n)=\left\lfloor \frac{n-1}{k-1} \right\rfloor$, where P_n is the path of order n, and $\lambda_c^{(k)}(K_{1,n-1})=n-k+1$. An algorithm for finding $\lambda_c^{(k)}$ of an arbitrary tree can be found in [6]. No formula or algorithm for finding $\lambda_c^{(k)}(G)$ of an arbitrary graph G has yet to be found. In this work we consider complete bipartite graphs $K_{p,q}$. Our modus operandi is to find a maximum k-component order edge-failure state of $K_{p,q}$; subtracting the number of edges in such a graph from pq would then yield the value of $\lambda_c^{(k)}(K_{p,q})$.

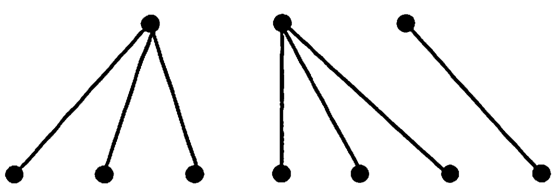

Fig. 1.3 : A maximum 5-component order edge-failure state for $K_{3,7}$

$$\lambda_c^{(5)}(K_{3,7})=21-7=14$$

2. Preliminary Results

We assume that the integers p, q, and k satisfy $1 \le p \le q$ and $4 \le k \le p+q$. We consider complete bipartite graphs $K_{p,q}$. It is clear that if $F \subseteq E$ is a minimum k-component order edge-failure set of $K_{p,q}$ then all components of the maximum failure state $K_{p,q}-F$ are complete bipartite graphs or isolated nodes. If $K_{a,b}$ is such a component, the nodes in the part of order a come from the part of order p of the $K_{p,q}$ and the nodes in the part of order b come from the part of order q of the $K_{p,q}$. Also it is possible that for the component $K_{a,b}$, $a>b$. Finally we use the notation $K_{a,0}$ or $K_{0,b}$ to denote a or b isolated nodes, from the appropriate part of the $K_{p,q}$.

We first establish some lemmas, from which we find possible forms of a maximum failure state. We only provide a proof of the first result as the others are established using a similar method.

Lemma 2.1: There exists a maximum failure state of $K_{p,q}$ with at most one nontrivial component with fewer than $k-1$ nodes.

Proof: Let H be a maximum failure state and suppose that H has two nontrivial components $C_1 = K_{a,b}$ and $C_2 = K_{c,d}$ of order less than k – 1. Assume $2 \le a+b \le c+d \le k-2$. If $a \le c$ replace the components C_1 and C_2 with $C_1' = K_{a,b-1}$ and $C_2' = K_{c,d+1}$, respectively, obtained by moving one node from the b part of C_1 to the d part of C_2; this can always be done since C_1 is nontrivial and thus $b \ge 1$. Note $n(K_{c,d+1}) \le k-1$. Let H' be the resulting failure state. Then $e(H')-e(H) = c-a \ge 0$. If $a > c$, then it follows that $b < d$. In this case replace the components C_1 and C_2 with $C_1' = K_{a-1,b}$ and $C_2' = K_{c+1,d}$, respectively, and let H' be the resulting failure state. Then $e(H')-e(H) = d-b > 0$. If the number of edges increases then it contradicts the assumption that H is maximum. If the number of edges remained the same, then set H equal to H' and repeat the process if H has two components of order less than k - 1. ■

Remark: From Lemma 2.1 we will assume that a maximum -failure state H has at most one nontrivial component of order less than k-1.

Lemma 2.2: If $K_{p_1,k-1-p_1}$ and $K_{p_2,k-1-p_2}$ are nontrivial components in a maximum failure state of $K_{p,q}$, then $|p_2 - p_1| \le 1$.

Remark: Lemma 2.2 implies that each component of order $k-1$ of a maximum failure state is of the form $K_{\beta,k-1-\beta}$ or $K_{\beta+1,k-2-\beta}$.

Lemma 2.3: If a maximum failure state contains an isolated node then all nontrivial components have order $k-1$. Moreover, all isolated nodes come from the same part of the $K_{p,q}$.

As a consequence of the previous lemmas we will assume that the components of a maximum failure state must be of one of the following three types.

Type 1: All components are of order $k-1$, each either of the form $K_{\beta,k-1-\beta}$ or $K_{\beta+1,k-2-\beta}$.

Type 2: All components except for one are of order $k-1$, each either of the form $K_{\beta,k-1-\beta}$ or $K_{\beta+1,k-2-\beta}$. The other component is complete bipartite of order at most $k-2$.

Type 3: All nontrivial components are of order $k-1$, each either of the form $K_{\beta,k-1-\beta}$ or $K_{\beta+1,k-2-\beta}$, along with at least one isolated node.

Lemma 2.4: Let $K_{a,b}$ be the component of order at most $k-2$ in a Type 2 maximum failure state. Then $a \le \beta$ and, if there exists a component of the form $K_{\beta+1,k-2-\beta}$, then $b \le k-2-\beta$; otherwise $b \le k-1-\beta$.

Our final result in this section concerns Type 3 maximum failure states with a "large" number of isolated nodes.

Lemma 2.5: Let H be a Type 3 maximum failure state with at least $k-2$ isolated nodes. Then the nontrivial components of H consists of p copies of $K_{1,k-2}$. Moreover, in this case $p \le \left\lfloor \frac{p+q}{k-1} \right\rfloor$.

We consider two cases for finding maximum failures state of $K_{p,q}$. The first case $p \le \left\lfloor \frac{p+q}{k-1} \right\rfloor$ is simpler and is covered in the next section. The second case $p > \left\lfloor \frac{p+q}{k-1} \right\rfloor$ is more extensive and will be covered in section 4.

3. Case 1: $p \le \left\lfloor \frac{p+q}{k-1} \right\rfloor$

Theorem 3.1: Let H be a maximum failure state of $K_{p,q}$, where $p \le \left\lfloor \frac{p+q}{k-1} \right\rfloor$. Then H consists of p copies of $K_{1,k-2}$ and $q-p(k-2)$ isolated nodes.

Proof: Since $p \le \left\lfloor \frac{p+q}{k-1} \right\rfloor$ it follows that $p(k-1) \le p+q$ or $p(k-2) \le q$. Observe that if t is the number of complete bipartite components of H then

$t \le p$. If $t = p$ each complete bipartite component must be of the specified form and the result follows. We show that $t \le p-1$ cannot occur. Suppose $t \le p-1$; then q_I, the number of nodes from the q-part not in a complete bipartite component, i.e. isolated nodes, satisfies $q_I \ge q - t(k-2) \ge q-(p-1)(k-2) = q - p(k-2) + (k-2) \ge k-2$. Thus the conditions of Lemma 2.5 are satisfied and we obtain the contradictory result that there are p nontrivial components. ■

Thus we see that when $p \le \left\lfloor \frac{p+q}{k-1} \right\rfloor$ maximum failure state must be either of Type 1, i.e., all components are of order $k-1$, or of Type 3, i.e., all nontrivial components are of order $k-1$ and there exists at least one isolated node. The former occurs when $q = p(k-2)$.

We can now give the formula for $\lambda_c^{(k)}(K_{p,q})$ in the case that $p \le \left\lfloor \frac{p+q}{k-1} \right\rfloor$.

Corollary 3.2: If $p \le \left\lfloor \frac{p+q}{k-1} \right\rfloor$ then $\lambda_c^{(k)}(K_{p,q}) = p(q-k+2)$.

4. Case 2: $p > \left\lfloor \frac{p+q}{k-1} \right\rfloor$

We now assume that $p > t = \left\lfloor \frac{p+q}{k-1} \right\rfloor$. Let H be a maximum failure state of $K_{p,q}$. If H is of Type 1 then all components are of order $k-1$ and there are exactly t such components. If H is of Type 2 then there is exactly one component of order less than $k-1$; thus there must be t components of order $k-1$. Finally, if H is of Type 3 then all nontrivial components are of order $k-1$. Since there is at least 1 but at most $k-3$ isolated nodes (by Lemma 2.5), once again there are t components of order $k-1$. Furthermore, each component of order $k-1$ is of the form $K_{\beta,k-1-\beta}$ or $K_{\beta+1,k-2-\beta}$.

Given p, q, and k, set $t = \left\lfloor \frac{p+q}{k-1} \right\rfloor$ and let s satisfy $1 \le s \le t$. Then there exists a maximum failure state of $K_{p,q}$ having the following form:

- s copies of $K_{\beta,k-1-\beta}$,
- $t-s$ copies of $K_{\beta+1,k-2-\beta}$
- possibly one copy of K_{p_l,q_l} where $p_l+q_l \le k-2$. If $p_l = 0$ $(q_l = 0)$, then $q_l \le k-3$ $(p_l \le k-3)$ and K_{p_l,q_l} consists of isolated nodes.

Definition 4.1: Given p, q, and k, the 4-tuple (β, s, p_l, q_l) is **realizable** provided the following conditions hold:

i. $1 \le \beta \le k-2$

ii. $1 \le s \le t = \left\lfloor \frac{p+q}{k-1} \right\rfloor$

iii. $p_l, q_l \ge 0$

iv. $p = s\beta + (t-s)(\beta+1) + p_l$, $q = s(k-1-\beta) + (t-s)(k-2-\beta) + q_l$.

Conditions i. through iv. ensure that $p_l + q_l \le k-2$. Hence the graph $sK_{\beta,k-1-\beta} \cup (t-s)K_{\beta+1,k-2-\beta} \cup K_{p_l,q_l}$ is a failure state of $K_{p,q}$. We will also use the 4-tuple notation to denote this associated failure state and $e((\beta,s,p_l,q_l))$ to denote the number of edges.

Definition 4.2: A realizable 4-tuple (β,s,p_l,q_l) is **potentially optimal** if it also satisfies:

v. if either $p_l = 0$ or $q_l = 0$ then $p_l + q_l \le k-3$

vi. $p_l \le \beta$; if $s = t$ then $q_l \le k-1-\beta$, otherwise $q_l \le k-2-\beta$.

Definition 4.3: A potentially optimal 4-tuple (β,s,p_l,q_l) is **optimal** if the associated failure state is a maximum failure state of $K_{p,q}$.

It is evident that if (β,s,p_l,q_l) is optimal, then $\lambda_c^{(k)}(K_{p,q}) = e(K_{p,q}) - e((\beta,s,p_l,q_l))$. Thus we obtain the following result:

Theorem 4.4: Suppose $p > \left\lfloor \frac{p+q}{k-1} \right\rfloor = t$. If (β,s,p_l,q_l) is an optimal 4-tuple, then $\lambda_c^{(k)}(K_{p,q}) = pq - (s \cdot \beta \cdot (k-1-\beta) + (t-s)(\beta+1)(k-2-\beta) + p_l \cdot q_l)$.

Our next lemma concerns realizability. We determine an ordering on the realizable 4-tuples to enable a concise statement of the lemma.

Definition 4.5: Suppose that (β, s, p_l, q_l) and (β', s', p_l', q_l') are two realizable 4-tuples for fixed $p, q,$ and k. We define $(\beta, s, p_l, q_l) \preceq (\beta', s', p_l', q_l')$ if and only if

i. $\beta = \beta'$ and $s' \le s$; or
ii. $\beta < \beta'$.

It is easy to show that $\preceq$ is a total ordering.

Remark: It should be noted that $(\beta, s, p_l, q_l) \preceq (\beta', s', p_l', q_l')$ does not imply that $e((\beta, s, p_l, q_l)) \le e((\beta', s', p_l', q_l'))$. Thus the optimality of a 4-tuple does not pertain to the ordering $\preceq$ but rather to the size of the associated failure state.

Theorem 4.6: Suppose (β, s, p_l, q_l) is a realizable 4-tuple fixed $p, q,$ and k. If

a) $p_l = 0$ then no larger realizable 4-tuple exists.
b) $q_l = 0$ then no smaller realizable 4-tuple exists.

This if the 4-tuple $(\beta, s, 0, 0)$ is realizable, it is the unique optimal 4-tuple.

Proof: The results follow immediately from the observation that if (β, s, p_l, q_l) $<$ (β', s', p_l', q_l') then $p_l > p_l'$ and $q_l < q_l'$. ■

In our next lemma we derive a necessary condition for an optimal 4-tuple to have $p_l > 0$ and a necessary condition for an optimal 4-tuple to have $q_l > 0$.

Lemma 4.7: Suppose (β, s, p_l, q_l) is optimal for fixed $p, q,$ and k. Then

a) $p_l > 0$ implies $k - 2 - 2\beta \le q_l - p_l + 1$; and
b) $q_l > 0$ and $s < t$ implies $q_l - p_l - 1 \le k - 2 - 2\beta$, while $q_l > 0$ and $s = t$ implies $q_l - p_l - 3 \le k - 2 - 2\beta$.

Proof: We only prove a) as b) is done similarly. Since $p_l > 0$, when $s = 1$, $(\beta + 1, t, p_l - 1, q_l + 1)$ is realizable, and when $s > 1$, $(\beta, s - 1, p_l - 1, q_l + 1)$ is realizable. Now subtracting the edge count of either of these 4-tuples from $e((\beta, s, p_l, q_l))$ yields $q_l - p_l + 1 - (k - 2 - 2\beta)$, which must be nonnegative by optimality. ■

The necessary conditions established above are the basis for the solutions of the optimality problems which are not covered by Theorem 4.6. Indeed our next two lemmas indicate that if a realizable 4-tuple exists with either $p_l > 0$ or $q_l > 0$, then it has the largest edge count over all larger or smaller, respectively, realizable 4-tuples.

Lemma 4.8: Let (β, s, p_l, q_l) be realizable for fixed p, q, and k. Suppose $p_l > 0$.

a) If $k-2-2\beta < q_l - p_l + 1$ then the given 4-tuple has a strictly larger edge count than all other larger realizable 4-tuples;
b) If $k-2-2\beta = q_l - p_l + 1$ then the 4-tuple $(\beta+1, t, p_l - 1, q_l + 1)$, when $s = 1$, and $(\beta, s-1, p_l - 1, q_l + 1)$, when $s > 1$, have the same edge count as the original 4-tuple, and this edge count exceeds the edge count of all other larger 4-tuples.

Proof: We begin with larger 4-tuples having the "same β" as the given 4-tuple. Suppose $0 < s' < s$ and (β, s', p_l', q_l') is realizable. Then $p_l' = p_l - (s - s')$ and $q_l' = q_l + (s - s')$. Thus $e((\beta, s, p_l, q_l)) - e((\beta, s', p_l', q_l'))$ = $(s-s')[q_l - p_l + (s-s') - (k-2-2\beta)]$, which immediately establishes both results. If no realizable 4-tuples exist of the form (β', s', p_l', q_l'), with $\beta' > \beta$ and $s' > 0$, then the proof is complete. Otherwise, realize that by letting $s' = 0$ the above argument establishes the entire result for the 4-tuple $(\beta+1, t, p_l - s, q_l + s)$, should it be realizable. Observe that $q_l' - p_l' + 1 = q_l - p_l + 2s + 1$ and $k-2-2\beta' = k-4-2\beta$. Thus $k-2-2\beta' < q_l' - p_l' + 1$ and so an induction argument handles all cases for which $\beta' - \beta \geq 1$. ■

We now consider the case when $q_l > 0$. The 4-tuple (β, s, p_l, q_l) is compared to smaller 4-tuples, i.e., those of the form $(\beta^*, s^*, p_l^*, q_l^*)$ where either $\beta^* = \beta$ and $s < s^* \leq t$, or $\beta^* < \beta$. A proof analogous to that given for Lemma 4.8 establishes the next result.

Lemma 4.9: Let (β, s, p_l, q_l) be realizable for fixed p, q, and k. Suppose $q_l > 0$.

If $1 \leq s < t$:

a) $q_l - p_l - 1 < k - 2 - 2\beta$ implies that the given 4-tuple has a strictly larger edge count than every smaller realizable 4-tuple.

b) $q_l - p_l - 1 = k - 2 - 2\beta$ implies that $(\beta, s+1, p_l+1, q_l-1)$ has the same edge count as the original 4-tuple and this edge count exceeds that of every smaller realizable 4-tuple.

If $s = t$:

c) $q_l - p_l - 3 < k - 2 - 2\beta$ implies no smaller realizable 4-tuple has larger edge count than the given 4-tuple.

d) $q_l - p_l - 3 = k - 2 - 2\beta$ implies that $(\beta - 1, 1, p_l + 1, q_l - 1)$ has the same edge count as the given 4-tuple and this edge count exceeds that of every other smaller realizable 4-tuple.

The previous lemmas are combined to establish our next theorem.

Theorem 4.10: Let (β, s, p_l, q_l) be realizable for fixed p, q, and k. Suppose $p > \left\lfloor \frac{p+q}{k-1} \right\rfloor$.

If $1 \le s \le t$:

a) $p_l > 0, q_l = 0$: $k - 2 - 2\beta < q_l - p_l + 1$ if and only if the 4-tuple is the unique optimal 4-tuple;

b) $p_l > 0, q_l = 0$: $k - 2 - 2\beta = q_l - p_l + 1$ if and only if $(\beta+1, t, p_l - 1, q_l + 1)$, when $s = 1$, or $(\beta, s-1, p_l - 1, q_l + 1)$, when $s > 1$, and the given 4-tuple are the only two optimal 4-tuples.

If $1 \le s < t$:

c) $p_l = 0, q_l > 0$: $q_l - p_l - 1 < k - 2 - 2\beta$ if and only if the 4-tuple is the unique optimal 4-tuple;

d) $p_l = 0, q_l > 0$: $q_l - p_l - 1 = k - 2 - 2\beta$ if and only if $(\beta, s+1, p_l + 1, q_l - 1)$ and the given 4-tuple are the only two optimal 4-tuples.

If $s = t$:

e) $p_l = 0, q_l > 0$: $q_l - p_l - 3 < k - 2 - 2\beta$ if and only if the 4-tuple is the unique optimal 4-tuple;

f) $p_l = 0, q_l > 0$: $q_l - p_l - 3 = k - 2 - 2\beta$ if and only if $(\beta - 1, 1, p_l + 1, q_l - 1)$ and the given 4-tuple are the only two optimal 4-tuples.

If $1 \le s < t$:

g) $p_l > 0, q_l > 0$: $q_l - p_l - 1 < k - 2 - 2\beta < q_l - p_l + 1$ if and only if the 4-tuple is the unique optimal 4-tuple;

h) $p_l > 0, q_l > 0$: $q_l - p_l - 1 = k - 2 - 2\beta$ if and only if $(\beta, s+1, p_l+1, q_l-1)$ and the given 4-tuple are the only two optimal 4-tuples; $k-2-2\beta = q_l - p_l + 1$ if and only if $(\beta+1, t, p_l-1, q_l+1)$, when $s = 1$, or $(\beta, s-1, p_l-1, q_l+1)$, when $s > 1$, and the given 4-tuple are the only two optimal 4-tuples.

If $s = t$:

i) $p_l > 0, q_l > 0$: $q_l - p_l - 3 < k - 2 - 2\beta < q_l - p_l + 1$ if and only if the 4-tuple is the unique optimal 4-tuple;

j) $p_l > 0, q_l > 0$: $q_l - p_l - 3 = k - 2 - 2\beta$ if and only if $(\beta-1, 1, p_l-1, q_l+1)$ and the given 4-tuple are the only two optimal 4-tuples; $k-2-2\beta = q_l - p_l + 1$ if and only if $(\beta, t-1, p_l-1, q_l+1)$ and the given 4-tuple are the only two optimal 4-tuples.

Remark: Actually when one of the conditions of h) or j) occur then the other one holds for the alternate optimal 4-tuple, e.g. in h) suppose (β, s, p_l, q_l) is realizable with $q_l - p_l - 1 = k - 2 - 2\beta$ then the 4-tuple (β', s', p_l', q_l') with $\beta' = \beta$, $s' = s+1$, $p_l' = p_l + 1$ and $q_l' = q_l - 1$ satisfies $k - 2 - 2\beta' = k - 2 - 2\beta = q_l - p_l - 1 = q_l' - p_l' + 1$.

There is one remaining question regarding the previous theorem, namely: Are any of the situations trivial or vacuous? We state a theorem that shows a) cannot occur and also refines b).

Theorem 4.11: The 4-tuple $(\beta, s, p_l, 0)$ is optimal for fixed p, q, and k, where $p > \left\lfloor \frac{p+q}{k-1} \right\rfloor = t$ and $p_l > 0$ if and only if

a) k is even and $\beta = \frac{k}{2} - 1$,

b) $p_l = 1$ (so $p + q = t(k-1)+1$), and

c) $s = t\left(\frac{k}{2}\right) + 1 - p = q - t\left(\frac{k}{2} - 1\right), 2s - t \geq 1$.

Also $(\beta, s-1, 0, 1)$ is the other optimal solution.

Before presenting an algorithm to determine optimal 4-tuples for given *p, q* and *k.*, we state a technical lemma.

Lemma 4.12: If (β, s, p_l, q_l) is a potentially optimal 4-tuple for $t = \left\lfloor \frac{p+q}{k-1} \right\rfloor$ and $p > t$, then $0 \le p_l \le \left\lfloor \frac{p}{t+1} \right\rfloor$. Conversely, if $0 \le p_l \le \min\left(\left\lfloor \frac{p}{t+1} \right\rfloor, p+q-t(k-1) \right)$, then there exists a unique pair of integers β, q such that $\beta \ge 1$, $1 \le s \le t$, and $p = \beta s + (\beta+1)(t-s) + p_l$. Also, with $q_l = q - \left[s(k-1-\beta) + (t-s)(k-2-\beta) \right]$, the 4-tuple (β, s, p_l, q_l) is realizable.

5. Algorithm for Determining Maximum Failure States

Lemma 4.12, along with the previous theorems on optimality, serves as the basis for the validity of the algorithm to follow. The algorithm begins by checking whether $p > t = \left\lfloor \frac{p+q}{k-1} \right\rfloor$. If not, the solution is immediately determined. If so, the possible values of p_l, from 0 to $\min\left(\left\lfloor \frac{p}{t+1} \right\rfloor, p+q-t(k-1) \right)$ are tested by determining $k-2-2\beta$, $q_l - p_l - 3$, $q_l - p_l - 1$ and $q_l - p_l + 1$, and checking the optimality conditions. It is clear that optimal 4-tuple must be uncovered by this procedure, as the optimality theorems cover all possibilities for p_l and q_l, and all possible values of p_l (and therefore q_l as well) are examined. In the event that $p > \left\lfloor \frac{p+q}{k-1} \right\rfloor$ the result is given as optimal 4-tuples.

Algorithm:
Input: Values p, q and k with $1 \le p \le q$, $4 \le k \le p+q$.
Output: All maximum failure states of $K_{p,q}$ having at most one nontrivial component of order less than $k-1$.

Step 1: Compute $t = \left\lfloor \frac{p+q}{k-1} \right\rfloor$, and check if $p \le t$. If $p \le t$, declare $pK_{1,k-2} \cup K_{0,q-p(k-2)}$ the unique maximal failure state; else, go to Step 2.

Step 2 : Set $p_l = 0$. Solve $t\beta - s = p - t$ for the unique $\beta \geq 1$ and $0 < s \leq t$. Set $q_l = q - \left[s(k-1-\beta) + (t-s)(k-2-\beta)\right]$. Compute $k-2-2\beta$.

Declare the 4-tuple $(\beta, s, 0, q_l)$ optimal if one of the following conditions holds:

2.1 $q_l = 0$;

2.2 $q_l > 0$, $s < t$ and $q_l - 1 \leq k-2-2\beta$;

2.3 $q_l > 0$, $s = t$ and $q_l - 3 \leq k-2-2\beta$.

If $q_l > 0$, $s < t$ and $q_l - 1 = k-2-2\beta$, then $(\beta, s+1, 1, q_l - 1)$ is also optimal. If $q_l > 0$, $s = t$ and $q_l - 3 = k-2-2\beta$ then $(\beta - 1, 1, p_l + 1, q_l - 1)$ is also optimal.

If optimal 4-tuple not yet found go to Step 3.

Step 3: Set $p_l = 1$. While $p_l \leq \left(\left\lfloor \frac{p}{t+1} \right\rfloor, p+q-t(k-1)\right)$ and optimal 4-tuple not yet found do the following: Solve $t\beta - s = p - p_l - t$ for $\beta \geq p_l$ and $0 < s \leq t$. Set $q_l = q - \left[s(k-1-\beta) + (t-s)(k-2-\beta)\right]$ Compute $k-2-2\beta$. Declare the 4-tuple (β, s, p_l, q_l) optimal if one of the conditions holds:

3.1 $p_l = 1$, $q_l = 0$ and $k-2-2\beta = 0$;

3.2 $p_l > 0$, $q_l > 0$, $s < t$ and $q_l - p_l - 1 \leq k-2-2\beta \leq q_l - p_l + 1$;

3.3 $p_l > 0, q_l > 0$, $s = t$ and $q_l - p_l - 3 \leq k-2-2\beta \leq q_l - p_l + 1$

If $p_l = 1$, $q_l = 0$ and $k-2-2\beta = 0$, then $(\beta, s-1, 0, 1)$ is also optimal. If $p_l, q_l > 0$ and $k-2-2\beta = q_l - p_l + 1$, then $(\beta, s-1, p_l - 1, q_l + 1)$ is also optimal. If $p_l, q_l > 0$, $s < t$ and $q_l - p_l - 1 = k-2-2\beta$, then $(\beta, s+1, p_l + 1, q_l - 1)$ is also optimal, while if $s = t$ and $q_l - p_l - 3 = k-2-2\beta$, then $(\beta - 1, 1, p_l + 1, q_l - 1)$ is also optimal. ∎

The following example demonstrates the algorithm.

Example 5.1: Let $p = 21$, $q = 79$ and $k = 16$.

Step 1: $t = \left\lfloor \frac{p+q}{k-1} \right\rfloor = \left\lfloor \frac{100}{15} \right\rfloor = 6$. Since 21 > 6 go to Step 2.

Step 2: Set $p_l = 0$. The solution of $6\beta - s = t\beta - s = p - t = 15$ is $\beta = 3$ and $s = 3$. Then

$q_l = q - [s(k-1-\beta) + (t-s)(k-2-\beta)] = 79 - [3 \cdot 12 + 3 \cdot 11] = 10$,

$k-2-2\beta = 8$. Since $q_l > 0$, $s < t$ but $q_l - 1 > k-2-2\beta$ go to Step 3.

Step 3: Compute $\min\left(\left\lfloor \frac{p}{t+1} \right\rfloor, p+q-t(k-1)\right) = \min\left(\left\lfloor \frac{21}{7} \right\rfloor, 100-6(15)\right) = 3$.

Let $p_l = 1$. The solution of $6\beta - s = t\beta - s = p - p_l - t = 14$ is $\beta = 3$ and $s = 4$. Then $q_l = q - [(k-1-\beta)s + (k-2-\beta)(t-s)] = 79 - [12 \cdot 4 + 11 \cdot 2] = 9$, and $k-2-2\beta = 8$. Since $p_l > 0,\ q_l > 0,\ s < t$ and $q_l - p_l - 1 < k-2-2\beta < q_l - p_l + 1$, the unique optimal 4-tuple is (3,4,1,9).

We conclude this section with a table of additional examples exhibiting some results of the algorithm . For each we also include the values of $e(K_{p,q})$ and $\lambda_c^{(k)}(K_{p,q})$.

Example 5.2:

p	q	k	optimality condition	optimal failure states	e	$\lambda_c^{(k)}$
5	95	16	$p \le t = \left\lfloor \frac{p+q}{k-1} \right\rfloor$	$5K_{1,14} \cup K_{0,25}$	475	405
43	61	14	$p_l = 0,\ q_l = 0$	(5,5,0,0)	2623	2297
45	46	16	$p_l = 1,\ q_l = 0$ $k-2-2\beta = 0$	(7,4,1,0) and (7,3,0,1)	2070	1734
42	58	12	$p_l = 0,\ q_l > 0$ $q_l - 1 < k-2-2\beta$	(4,3,0,1)	2436	2172
42	58	13	$p_l > 0,\ q_l > 0,\ s < t$ $q_l - p_l - 1 = k-2-2\beta$	(5,7,1,3) and (5,8,2,2)	2436	2152
21	79	16	$p_l > 0, q_l > 0$ $q_l - p_l - 1 < k-2-2\beta < q_l - p_l + 1$	(3,4,1,9)	1659	1418
22	79	16	$p_l > 0,\ q_l > 0,\ s < t$ $k-2-2\beta = q_l - p_l + 1$	(3,4,2,9) and (3,3,1,10)	1738	1488

6. Maximum Failure States With Two Nontrivial Component of Order Less Than $k-1$.

In the previous sections we assumed that a maximum failure state had at most one nontrivial component of order less than $k-1$. For the sake of completeness we state a theorem which gives an all-inclusive set of scenarios for when there exists a maximum failure state of $K_{p,q}$ with two nontrivial components of order less than $k-1$.

Theorem 6.1. Suppose (β, s, p_l, q_l) is optimal for fixed p, q, and k and suppose $p_l + q_l = k-3$.

a) If $p_l = \beta$ then $(s-1)K_{\beta,k-1-\beta} \cup (t-s)K_{\beta+1,k-2-\beta} \cup 2K_{\beta,k-2-\beta}$ is also a maximum failure state. In this case $(\beta+1, t, p_l-1, q_l+1)$, when $s=1$, or $(\beta, s-1, p_l-1, q_l+1)$, when $s>1$, is also optimal.

b) If $p_l = \beta - 1$ and $s < t$ then $sK_{\beta,k-1-\beta} \cup (t-s-1)K_{\beta+1,k-2-\beta} \cup 2K_{\beta,k-2-\beta}$ is also a maximum failure state. In this case $(\beta, s+1, p_l+1, q_l-1)$ is also optimal.

c) If $p_l = \beta - 2$, then $s = t$ and $(s-1)K_{\beta,k-1-\beta} \cup 2K_{\beta-1,k-1-\beta}$ is also a maximum failure state. In this case $(\beta-1, 1, p_l+1, q_l-1)$ is also optimal.

One consequence of Theorem 6.1 is that if there exists a maximum failure state of $K_{p,q}$ with two nontrivial components of order less than $k-1$, then there also exists two other maximum failure states, each with one nontrivial component of order less than $k-1$. Figure 1.3 depicted a maximum 5-component order edge-failure state for $K_{3,7}$, i.e., the 4-tuple (1,2,1,1). Since $p_l = \beta$, (1,1,0,2) and $K_{1,3} \cup 2K_{1,2}$ are also maximum 5-component order edge-failure states.

7. Conclusions

The k-component order edge connectivity of a graph G is the minimum cardinality of a set of edges F such that the subgraph $G-F$ contains no component of order at least k. We refer to any subgraph of the form $G-F$ containing no component of order at least k as a failure state of G. In this work we studied this parameter for complete bipartite graphs $K_{p,q}$. Our method for computing the parameter is to find a maximum failure state of $K_{p,q}$, and then subtracting its size from pq. Under the assumption that $p \leq q$, the result is straight forward for "small" values of p, but it requires an extensive case study and an algorithm to find maximum failure states for "large" values of p.

8. References

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