Primality of Some graphs

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Abstract
We discuss the primality of some corona graphs and some families of graphs.
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1 Introduction

The notion of a prime labelling originated with Entringer and was introduced in a paper by Tout, Dabboucy, and Howalla (see [6]). A graph with vertex set V is said to have a prime labelling if its vertices can be labelled with distinct integers 1,2,...,$|V|$ such that for each edge xy the labels assigned to x and y are relatively prime.
In [7] Seoud,Elsonbaty and Mahran discussed some necessary and sufficient conditions for a graph to be prime. They gave also a procedure to determine whether or not a graph is prime.
In [8] Seoud and Youssef presented some new families of graphs which have a prime labelling. They gave a closed formula for the maximum number of edges in a graph of order n having a prime labelling. They conjectured that all unicycle graphs are prime.
In [9] Youssef gave some necessary conditions for a prime graph. He gave also necessary and sufficient conditions for some disconnected graphs to be prime.
A dual of prime labelling has been introduced by Deretsky, Lee, and Mitchem (see [2]). They defined a graph with edge set E has a vertex prime labelling if its edges can be labelled with distinct integers 1,...,$|E|$ such that for each vertex of degree at least 2 the greatest common divisor of the labels on its incident edges is 1. They showed that certain families of graphs are vertex prime. They further prove that a graph with exactly two components, one of which is not an odd cycle, has a vertex prime labelling and a 2-regular graph with at least two odd cycles does not have a vertex prime labelling. They conjecture that a 2-regular graph has a vertex prime labelling if and only if it does not have two odd cycles.
Here, we discuss the primality of some corona graphs $G \odot H$ and conjecture that $K_n \odot \overline{K_m}$ is prime if and only if $n \leq \pi(nm+n)+1$. And as an application we give the exact values of n for each $m \leq 20$ for which $K_n \odot \overline{K_m}$ is prime. We

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show also that $K_{n,m}$ is prime if and only if $\min\{m,n\} \leq \pi(m+n)-\pi(\frac{m+n}{2})+1$, where $\pi(x) :=\mid \{p : p \; prime, 2 \leq p \leq x\} \mid$. We use $\mid A \mid$ to denote the order of the set A, i.e., its number of elements. All graphs here are finite and simple. Throughout this paper, we use the standard notations and conventions in graph theory as in [3] and [4], and in number theory as in [1] and [5].

2 Some families of prime and non-prime graphs

Definition 2.1. A simple graph with vertex set V is said to be prime if its vertices can be labelled with distinct integers 1, 2, ...,$\mid V \mid$ such that for each edge xy the labels assigned to x and y are relatively prime, i.e., their greatest common divisor equals 1. A graph which is not prime is called a non-prime graph.

Definition 2.2. If G is a graph with vertex set V, then a set $W \subset V$ is called independent if each two vertices in W are not adjacent.

Definition 2.3. $\beta(G)$ is defined as the order of the largest independent set of vertices of the graph G, i.e., $\beta(G) := \max_{A\subset V} \mid A \mid$, where A is an independent set of vertices.

Theorem 2.1. [9]

Condition 1: If G is a graph of n vertices and m edges and $m > \sum_{i=2}^{i=n} \phi(i)$, then G is not prime, where $\phi(i) :=\mid \{k : k \leq i, (k,i) = 1\} \mid$.

Condition 2: If G is a simple graph of n vertices, which has a complete subgraph of order more than $\pi(n) + 1$, then the graph G is not prime.

Condition 3: If G is a graph of n vertices, which has more than $\pi(n) - \pi(\frac{n}{2}) + 1$ vertices of degree $n-1$, then G is not prime.

Condition 4: If the minimum degree of the graph is greater than the minimum degree of the corresponding maximal prime graph, then the graph is not prime (we denote the minimum degree of the maximal prime graph of n vertices by $\delta(n) = n + \sum_{s=1}^{m}(-1)^s(\sum_{1\leq j_1<j_2<...<j_s\leq m}\lfloor\frac{n}{p_{j_1}p_{j_2}...p_{j_s}}\rfloor)$, where $p_1 < p_2 < ... < p_{\pi(n)}$ is the list of all prime numbers not exceeding n and $m =: \max\{1 \leq k \leq \pi(n) : p_1p_2...p_k \leq n\}$.

Condition 5: If a graph G of n vertices and $\beta(G) < \lfloor\frac{n}{2}\rfloor$, then the graph G is not prime.

Examples 2.1.

(a) The helm H_n is the graph obtained from the wheel W_n by adjoining a pendant vertex to each vertex of the circle of the wheel W_n.
We notice from the geometry of H_n that $\beta(H_n) = n + 1$.

(b) The flower H_n^* is the graph obtained from H_n by connecting each pendent vertex to the central vertex. $\beta(H_n^*) = n$.

(c) The gear G_n is the graph obtained from W_n by adding a vertex between any two adjacent vertices in its circle C_n. $\beta(G_n) = n + 1$.

(d) The fan F_n is defined as follows : $F_n := K_1 \odot P_n$. $\beta(F_n) = \lfloor\frac{n+1}{2}\rfloor$.

(e) The closed helm $\overline{H_n}$ is the graph obtained from H_n by connecting each two consecutive pendant vertices forming the outer circle. $\beta(\overline{H_n}) = 2\lfloor\frac{n}{2}\rfloor$.

(f) The web $\overline{W_n}$ is the graph obtained from $\overline{H_n}$ by adjoining a pendant vertex to each vertex of the outer circle of $\overline{H_n}$. $\beta(\overline{W_n}) = n + \lfloor \frac{n}{2} \rfloor$.
(g) The triangular snake T_n is the graph obtained from the path P_n having the vertices $v_1, v_2, ..., v_n$ by adding a new vertex w_i and connect it to the vertices v_i, v_{i+1} for each i. $\beta(T_n) = n - 1$.
(h) The Möbius ladder M_n is the graph obtained from $P_n \times P_2$ by connecting the first vertex of the first copy of P_n to the last vertex of the second copy of P_n and the first vertex of the second copy of P_n to the last vertex of the first copy of P_n.

$$\beta(M_n) = \begin{cases} n & ,if\ n\ is\ odd \\ n-1 & ,if\ n\ is\ even \end{cases}$$

(i) The book B_n is defined as follows $B_n := P_2 \times S_n$, $S_n = K_{1,n}$, $\beta(B_n) = n+1$.
(j) The graph $D_{n,m}$ is the graph obtained from the circle C_n by adjoining the path P_m to any vertex of C_n. $\beta(D_{n,m}) = \lfloor \frac{n}{2} \rfloor + \lfloor \frac{m+1}{2} \rfloor$.
(k) The graph C_n^m is the graph obtained from C_n by connecting each two vertices having distance not exceeding m.
Since C_n is a subgraph of C_n^m, $\beta(C_n^m) < \lfloor \frac{n}{2} \rfloor$ for $m > 1$.

Theorem 2.2.
$K_{n,m}$ is prime if and only if $\min\{m,n\} \leq \pi(m+n) - \pi(\frac{m+n}{2}) + 1$.

Proof
Without any loss of generality suppose that $n < m$.
"$\Rightarrow$" by contrapositive principle, we suppose that $n = \min\{m,n\} > \pi(m+n) - \pi(\frac{m+n}{2}) + 1$, then there exists a label k of one of the n vertices whose prime factor $p \leq \frac{m+n}{2}$. Also, the even numbers must label some of the m vertices, then the two vertices having the labels $k, 2p$ are adjacent, and $(k, 2p) = p > 1$, hence $K_{n,m}$ is non-prime.
"$\Leftarrow$" Let $n = \min\{m,n\} \leq \pi(m+n) - \pi(\frac{m+n}{2}) + 1$. Now we get a prime labelling of $K_{n,m}$ by labelling the n vertices by the labels "1" and the primes exceeding $\frac{m+n}{2}$ and less than or equal to $m+n$, and give the other labels to the m vertices. Now we get a prime labelling of $K_{n,m}$ since for every prime number $p > \frac{m+n}{2}$, $(p, i) = 1$ for each $i \leq m+n$ and $i \neq p$.

Corollary 2.3. If G and H are two graphs of m and n vertices respectively, and $\min\{m,n\} > \pi(n+m) - \pi(\frac{n+m}{2}) + 1$ then the sum graph $G + H$ is a non-prime graph.
Proof is a direct consequence of Theorem 2.2.

3 Primality of some corona graphs

Definition 3.1. If G and H are two graphs then the corona graph $G \odot H$ is defined as the graph obtained by taking one copy of G (which has n vertices) and n copies of H and joining the ith vertex of G to every vertex in the ith copy of H.

Example 3.1.
If $m > \pi((m+1)n)+1$ then the graphs $C_n \odot K_m$ and $P_n \odot K_m$ are non-prime graphs.
Proof
These graphs contain complete subgraphs of order m which are greater than $\pi((m+1)n)+1$, hence by Condition 2 Theorem 2.1 they are not prime.
Remark 3.1.
$K_1 \odot G$ is prime if G is a prime graph of n vertices and $n+1$ is a prime number.
Proof
Since G is prime then there exists a prime labelling to G, we give the vertex of K_1 the label $n+1$ which is a prime number, then $(n+1, i) = 1$ for each $i < n+1$, hence there exists a prime labelling of $K_1 \odot G$.
Lemma 3.1.
If G and H are two graphs of n and m vertices respectively, then $\beta(G \odot H) = n\beta(H)$.
Proof.
By induction on n.
If $n = 1$, then the statement clearly holds. Let the assertion be true for $n = k$. For $n = k+1$. Remove a vertex from the graph G and its corresponding copy of H, get the graph $G \odot H$ where G is of k vertices, then from our assumption $\beta(G \odot H) = k\beta(H)$, since none of the counted vertices is adjacent to any of the removed vertices, then it remains to calculate the order of the maximal independent set of vertices of the removed part, which equals $\beta(H)$ from the first step of induction, hence $\beta(G \odot H) = k\beta(H) + \beta(H) = (k+1)\beta(H)$.
Lemma 3.2.
If G and H are two graphs of m and n vertices respectively, $m \neq 1$ and $\beta(H) \leq \lfloor \frac{n}{2} \rfloor$, then $G \odot H$ is non-prime.
Proof
From Lemma 3.1 $\beta(G \odot H) = m\beta(H) \leq m\lfloor \frac{n}{2} \rfloor$, since $m \neq 1$, then $\beta(G \odot H) \leq m\lfloor \frac{n}{2} \rfloor \leq \lfloor \frac{mn}{2} \rfloor < \lfloor \frac{mn+m}{2} \rfloor$, hence by Condition 5 Theorem 2.1 we get $G \odot H$ is non-prime.
We notice that if $m = 1$, the assertion is not true. e.g. The graph $K_1 \odot P_4$ is prime in which $G = K_1$, i.e., $m = 1$ and $\beta(P_4) = \lfloor \frac{4}{2} \rfloor = 2$.
Corollaries 3.3.
(1)$\overline{K_m} \odot C_n$ is non-prime for $m \neq 1$ and for all n, or for $m = 1$ and for all odd n.
(2)$\overline{K_m} \odot H_n^*$ is non-prime for all m, n.
(3)$\overline{K_m} \odot F_n$ is non-prime for $m \neq 1$ and for all n, or for $m = 1$ and for all even n.
(4)$\overline{K_m} \odot T_n$ is non-prime for all m, n.
(5)$\overline{K_m} \odot M_n$ is non-prime for $m \neq 1$ and for all n, or for $m = 1$ and for all even n.
(6)$\overline{K_m} \odot C_n^k$ is non-prime for $k \neq 1$ and for all n, m.
(7)$\overline{K_m} \odot K_{n,n}$ is non-prime for $m \neq 1$ and for all n.
(8)$\overline{K_m} \odot (P_n + \overline{K_2})$ is non-prime for $m \neq 1$ and for all $n \neq 1$. In case of $m = 1$ we have $\overline{K_m} \odot (P_n + \overline{K_2})$ is non-prime for all $n > 2$.

(9)$\overline{K_m} \odot D_{2n,2k}$, $\overline{K_m} \odot D_{2n+1,2k}$, and $\overline{K_m} \odot D_{2n+1,2k+1}$ are non-prime for $m \neq 1$ and for all n. In case of $m = 1$ we have $\overline{K_m} \odot D_{2n+1,2k}$ is non-prime.
(10)$\overline{K_m} \odot \overline{H_n}$ is non-prime for $m \neq 1$ and for all n.
(11)$\overline{K_m} \odot \overline{W_n}$ is non-prime for all n, m.
(12)$\overline{K_m} \odot B_n$ is non-prime for $m \neq 1$ and for all n.

Proof

(1) In case of $m = 1$ and n odd, $\beta(\overline{K_1} \odot C_n) = \beta(C_n) = \lfloor \frac{n}{2} \rfloor < \lfloor \frac{n+1}{2} \rfloor$, hence $\overline{K_1} \odot C_n$ is a non-prime graph. In case of $m \neq 1$, it is a direct consequence of Lemma 3.2.
(2) Since $\beta(H_n^*) = n = \lfloor \frac{2n+1}{2} \rfloor$, and in case of $m = 1$, we have $\beta(\overline{K_1} \odot H_n^*) = \beta(H_n^*) = n < n+1 = \lfloor \frac{2n+2}{2} \rfloor$. Hence $\overline{K_1} \odot H_n^*$ is a non-prime graph. In case of $m \neq 1$, it is a direct consequence of Lemma 3.2.
(3) In case of $m = 1$ and n even, $\beta(\overline{K_1} \odot F_n) = \beta(F_n) = \lfloor \frac{n+1}{2} \rfloor < \lfloor \frac{n+2}{2} \rfloor$, hence $\overline{K_1} \odot C_n$ is a non-prime graph. In case of $m \neq 1$, it is a direct consequence of Lemma 3.2.
(4) Since $\beta(T_n) = n - 1 = \lfloor \frac{2n-1}{2} \rfloor$, and in case of $m = 1$, $\beta(\overline{K_1} \odot T_n) = \beta(T_n) = n - 1 < n = \lfloor \frac{2n}{2} \rfloor$, hence $\overline{K_1} \odot T_n$ is a non-prime graph. In case of $m \neq 1$, it is a direct consequence of Lemma 3.2.
(5) In case of $m = 1$ and n even, $\beta(\overline{K_1} \odot M_n) = \beta(M_n) = n - 1 < n = \lfloor \frac{2n+1}{2} \rfloor$, hence $\overline{K_1} \odot M_n$ is a non-prime graph. In case of $m \neq 1$, it is a direct consequence of Lemma 3.2.
(6) In case of $m = 1$, $\beta(\overline{K_1} \odot C_n^k) = \beta(C_n^k) < \lfloor \frac{n}{2} \rfloor \leq \lfloor \frac{n+1}{2} \rfloor$, hence $\overline{K_1} \odot C_n^k$ is a non-prime graph. In case of $m \neq 1$, it is a direct consequence of Lemma 3.2.
(8) In case of $m = 1$ and $n > 2$, $\beta(\overline{K_1} \odot (P_n + \overline{K_2})) = \beta(P_n + \overline{K_2}) = \max\{2, \lfloor \frac{n+1}{2} \rfloor\} < \lfloor \frac{n+3}{2} \rfloor$, hence $\overline{K_1} \odot (P_n + \overline{K_2})$ is a non-prime graph. In case of $m \neq 1$ and $n \neq 1$, it is a direct consequence of Lemma 3.2.
(9) In case of $m = 1$, $\beta(\overline{K_1} \odot D_{2n+1,2k}) = \beta(D_{2n+1,2k}) = \lfloor \frac{2n+1}{2} \rfloor + \lfloor \frac{2k+1}{2} \rfloor = n + k < n + k + 1 = \lfloor \frac{2n+2k+2}{2} \rfloor$, hence $\overline{K_1} \odot D_{2n+1,2k}$ is a non-prime graph. In case of $m \neq 1$, it is a direct consequence of Lemma 3.2.
(11) In case of $m = 1$, $\beta(\overline{K_1} \odot \overline{W_n}) = \beta(\overline{W_n}) = n + \lfloor \frac{n}{2} \rfloor < n + \lfloor \frac{n}{2} \rfloor + 1 = \lfloor \frac{3n+2}{2} \rfloor$, hence $\overline{K_1} \odot \overline{W_n}$ is a non-prime graph. In case of $m \neq 1$, it is a direct consequence of Lemma 3.2.
(7), (10) and (12) are direct consequences of Lemma 3.2.

Corollary 3.4.

If $G_1, G_2, ..., G_n$ are graphs, $n > 2$, and G_n is of m vertices satisfying that $\beta(G_n) \leq \lfloor \frac{m}{2} \rfloor$, then $(...((G_1 \odot G_2) \odot ... \odot) G_{n-1}) \odot G_n$ is non-prime.

Proof

We consider the graph $H = (...((G_1 \odot G_2) \odot ...) \odot G_{n-1})$, then $(...(G_1 \odot G_2) \odot ... \odot G_{n-1}) \odot G_n = H \odot G_n$, and by Theorem 3.2 we get $(...((G_1 \odot G_2) \odot ...) \odot G_{n-1}) \odot G_n$ is non-prime.

Theorem 3.5.

If $G_1, G_2, ..., G_n$ are graphs of $m_1, m_2, ..., m_n$ vertices satisfying that $m_{n-1} \neq 1$, and $\beta(G_n) \leq \lfloor \frac{m_n}{2} \rfloor$, then $G_1 \odot (G_2 \odot (... \odot (G_{n-1} \odot G_n))...)$ is non-prime.

Proof

Let $H_1 = G_{n-1} \odot G_n$. From Lemma 3.1 we have $\beta(H_1) = m_{n-1}\beta(G_n)$, since

$\beta(G_n) \leq \lfloor \frac{m_n}{2} \rfloor$, and $m_{n-1} \neq 1$, we get $\beta(H_1) = m_{n-1}\beta(G_n) < \lfloor \frac{m_{n-1}(m_n+1)}{2} \rfloor$. Now let $H_2 = G_{n-2} \odot H_1$, since $\beta(H_1) < \lfloor \frac{m_{n-1}(m_n+1)}{2} \rfloor$, we get by Lemma 3.1 $\beta(H_2) = m_{n-2}\beta(H_1) < m_{n-2}\lfloor \frac{m_{n-1}(m_n+1)}{2} \rfloor \leq \lfloor \frac{m_{n-2}(m_{n-1}(m_n+1)+1)}{2} \rfloor$. Continuing recursively we get:
$\beta(H_i) < \lfloor \frac{m_{n-i}(m_{n-i+1}(m_{n-i+2}(...(m_{n-1}(m_n+1)...)+1)+1)}{2} \rfloor$, where $H_i = G_{n-i} \odot H_{i-1}$, and consequently $\beta(H_{n-1}) < \lfloor \frac{m_1(m_2(m_3(...(m_{n-1}(m_n+1)...)+1)+1)+1)}{2} \rfloor$. Hence by Condition 5 Theorem 2.1 we get:
$H_{n-1} = G_1 \odot (G_2 \odot (... \odot (G_{n-1} \odot G_n))...)$ is non-prime.
We are in need of the following lemma (see [1]).

Lemma 3.6.
For $n \geq 2$ we have $\frac{1}{6}\frac{n}{\ln(n)} \leq \pi(n) \leq 6\frac{n}{\ln(n)}$.

Corollary 3.7.
If $n > \frac{1}{m+1}e^{7(m+1)}$ then $K_n \odot \overline{K_m}$ is non-prime.

Proof
If $n > \frac{1}{m+1}e^{7(m+1)}$, then $1 > \frac{7(m+1)}{\ln(n(m+1))}$ this implies $n > \frac{7n(m+1)}{\ln(n(m+1))} = \frac{6n(m+1)}{\ln(n(m+1))} + \frac{n(m+1)}{\ln(n(m+1))}$. From Lemma 3.6 it follows that $n > \frac{6n(m+1)}{\ln(n(m+1))} + \frac{n(m+1)}{\ln(n(m+1))} > \pi(n(m+1)) + \frac{n(m+1)}{\ln(n(m+1))}$, and since $\frac{N}{\ln(N)} > 1$ for every $N > 1$, it follows that $n > \pi(n(m+1)) + 1$, hence from Condition 2 Theorem 2.1 we get $K_n \odot \overline{K_m}$ is not prime.

Corollary 3.8.
$K_n \odot \overline{K_m}$ is non-prime if $n > \pi(n(m+1)) + 1$.

Proof
Let $n > \pi(n(m+1)) + 1$. Then the graph $K_n \odot \overline{K_m}$ contains a complete subgraph of order $n > \pi(n(m+1)) + 1$, hence by Condition 2 Theorem 2.1 the graph $K_n \odot \overline{K_m}$ is non-prime.

Conjecture 3.9.
$K_n \odot \overline{K_m}$ is prime if $n \leq \pi(n(m+1)) + 1$.

Suppose that $n \leq \pi(nm+n)+1$ and that the n-vertices of K_n are $v_1, v_2, ..., v_n$ and the m vertices of $\overline{K_m}$ attached to v_i are $v_{i1}, v_{i2}, ..., v_{im}$. We will get a labelling of the graph $K_n \odot \overline{K_m}$ by the following way:
We label the vertex v_1 by the label 1 and the vertices $v_2, ..., v_n$ by the primes not exceeding $n(m+1)$ in a descending order then the minimum prime label of the vertices of K_n is $p_{\pi(n(m+1))-n+2}$, where p_i is the i th prime, where $p_1 = 2$. Depending on the fact that if p is a prime number and $p > \frac{N}{2}$ then $(p, r) = 1$ for all $r \leq N$, it suffices to label only the vertices attached to the vertices of K_n having labels less than or equal to $\frac{n(m+1)}{2}$. Since the minimum prime label on these n vertices is $p_{\pi(n(m+1))-n+2}$. We have two cases:
First if $p_{\pi(n(m+1))-n+2} > p_{\pi(\frac{n(m+1)}{2})}$, then by any distribution of the remaining numbers we get a prime labelling.
Second if $p_{\pi(n(m+1))-n+2} \leq p_{\pi(\frac{n(m+1)}{2})}$, then we will introduce a method to label those remaining vertices.
We list the remaining labels in a sorted sequence T_1. We start the labelling from

the vertices attached to the vertex of the label $p_{\pi(n(m+1))-n+2}$ till the vertices attached to the vertex of the label $p_{\pi(\frac{n(m+1)}{2})}$. We label the vertices attached to the vertex of the label $p_{\pi(n(m+1))-n+2}$ by the first m-labels in T_1, that are relatively prime to $p_{\pi(n(m+1))-n+2}$, and delete them from the sequence T_1 to get a new sequence T_2. Now we label the vertices attached to the vertex of the label $p_{\pi(n(m+1))-n+3}$ by the first m-labels in T_2, that are relatively prime to $p_{\pi(n(m+1))-n+3}$, and delete them from the sequence T_2 to get a new sequence T_3, and continue in the same manner.

To show that this procedure gives a prime labelling of $K_n \odot \overline{K_m}$, we suppose to the contrary that there exists a prime number $p_{\pi(n(m+1))-n+i+1} \leq \frac{n(m+1)}{2}$, which has no m-relatively prime labels in T_i. Without any loss of generality, suppose that there are $m-1$ relatively prime labels to $p_{\pi(n(m+1))-n+i+1}$ in T_i, then after removing these $m-1$ labels from T_i we get a new sequence T_{i+1} of order $L > m(\pi(n(m+1)) - \pi(\frac{n(m+1)}{2}) + 1)$. From Bertrand's postulate theorem (see [5]) we have $\pi(N) - \pi(\frac{N}{2}) \geq 1$, so we get $L > 2m$. Now we conjecture that there exist two numbers $r, r+j$ in T_{i+1} and $j < p_{\pi(n(m+1))-n+i+1}$, then we get a contradiction.

Corollary 3.10.

If $n < \frac{1}{m+1} e^{\frac{(m+1)}{6}}$ then $K_n \odot \overline{K_m}$ is prime.

Proof

If $n < \frac{1}{m+1} e^{\frac{(m+1)}{6}}$ then $1 < \frac{1}{6}\frac{(m+1)}{\ln(n(m+1))}$ this implies $n < \frac{1}{6}\frac{n(m+1)}{\ln(n(m+1))}$. From Lemma 3.6 it follows that $n < \frac{1}{6}\frac{n(m+1)}{\ln(n(m+1))} \leq \pi(n(m+1)) < \pi(n(m+1)) + 1$, then from Conjecture 3.9 we get $K_n \odot \overline{K_m}$ is prime.

By performing the calculations using the computer, we get the exact values of n for each $m \leq 20$, for which $K_n \odot \overline{K_m}$ is prime according to the conjecture (i.e., $n \leq \pi(n(m+1)) + 1$). They are shown in the following table.

m	n
1	$n \leq 7$
2	$n \leq 16$
3	$n \leq 33$, $n = 35$
4	$n \leq 78$
5	$n \leq 190$
6	$n \leq 443$, $n = 446$
7	$n \leq 1060$
8	$n \leq 2702$
9	$n \leq 6473$
10	$n \leq 15930$, $n = 16056$
11	$n \leq 40074$, $n = 40079, 40080, 40081, 40082,$ 40083, 40084, 40085, 40086, 40087, 40088, 40089, 40090, 40091, 40092, 40093, 40094, 40095, 40096, 40097,

	40098, 40099, 40100, 40101, 40102, 40103, 40104, 40105, 40106, 40107, 40108, 40109, 40110, 40111, 40112, 40113, 40114, 40115, 40116, 40117, 40118, 40119, 40120, 40121, 40122, 40123,n=40157
12	$n \leq 100366$
13	$n \leq 251710$, $n = 251712, 251713, 251718, 251720, 251721,$ 251722, 251723, 251724, 251726, 251732, 251733, 251734, 251735,251736 251737, 251738, 251741, 251742, 251743, 251744, 251745, 251746 251747, 251748, 251749, 251750, 251751, 251752, 251753, 251754 251755, 251756, 251757, 251758, 251759, 251760, 251761, 251762, 251766, 251767, 251770, 251771, 251788, 251789.
14	$n \leq 637345$
15	$n \leq 1617175$
16	$n \leq 4124437$, $n = 4124456, 4124457, 4124458, 4124459, 4124460,$ 4124461, 4124462, 4124463, 4124464, 4124465, 4124466, 4124467, 4124468, 4124469, 4124470, 4124471, 4124472, 4124473, 4124474, 4124583, 4124584, 4124585, 4124588,4124589, 4124590, 4124591, 4124592, 4124593, 4124594, 4124595, 4124596, 4124597, 4124598, 4124599, 4124600, 4124601, 4124602, 4124603, 4124604, 4124605, 4124606, 4124607, 4124608, 4124609, 4124610, 4124611,4124612, 4124613, 4124614, 4124615, 4124616, 4124617, 4124618, 4124619, 4124620, 4124621, 4124622, 4124623,4124624, 4124625, 4124626, 4124627, 4124628, 4124629, 4124630, 4124631, 4124632, 4124633, 4124634, 4124635, 4124636, 4124637, 4124638, 4124639, 4124640, 4124641, 4124642, 4124643, 4124644, 4124645, 4124646, 4124647, 4124648, 4124649, 4124650, 4124651, 4124652, 4124653, 4124654, 4124655, 4124656, 4124657, 4124658,4124659, 4124660, 4124661, 4124662, 4124663, 4124664, 4124665, 4124666, 4124667, 4124668, 4124669, 4124670, 4124671, 4124672, 4124673, 4124674, 4124675, 4124676, 4124677, 4124678, 4124679, 4124680, 4124681, 4124682, 4124683, 4124684, 4124685, 4124686, 4124687, 4124688, 4124689, 4124690, 4124691, 4124692, 4124693,4124694, 4124695, 4124696, 4124697, 4124698, 4124699, 4124700, 4124701, 4124702, 4124703, 4124704, 4124705, 4124706.

17	$n \le 10553415$, $n = 10553425, 10553426, 10553433, 10553434, 10553435$, 10553436, 10553438, 10553440, 10553441, 10553442, 10553443, 10553444, 10553445, 10553446, 10553447, 10553448, 10553449, 10553450, 10553451, 10553452, 10553453, 10553454, 10553455, 10553456, 10553457, 10553458, 10553459,10553460, 10553461, 10553462, 10553463, 10553464, 10553465, 10553466, 10553467, 10553468, 10553469, 10553470, 10553471, 10553472, 10553473, 10553474, 10553475, 10553476, 10553478, 10553479, 10553480, 10553481, 10553482, 10553483, 10553484, 10553485, 10553486, 10553487, 10553488, 10553489, 10553490, 10553491, 10553492, 10553493, 10553494, 10553495, 10553496, 10553497, 10553498, 10553499, 10553500, 10553502, 10553503, 10553504, 10553505, 10553506, 10553507, 10553508, 10553509, 10553510, 10553511, 10553512, 10553513, 10553514, 10553515, 10553516, 10553517, 10553518, 10553519, 10553520, 10553521, 10553522, 10553523, 10553524, 10553525, 10553526, 10553527, 10553528, 10553529, 10553530, 10553531, 10553532, 10553533, 10553534, 10553535, 10553536, 10553537, 10553538, 10553539, 10553540, 10553541, 10553542, 10553543, 10553544, 10553545, 10553546, 10553547, 10553548, 10553549, 10553550, 10553551, 10553552, 10553553, 10553554, 10553569, 10553570, 10553571, 10553572, 10553574, 10553575, 10553576, 10553577, 10553578, 10553646, 10553647, 10553816, 10553818, 10553819, 10553820, 10553825, 10553827, 10553828, 10553829, 10553830, 10553831, 10553832, 10553833, 10553834, 10553835, 10553836, 10553837, 10553838, 10553839, 10553840, 10553841, 10553842, 10553843, 10553844, 10553845, 10553846, 10553847, 10553848, 10553849, 10553850, 10553851, 10553852, 10553853, 10553854, 10553855.
18	$n \le 27066974$, $n = 27066987, 27066988, 27066989, 27066990, 27066991$, 27066992, 27067012, 27067013, 27067014, 27067015, 27067016, 27067017, 27067018, 27067019, 27067020, 27067021, 27067022, 27067023, 27067024, 27067025, 27067026, 27067027, 27067028, 27067029, 27067030, 27067031, 27067032, 27067033, 27067034, 27067035, 27067036, 27067037, 27067038, 27067043, 27067044, 27067045, 27067046, 27067047, 27067048, 27067049, 27067050, 27067051, 27067052, 27067053, 27067054, 27067055, 27067056, 27067057, 27067058, 27067059, 27067060, 27067061, 27067062, 27067063, 27067064, 27067065, 27067066, 27067067, 27067068, 27067069,

	27067070, 27067071, 27067072, 27067073, 27067074, 27067075, 27067076, 27067077, 27067078, 27067079, 27067080, 27067081, 27067082, 27067083, 27067084, 27067085, 27067086, 27067087, 27067088, 27067089, 27067090, 27067091, 27067092, 27067093, 27067094, 27067095, 27067096, 27067097, 27067099, 27067100, 27067101, 27067102, 27067103, 27067104, 27067105, 27067106, 27067107, 27067108, 27067109, 27067110, 27067111, 27067112, 27067113, 27067114, 27067115, 27067116, 27067117, 27067118, 27067119, 27067120, 27067121, 27067125, 27067126, 27067127, 27067128, 27067133, 27067134, 27067135, 27067136, 27067137, 27067138, 27067139, 27067140, 27067141, 27067230, 27067262, 27067263, 27067264, 27067265, 27067266, 27067267, 27067268, 27067269, 27067270, 27067271, 27067272, 27067273, 27067274, 27067275, 27067276, 27067277, 27067278, 27067520, 27067521, 27067522, 27067524, 27067525.
19	$n \leq 69709707$, $n =$ 69709709, 69709710, 69709711, 69709712, 69709713, 69709714, 69709715, 69709716, 69709717, 69709718, 69709719, 69709720, 69709721, 69709723, 69709724, 69709725, 69709726, 69709727, 69709728, 69709729, 69709730, 69709731, 69709732, 69709733, 69709734, 69709735, 69709737, 69709869, 69709870, 69709871, 69709872, 69709873, 69709874, 69709877, 69709878, 69709879, 69709882, 69709918, 69709922, 69709932, 69709933, 69709934, 69709935, 69709936, 69709941, 69709942, 69709943, 69709944, 69709945, 69709954, 69709955, 69709956, 69709957, 69709958, 69709959, 69709960, 69709961, 69709962, 69709963, 69709964, 69709965, 69709966.
20	$n \leq 179993025$, $n =$ 179993029, 179993141, 179993146, 179993160, 179993161, 179993162, 179993167, 179993170, 179993171, 179993172, 179993173, 179993174.

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