Injectively $(\Delta + 1)$-choosable graphs

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Abstract

An injective coloring of a graph G is an assignment of colors to the vertices of G so that any two vertices with a common neighbor receive distinct colors. A graph G is said to be *injectively k-choosable* if any list $L(v)$ of size at least k for every vertex v allows an injective coloring $\phi(v)$ such that $\phi(v) \in L(v)$ for every $v \in V(G)$. The least k for which G is injectively k-choosable is *the injective choosability number* of G, denoted by $\chi_i^l(G)$. In this paper, we obtain new sufficient conditions to be $\chi_i^l(G) \leq \Delta(G) + 1$. We prove that if $mad(G) < \frac{12k}{4k+3}$, then $\chi_i^l(G) = \Delta(G)+1$ where $k = \Delta(G)$ and $k \geq 4$. Typically proofs using discharging technique are different depending on maximum average degree $mad(G)$ or maximum degree $\Delta(G)$. The main objective of this paper is finding a function $f(\Delta(G))$ such that $\chi_i^l(G) \leq \Delta(G) + 1$ if $mad(G) < f(\Delta(G))$, which can be applied to every $\Delta(G)$.

Keywords: Injective coloring, list coloring, maximum average degree, discharging

1 Introduction

All graphs considered in this paper are simple, finite, and undirected. We use $V(G), E(G)$ and $\Delta(G)$ to denote the vertex set, the edge set and the

*This paper was supported by Konkuk University in 2012.

maximum degree of G, respectively, and let $n(G)$ and $e(G)$ denote the cardinal number of $V(G)$ and $E(G)$, respectively.

An injective coloring of a graph G is an assignment of colors to the vertices of G so that any two vertices with a common neighbor receive distinct colors. The *injective chromatic number* $\chi_i(G)$ is the least number of colors needed for an injective coloring of G. Note that injective coloring is not necessarily proper, and $\chi_i(G) = \chi(G^{(2)})$ where the neighboring graph $G^{(2)}$ is defined by $V(G^{(2)}) = V(G)$ and $E(G^{(2)}) = \{uv :$ u and v have a common neighbor in $G\}$.

A graph G is said to be *injectively k-choosable* if any list $L(v)$ of size at least k for every vertex v allows an injective coloring $\phi(v)$ such that $\phi(v) \in L(v)$ for every $v \in V(G)$. The least k for which G is injectively k-choosable is *the injective choosability number* of G, denoted by $\chi_i^l(G)$.

The *girth* $g(G)$ of G is the length of a shortest cycle in G. Maximum average degree, mad(G), is defined by $\text{mad}(G) = \max\{\frac{2e(H)}{n(H)} : H \subset G\}$. Note that $\text{mad}(G) < \frac{2g}{g-2}$ for every planar graph G with girth at least g.

Let G^2 be the *square of* G, that is, $V(G^2) = V(G)$ and $uv \in E(G^2)$ whenever $d_G(u,v) \leq 2$. As one can see, injective coloring is closely related with the coloring of square of a graph (or called 2-distance coloring), which is a proper coloring and an injective coloring. The study of $\chi(G^2)$ has been largely focused on the well-known Wenger's Conjecture [10]. Note that $\Delta(G) + 1 \leq \chi(G^2)$ for every graph G. Also a lot of study has been done to find sufficient conditions to have $\chi(G^2) \leq \Delta(G) + c$ for some small constanct c in terms of girth $g(G)$ or in terms of maximum average degree; see [2] for a good survey.

Note that $\Delta \leq \chi_i(G) \leq \chi_i^l(G) \leq \Delta^2 - \Delta + 1$ for every G where Δ is the maximum degree of G. A natural interesting problem is to decide which graphs have small injective chromatic numbers such that $\chi_i(G) \leq \Delta(G) + t$ for some small constant t (see [1, 4, 6, 8, 9]). The case when $t = 0$ was studied in [1, 4, 8], the case when $t = 1$ was studied in [1, 4], the case when $t = 2$ was sdudied in [5, 9], and the case when $t \geq 3$ was studied in [6],

In this paper, we consider the case when $t = 1$. Borodin and Ivanova [1] showed that $\chi_i^l(G) \leq \Delta(G) + 1$ if G is a planar graph with $g(G) \geq 6$ and $\Delta(G) \geq 24$. Cranston et al. [4] showed that $\chi_i^l(G) \leq \Delta(G) + 1$ if $mad(G) \leq \frac{5}{2}$ and $\Delta(G) \geq 3$. On the other hand, Li and Xu [9] showed that $\chi_i^l(G) \leq \Delta(G) + 2$ if $mad(G) < 3$ and $\Delta(G) \geq 12$.

An interesting problem is finding the optimal value of the upper bound of $mad(G)$ to have that $\chi_i^l(G) \leq \Delta(G) + 1$. In most cases, proofs are different depending on $mad(G)$ or $\Delta(G)$. In this paper, we are interested in finding a function $f(\Delta(G))$ such that $\chi_i^l(G) \leq \Delta(G) + 1$ if $mad(G) < f(\Delta(G))$. Our main result is as follows.

Theorem 1.1. *Let* $\Delta(G) = k \geq 4$. *If* $mad(G) < \frac{12k}{4k+3}$, *then* G *is injectively* $(k+1)$*-choosable.*

Thereom 1.1 improves the results in [4] when $\Delta(G) \geq 4$. Note that $\frac{12k}{4k+3}$ closes to 3 when k is sufficiently large. Hence by considering the results in [9] that $\chi_i^l(G) \leq \Delta(G) + 2$ if $mad(G) < 3$ and $\Delta(G) \geq 12$, Theorem 1.1 gives new information when $\Delta(G)$ is sufficiently large.

2 Proof of Theorem 1.1

Here we introduce some notation. A k-vertex is a vertex of degree k; a k^+-vertex is a vertex whose degree is greater or equal to k. A vertex y is called k-neighbor of x if y is a neighbor of x with degree k; a vertex y is called k^+-neighbor of x if y is a neighbor of x and the degree of y is greater or equal to k. A thread is a path with 2-vertices in its interior and 3^+-vertices as its endpoints. For $k \geq 1$, a k-thread has k interior 2-vertices. If u and v are the endpoints of a thread, then we say that u and v are pseudo-adjacent. If a 3^+-vertex u is the endpoint of a thread containing a 2-vertex v, then we say that v is a *nearby vertex* of u and vice versa.

In this section, let G be a minimal counterexample to Theorem 1.1. We have the following Remark which is simple, but important.

Remark 2.1. For every edge uv of G, at least one of u and v has at least $\Delta(G) + 1$ vertices at distance 2.

To prove the theorem, we will use discharging method. We have following reducible configurations.

Lemma 2.2. The following reducible configurations are straightfoward from Remark 2.1

(C1) G has no vertex of degree 1.

(C2) G has no 2^+-thread.

(C3) If a vertex v is adjacent to $d(v)$ 2-vertices, then every nearby vertice of v has degree at least $\Delta(G) - d(v) + 3$.

(C4) If a vertex v is adjacent to $(d(v) - 1)$ 2-vertices, then every nearby vertice of v has degree at least $\Delta(G) - d(v) + 3$ or the 3^+-neighbor of v has degree at least $\Delta - d(v) + 3$.

(C5) If a vertex v is adjacent to $(d(v) - 2)$ 2-vertices, then every nearby vertice of v has degree at least $\Delta(G) - d(v) + 3$ or the sum of degrees of the two 3^+-neighbors is at least $\Delta(G) - d(v) + 5$.

We divide the proof of Theorem 1.1 into two cases. First, we consider when $\Delta(G) \geq 7$ and then we consider the case when $4 \leq \Delta(G) \leq 6$.

Theorem 2.3. *Let $\Delta(G) = k$ and $\Delta(G) \geq 7$. If $mad(G) < \frac{12k}{4k+3}$, then G is injectively $(k+1)$-choosable.*

Proof. To prove this theorem, we will use discharging method with initial charge $\mu(v) = d(v)$. For convenience, let $\beta = \frac{4k-6}{4k+3}$. We have the following discharging rules.

(R1) If $3 \leq d(v) \leq 4$, then v gives charge $\frac{\beta}{2}$ to each of its 2-neighbors.

(R2) If $d(v) = 5$ and $7 \leq \Delta(G) \leq 15$, then v gives charge $\frac{\beta}{2}$ to each of its neighbors..

If $d(v) = 5$ and $\Delta(G) \geq 16$, then v gives charge $\frac{\beta}{2}$ to each of its 2-neighbors, but does not give any charge to its 3^+-neighbors.

(R3) If $6 \leq d(v) \leq \Delta(G) - 3$, then v gives charge $\frac{\beta}{2}$ to each of its neighbors.

(R4) If $d(v) = \Delta(G) - 2$ and $7 \leq \Delta(G) \leq 15$, then v gives charge $\frac{\beta}{2}$ to each of its neighbors.

If $d(v) = \Delta(G) - 2$ and $\Delta(G) \geq 16$, then v gives charge $\frac{\beta}{2}$ to each of its 2-neighbors and gives charge $\frac{1}{5} \cdot \frac{2k-30}{4k+3}$ to each of its nearby vertices, and gives charge $\frac{\beta}{2} + \frac{1}{5} \cdot \frac{2k-30}{4k+3}$ to each its 3^+-neighbors.

(R5) If $d(v) = \Delta(G) - 1$, then v gives charge $\frac{\beta}{2}$ to each of its 2-neighbors and gives charge $\frac{k-6}{4k+3}$ to each of its nearby vertices, and gives charge $\frac{\beta}{2} + \frac{k-6}{4k+3}$ to each of its 3^+-neighbors.

(R6) If $d(v) = \Delta(G)$, then v gives charge $\frac{\beta}{2}$ to each of its 2-neighbors and gives charge $\frac{2k-6}{4k+3}$ to its nearby vertices, and gives charge $\frac{\beta}{2} + \frac{2k-6}{4k+3}$ to its 3^+-neighbors.

Let $\mu^*(v)$ be the new charge of v after discharging such that

$$\sum_{v \in V(G)} \mu(v) = \sum_{v \in V(G)} \mu^*(v).$$

We will show that $\mu^*(v) \geq \frac{12k}{4k+3}$ for each vertex v after discharging. From now on, we denote the maximum degree of G by Δ.

Case 1: $d(v) = 2$.

If $d(v) = 2$, then v receives $\frac{\beta}{2}$ from each of its neighbors. Hence $\mu^*(v) \geq 2 + \frac{\beta}{2} + \frac{\beta}{2} = \frac{12k}{4k+3}$.

Case 2: $d(v) = 3$.

In this case, we have four subcases.

Subcase 2.1 : v is incident to three 1-threads.

v gives total $\frac{3\beta}{2}$ to its neighbors. Since all nearby vertices of v have degree Δ by Lemma 2.2, v receives charge $\frac{2k-6}{4k+3}$ from each of nearby vertices (R6). Therefore

$$\mu^*(v) = d(v) - \frac{3\beta}{2} + 3 \cdot \frac{2k-6}{4k+3} = 3 - \frac{6k-9}{4k+3} + \frac{6k-18}{4k+3} = \frac{12k}{4k+3}.$$

Subcase 2.2 : v is incident to two 1-threads.

Let x and y be 2-vertices adjacent to v and x' and y' be the neighbors of x and y which is not v, respectively. Let z be the 3^+-neighbor of v. Then, by Lemma 2.2, $d(x') = d(y') = \Delta$ or $d(z) = \Delta$. If $d(x') = d(y') = \Delta$, then both of x' and y send $\frac{2k-6}{4k+3}$ to v by (R6). Hence

$$\mu^*(v) \geq d(v) - \beta + 2 \cdot \frac{2k-6}{4k+3} = 3 - \frac{4k-6}{4k+3} + \frac{4k-12}{4k+3} = \frac{12k+3}{4k+3} > \frac{12k}{4k+3}.$$

And if $d(z) = \Delta$, then z sends $\frac{\beta}{2} + \frac{2k-6}{4k+3}$ to v by (R6). Hence

$$\mu^*(v) \geq d(v) - \beta + \frac{\beta}{2} + \frac{2k-6}{4k+3} = 3 - \frac{2k-3}{4k+3} + \frac{2k-6}{4k+3} = \frac{12k+6}{4k+3} > \frac{12k}{4k+3}.$$

Subcase 2.3 : v is incident to one 1-thread.

Let x be the 2-vertex adjacent to v and x' be the neighbor of x which is not v. Let y and z be other neighbors of v. By Lemma 2.2, $d(x') = \Delta$ or $d(y) + d(z) \geq \Delta + 2$. If $d(x') = \Delta$, then x' sends $\frac{2k-6}{4k+3}$ to v by (R6). Hence

$$\mu^*(v) \geq d(v) - \frac{\beta}{2} + \frac{2k-6}{4k+3} = 3 - \frac{2k-3}{4k+3} + \frac{2k-6}{4k+3} = \frac{12k+6}{4k+3} > \frac{12k}{4k+3}.$$

Now suppose that $d(y)+d(z) \geq \Delta+2$. We may assume that $d(z) \leq d(y)$. If $\Delta \geq 16$, then $d(y) \geq 6$. Thus v receives charge $\frac{\beta}{2}$ from y by (R3). If $7 \leq \Delta \leq 15$, then $d(y) \geq 5$ and v receives charge $\frac{\beta}{2}$ from y by (R2). Hence $\mu^*(v) \geq d(v) - \frac{\beta}{2} + \frac{\beta}{2} = 3 > \frac{12k}{4k+3}$.

Subcase 2.4 : v is not incident to any 1-thread.

In this case, v does not lose any charge. Therefore $\mu^*(v) \geq 3 > \frac{12k}{4k+3}$.

Case 3: $d(v) = 4$.

We have three subcases.

Subcase 3.1 : v is incident to four 1-threads.

v gives total 2β charges to its neighbors. Since each nearby vertex to v has degree at least $\Delta - 1$ by Lemma 2.2, v receives charge at least $4 \cdot \frac{k-6}{4k+3}$ by (R5). Hence

$$\mu^*(v) \geq d(v) - 2\beta + 4 \cdot \frac{k-6}{4k+3} = 4 - \frac{8k-12}{4k+3} + \frac{4k-24}{4k+3} = \frac{12k}{4k+3}.$$

Subcase 3.2 : v is incident to three 1-threads.

Let x, y and z be 2-vertices adjacent to v and x', y' and z' be the neighbors of x, y and z which is not v, respectively. Let w be the 3^+-neighbor of v. Then, by Lemma 2.2, $\min\{d(x'), d(y'), d(z')\} \geq \Delta - 1$ or $d(w) \geq \Delta - 1$. If $\min\{d(x'), d(y'), d(z')\} \geq \Delta - 1$, then by (R5) and (R6)

$$\mu^*(v) \geq d(v) - \frac{3\beta}{2} + 3 \cdot \frac{k-6}{4k+3} = 4 - \frac{6k-9}{4k+3} + \frac{3k-18}{4k+3} = \frac{13k+3}{4k+3} > \frac{12k}{4k+3}.$$

And if $d(w) \geq \Delta - 1$, then by (R5) and (R6)

$$\mu^*(v) \geq d(v) - \frac{3\beta}{2} + \frac{\beta}{2} + \frac{k-6}{4k+3} = 4 - \frac{6k-9}{4k+3} + \frac{2k-3}{4k+3} + \frac{k-6}{4k+3} > \frac{12k}{4k+3}.$$

Subcase 3.3 : v is incident to at most two 1-threads.

In this case, v loses charge at most β. Therefore, $\mu^*(v) \geq d(v) - \beta = 4 - \frac{4k-6}{4k+3} > \frac{12k}{4k+3}$.

Case 4: $d(v) = 5$.

First, we consider the case when $7 \leq k \leq 15$. In this case, by (R2), $\mu^*(v) \geq d(v) - d(v)\frac{\beta}{2} = d(v)\frac{2k+6}{4k+3} \geq \frac{12k}{4k+3}$, since $d(v) \geq \frac{12k}{2k+6}$ when $7 \leq k \leq 15$.

Next we consider the case when $k \geq 16$. In this case, we have two subcases.

Subcase 4.1 : v is incident to five 1-threads.

In this case, every nearby vertices of v have degree at least $\Delta - 2$. Hence every nearby vertex of v gives charge $\frac{1}{5} \cdot \frac{2k-30}{4k+3}$ to v by (R4). Thus

$$\mu^*(v) \geq d(v) - \frac{5\beta}{2} + 5 \cdot \frac{1}{5} \cdot \frac{2k-30}{4k+3} = 5 - \frac{10k-15}{4k+3} + \frac{2k-30}{4k+3} = \frac{12k}{4k+3}.$$

Subcase 4.2 : v is incident to at most four 1-threads.

v loses charge at most 2β. Therefore $\mu^*(v) \geq 5 - 2\beta = \frac{12k+27}{4k+3} > \frac{12k}{4k+3}$.

Case 5: $6 \leq d(v) \leq \Delta - 3$.

Note that v gives charge at most $d(v) \cdot \frac{\beta}{2}$ to its neighbors. Hence $\mu^*(v) \geq d(v) - d(v)\frac{\beta}{2} = d(v)\frac{2k+6}{4k+3} \geq \frac{12k}{4k+3}$, since $d(v) \geq 6$.

Case 6: $d(v) = \Delta - 2$.

When $k \geq 16$, v loses charge at most $d(v) \cdot (\frac{\beta}{2} + \frac{1}{5} \cdot \frac{2k-30}{4k+3})$ by (R4). Hence $\mu^*(v) \geq d(v) - d(v) \cdot (\frac{\beta}{2} + \frac{1}{5} \cdot \frac{2k-30}{4k+3}) = \frac{1}{5} \cdot \frac{8k^2+44k-120}{4k+3} \geq \frac{12k}{4k+3}$. When $7 \leq k \leq 15$, by (R4) we have that $\mu^*(v) \geq d(v) - d(v) \cdot \frac{\beta}{2} = d(v)(1 - \frac{\beta}{2}) = (k-2)\frac{2k+6}{4k+3} \geq \frac{12k}{4k+3}$

Case 7: $d(v) = \Delta - 1$.

In this case, v loses charge at most $d(v) \cdot (\frac{\beta}{2} + \frac{k-6}{4k+3})$ by (R5). Hence $\mu^*(v) \geq d(v) - d(v) \cdot (\frac{\beta}{2} + \frac{k-6}{4k+3}) = \frac{k^2+11k-12}{4k+3} \geq \frac{12k}{4k+3}$. Note that $\frac{k^2+11k-12}{4k+3} \geq \frac{12k}{4k+3}$ when $k \geq 4$.

Case 8: $d(v) = \Delta$.

In this case, we have that $\mu^*(v) \geq d(v) - d(v) \cdot (\frac{\beta}{2} + \frac{2k-6}{4k+3}) = \frac{12k}{4k+3}$.

Therefore $\mu^*(v) \geq 0$ for every vertex v in $V(G)$. This contradiction completes the proof of Theorem 2.3. □

Next we consider the case when $4 \leq \Delta(G) \leq 6$.

Theorem 2.4. *Let $\Delta(G) = k$ and $4 \leq \Delta(G) \leq 6$. If $mad(G) < \frac{12k}{4k+3}$, then G is injectively $(k+1)$-choosable.*

Proof. Let G be a minimal counterexample. We use discharging method with initial charge $\mu(v) = d(v)$. We have the following discharging rules.

(R1) If $d(v) = 3$, then v gives charge $\frac{2k-3}{4k+3}$ to each of its 2-neighbors.

(R2) If $4 \leq d(v) \leq \Delta(G) - 1$, then v gives charge $\frac{2k-3}{4k+3}$ to each of its 2-neighbors and gives charge $\frac{2k-3}{4k+3}$ to each of its 3-neighbors.

(R3) If $d(v) = \Delta(G)$, then v gives charge $\frac{2k-3}{4k+3}$ to each of its 2-neighbors and gives charge $\frac{2k-6}{4k+3}$ to its nearby vertices and gives charge $\frac{4k-9}{4k+3}$ to each of its 3^+-neighbors.

We will show that $\mu^*(v) \geq \frac{12k}{4k+3}$ for each vertex v after discharging.

Case 1: $d(v) = 2$.

$\mu^*(v) = 2 + 2 \cdot \frac{2k-3}{4k+3} = \frac{12k}{4k+3}$

Case 2: $d(v) = 3$.

If v is adjacent to three 2-vertices, then all of its nearby vertices has degree Δ. Then $\mu^*(v) = 3 - 3 \cdot \frac{2k-3}{4k+3} + 3 \cdot \frac{2k-6}{4k+3} = \frac{12k}{4k+3}$ by (R3). If v is adjacent to two 2-vertices, then its nearby vertices have degree Δ or its 3^+-neighbor has degree $\Delta(G)$. In any case, $\mu^*(v) \geq 3 - 2 \cdot \frac{2k-3}{4k+3} + \frac{4k-12}{4k+3} = \frac{12k+3}{4k+3} > \frac{12k}{4k+3}$ by (R3). If v is adjacent at most one 2-vertex, then $\mu^*(v) \geq 3 - \frac{2k-3}{4k+3} = \frac{10k+12}{4k+3} \geq \frac{12k}{4k+3}$ when $k = 4, 5$ or 6.

Case 3: $4 \leq d(v) \leq \Delta(G) - 1$.

In this case, we only consider when $\Delta(G) = 5$ or 6. By (R2), v sends charge at most $\frac{2k-3}{4k+3}$ to each of its neighbors. Hence $\mu^*(v) \geq d(v) - d(v) \cdot \frac{2k-3}{4k+3} = d(v) \cdot \frac{2k+6}{4k+3} \geq \frac{12k}{4k+3}$, since $k = 5, 6$.

Case 4: $d(v) = \Delta$.

$\mu^*(v) \geq k - k \cdot \frac{4k-9}{4k+3} = \frac{12k}{4k+3}$.

□

Remark 2.5. It is proved in [1] that $\chi_i^l(G) \leq \Delta(G)+1$ if G is a planar graph with $g(G) \geq 6$ and $\Delta(G) \geq 24$, and it is proved in [9] that $\chi_i^l(G) \leq \Delta(G)+2$ if $mad(G) < 3$ and $\Delta(G) \geq 12$. Note that if the girth of G is at least 6, then $mad(G) < 3$. A natural question is as follows.

Question 2.6. Is it true that $\chi_i^l(G) \leq \Delta(G) + 1$ if $mad(G) < 3$?

Or we can ask the following weaker question.

Question 2.7. Is there a small constant $\epsilon > 0$ such that $\chi_i^l(G) \leq \Delta(G) + 1$ for every graph G with $mad(G) < 3 - \epsilon$?

Our result implies that given any sufficiently small $\epsilon > 0$, $\chi_i^l(G) \leq \Delta(G) + 1$ for every graph G with $mad(G) < 3 - \epsilon$ and $\Delta(G) \geq \frac{9-3\epsilon}{4\epsilon}$. Question is whether there is a sufficiently small ϵ which is independent of $\Delta(G)$.

Acknowledgments. We thank an anonymous referee for helpful comments.

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