On the signed edge domination numbers of $K_{m,n}$ *

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Abstract

Let $G = (V, E)$ be a simple undirected graph. For an edge e of G, the closed edge-neighborhood of e is the set $N[e] = \{e' \in E | e' \text{ is adjacent to } e\} \cup \{e\}$. A function $f : E \to \{1, -1\}$ is called *a signed edge domination function* (SEDF) of G if $\sum_{e' \in N[e]} f(e') \geq 1$ for every edge e of G. *The signed edge domination number* of G is defined as $\gamma'_s(G) = \min \{\sum_{e \in E} f(e) | f \text{ is an SEDF of } G\}$. In this paper, we determine the signed edge domination numbers of all complete bipartite graph $K_{m,n}$, and therefore determine the signed domination numbers of $K_m \times K_n$.

Keywords: Signed domination function; Signed domination number; Signed edge domination function; Signed edge domination number.

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1 Introduction

Let $G = (V, E)$ be a simple undirected graph with vertex set V and edge set E. For a vertex u of G, let $N_G(u)$ denote the open neighborhood of u in G and the closed neighborhood of u is the set $N_G[u] = N_G(u) \cup \{u\}$. The degree of u is denoted by $d_G(u)$. For a subset S of V(or E), let $G[S]$

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be the subgraph of G induced by S and $G - S$ be the subgraph induced by $V - S$(or $E - S$). For an edge e of G, the open edge-neighborhood of e is the set $N_G(e) = \{e' \in E(G) | e' \textit{ is adjacent to } e\}$ and the closed edge-neighborhood of e is the set $N_G[e] = N_G(e) \cup \{e\}$. Let $\lfloor x \rfloor$ be the integer part of a non-negative real number x. Other terminologies used in this paper will follow [1].

A signed domination function of G is defined in [2] as a function $f : V \to \{+1, -1\}$ such that $f(N_G[v]) \geq 1$ for every $v \in V$. The signed domination number for a graph G is $\gamma_s(G) = \min\{\sum_{v \in V(G)} f(v) | f$ is a signed domination function of $G\}$. A function $f : E(G) \to \{+1, -1\}$ is called *a signed edge domination function* (SEDF) of G if $\sum_{e' \in N_G[e]} f(e') \geq 1$ for every edge e of G. *The signed edge domination number* of G is defined as $\gamma'_s(G) = \min \{\sum_{e \in E(G)} f(e) | f$ is an SEDF of $G\}$. Denote $f(G) = \sum_{e \in E(G)} f(e)$.

The cartesian product $G \times H$ of graphs G and H is a graph such that the vertex set of $G \times H$ is the cartesian product $V(G) \times V(H)$ and any two vertices (u, u') and (v, v') are adjacent in $G \times H$ if and only if either $u = v$ and u' is adjacent to v' in H, or $u' = v'$ and u is adjacent to v in G.

The line graph $L(G)$ of graph G is a graph such that each vertex of $L(G)$ represents an edge of G and two vertices of $L(G)$ are adjacent if and only if their corresponding edges share a common endpoint ("are adjacent") in G. Clearly an SEDF of a graph G is a signed domination function of $L(G)$.

Xu[4] showed that

Proposition 1.1 *([4]). Let $K_{m,n}$ be a complete bipartite graph. Then*

$$\gamma'_s(K_{m,n}) \geq \begin{cases} \frac{2mn}{m+n-1} & m+n \text{ is odd} \\ \frac{mn}{m+n-1} & m+n \text{ is even} \end{cases}$$

For other results on the signed edge domination number, the readers may refer to the survey papers of Xu[3,4].

In this paper, we determine the signed edge domination numbers of all complete bipartite graph $K_{m,n}$, and therefore determine the signed domination numbers of $K_m \times K_n$.

2 Main results

In this paper, for a complete bipartite graph $K_{m,n}$, we always assume that $m \leq n$ and $V_1 = \{v_1, v_2, \cdots, v_m\}$ and $V_2 = \{u_1, u_2, \cdots, u_n\}$ are the partition sets of $K_{m,n}$. It is well known that $K_{m,m}$ has m pairwise edge-disjoint perfect matchings.

For convenience, write $E_1 = \{e \in E | f(e) = 1\}$, $E_2 = \{e \in E | f(e) = -1\}$, and for a vertex u, denote $d^*_f(u) = d_{G_1}(u) - d_{G_2}(u)$, where $G_i = G[E_i], i = 1, 2$.

If $g = \lfloor \frac{n}{m} \rfloor$, we let $G'_j = K_{m,n}[V_1, V'_{2,j}]$, then $G'_j \simeq K_{m,m}$, where $V'_{2,j} = \{u_{1+jm}, u_{2+jm}, \cdots, u_{m+jm}\}$ and $j \in \{0, 1, \cdots, (g-1)\}$.

Theorem 1 *Let $K_{m,n}$ be a complete bipartite graph. Then*

$$\gamma'_s(K_{m,n}) = \begin{cases} n, & m \equiv 1 \pmod 2, n \equiv 1 \pmod 2, \\ & n \leq 2m-1; \\ 2m-1, & m \equiv 1 \pmod 2, n \equiv 1 \pmod 2, \\ & n \geq 2m+1; \\ n, & m \equiv 0 \pmod 2, n \equiv 0 \pmod 2, \\ & n \leq 2m-1; \\ 2m, & m \equiv 0 \pmod 2, n \equiv 0 \pmod 2, \\ & n \geq 2m; \\ 2m, & m \equiv 1 \pmod 2, n \equiv 0 \pmod 2, \\ & n \leq 2m; \\ n, & m \equiv 1 \pmod 2, n \equiv 0 \pmod 2, \\ & 2m+2 \leq n \leq 3m-1; \\ 3m-1, & m \equiv 1 \pmod 2, n \equiv 0 \pmod 2, \\ & n \geq 3m+1; \\ 2m, & m \equiv 0 \pmod 2, n \equiv 1 \pmod 2, \\ & n \leq 2m; \\ n+1, & m \equiv 0 \pmod 2, n \equiv 1 \pmod 2, \\ & 2m+1 \leq n \leq 3m-1; \\ 3m, & m \equiv 0 \pmod 2, n \equiv 1 \pmod 2, \\ & n \geq 3m+1. \end{cases}$$

Proof. We only prove items 1,2,3 and 4, as the proofs of items 5-10 are similar. For items 1 and 2, define a function $f : E(K_{m,n}) \to \{+1, -1\}$ as follows:

Let the value of e in the $\frac{m-1}{2}$ pairwise edge-disjoint perfect matchings of G'_0 be -1, and the value of e in the others be 1.

If $n \leq 2m-1$, then the value of other edge in $K_{m,n}$ is

$$f(v_i u_{m+j}) = \begin{cases} 1 & i+j \equiv 0 \pmod 2, i = 1, 2, \cdots, m-1; \\ & j = 1, 2, \cdots, n-m \\ -1 & i+j \equiv 1 \pmod 2, i = 1, 2, \cdots, m-1; \\ & j = 1, 2, \cdots, n-m \\ 1 & i = m; j = 1, 2, \cdots, n-m \end{cases}$$

If $n \geq 2m+1$, then for every vertex $v_i \in V_1$, $f(v_i u_n) = 1$ and the value of other edge in $K_{m,n}$ is

$$f(v_i u_{m+j}) = \begin{cases} 1 & i+j \equiv 1 \pmod 2, i = 1, 2, \cdots, m; \\ & j = 1, 2, \cdots, n-m-1 \\ -1 & i+j \equiv 0 \pmod 2, i = 1, 2, \cdots, m; \\ & j = 1, 2, \cdots, n-m-1 \end{cases}$$

Clearly, f is an SEDF of $K_{m,n}$, and

$$f(K_{m,n}) = \begin{cases} n, & n \le 2m-1 \\ 2m-1, & n \ge 2m+1 \end{cases}$$

Thus

$$\gamma'_s(K_{m,n}) \le \begin{cases} n & n \le 2m-1 \\ 2m-1, & n \ge 2m+1 \end{cases}$$

Now let f' be an SEDF of $K_{m,n}$, and $f'(K_{m,n}) = \gamma'_s(K_{m,n})$. Observe that $d^*_{f'}(v)$ is odd for all v of $K_{m,n}$. Assume that there exists a vertex u of $K_{m,n}$ such that $d^*_{f'}(u) \le -3$, then every vertex v of $N_{K_{m,n}}(u)$ satisfies

$$d^*_{f'}(v) \ge 3 \quad or \quad d^*_{f'}(v) \ge 5.$$

Hence $f'(K_{m,n}) \ge 3m$, a contradiction. So $d^*_{f'}(u) \ge -1$ for every vertex u of $K_{m,n}$.

Now we consider two cases.

Case 1. $n \le 2m-1$. If there exists a vertex u of V_2 such that $d^*_{f'}(u) = -1$, then every vertex v of V_1 satisfies

$$d^*_{f'}(v) \ge 1 \quad or \quad d^*_{f'}(v) \ge 3,$$

and the numbers of vertices in V_1 satisfying the former and the latter of the above inequalities are equal to $\frac{m+1}{2}$ and $\frac{m-1}{2}$, respectively.

When $n = 2m-1$, we have

$$f'(K_{m,n}) \ge \frac{m+1}{2} \cdot 1 + \frac{m-1}{2} \cdot 3 = 2m-1.$$

By $\gamma'_s(K_{m,n}) \le n$, we obtain $\gamma'_s(K_{m,n}) = n$.

When $n \le 2m-3$, we have

$$f'(K_{m,n}) \ge \frac{m+1}{2} \cdot 1 + \frac{m-1}{2} \cdot 3 = 2m-1 > n,$$

which is a contradiction. Hence for every vertex u of V_2, we have $d^*_{f'}(u) \ge 1$. Therefore $f'(K_{m,n}) \ge n$. Also by $\gamma'_s(K_{m,n}) \le n$, we obtain $\gamma'_s(K_{m,n}) = n$.

Case 2. $n \ge 2m+1$. If for every vertex u of V_2, $d^*_{f'}(u) \ge 1$, then

$$f'(K_{m,n}) = n > 2m-1,$$

which is a contradiction. Therefore there exists a vertex u of V_2 such that $d^*_{f'}(u) = -1$, then every vertex v of V_1 satisfies

$$d^*_{f'}(v) \ge 1 \quad or \quad d^*_{f'}(v) \ge 3,$$

and the numbers of vertices in V_1 satisfying the former and the latter of the above inequalities are equal to $\frac{m+1}{2}$ and $\frac{m-1}{2}$, respectively. Therefore

$$f'(K_{m,n}) \geq \frac{m+1}{2} \cdot 1 + \frac{m-1}{2} \cdot 3 = 2m - 1.$$

It follows that $\gamma_s'(K_{m,n}) = 2m - 1$.

Now we prove items 3 and 4. We consider two cases.

Case 1. $n \leq 2m - 2$. Define a function $f : E(K_{m,n}) \to \{+1, -1\}$ as follows:

For each $e = v_i u_j$ of $E(G_0')$,

$$f(v_i u_j) = \begin{cases} -1 & i = 1, 2, \cdots, \frac{1}{2}m; j = 1, 2, \cdots, \frac{1}{2}m, \\ 1 & i = 1, 2, \cdots, \frac{1}{2}m; j = \frac{1}{2}m + 1, \frac{1}{2}m + 2, \cdots, m, \\ 1 & i = \frac{1}{2}m + 1, \frac{1}{2}m + 2, \cdots, m; j = 1, 2, \cdots, \frac{1}{2}m, \\ 1 & i = j = \frac{1}{2}m + 1, \frac{1}{2}m + 2, \cdots, m, \\ -1 & \text{otherwise}, \end{cases}$$

where for $e = v_i u_{m+j}$ of $E(K_{m,n}) - E(G_0')$,

$$f(v_i u_{m+j}) = \begin{cases} -1 & i = 1, 2, \cdots, \frac{1}{2}m; \text{ j is odd}, \\ 1 & i = \frac{1}{2}m + 1, \frac{1}{2}m + 2, \cdots, m; \text{ j is odd}, \\ 1 & i = 1, 2, \cdots, \frac{1}{2}m + 1; \text{ j is even}, \\ -1 & i = \frac{1}{2}m + 2, \frac{1}{2}m + 3, \cdots, m; \text{ j is even}. \end{cases}$$

Clearly, f is an SEDF of $K_{m,n}$ such that $f(K_{m,n}) = n$. Thus, $\gamma_s'(K_{m,n}) \leq n$.

Now let f' be an SEDF of $K_{m,n}$ such that $f'(K_{m,n}) = \gamma_s'(K_{m,n})$. Suppose that there exists a vertex v of V_1 such that $d_{f'}^*(v) \leq -2$, then for every vertex u of V_2,

$$d_{f'}^*(u) \geq 2 \quad or \quad d_{f'}^*(u) \geq 4.$$

Therefore

$$f'(K_{m,n}) \geq 2n,$$

which is a contradiction. Thus, for every vertex v of V_1, $d_{f'}^*(v) \geq 0$. Suppose that $d_{f'}^*(v) \geq 2$ for every vertex v of V_1. Then

$$f'(K_{m,n}) \geq 2m > n,$$

which is also a contradiction. Hence there exists a vertex v of V_1 such that $d_{f'}^*(v) = 0$. Then for every vertex u of V_2,

$$d_{f'}^*(u) \geq 0 \quad or \quad d_{f'}^*(u) \geq 2,$$

and the number of vertices in V_2 satisfying each of the above inequalities is equal to $\frac{n}{2}$. Therefore

$$f'(K_{m,n}) \geq \frac{n}{2} \cdot 0 + \frac{n}{2} \cdot 2 = n.$$

It follows that $\gamma_s'(K_{m,n}) = n$ when $n \leq 2m-2$.

Case 2. $n \geq 2m$. Define a function $f : E(K_{m,n}) \to \{+1,-1\}$ as follows:

Assign $f(e) = -1$ for each edge e of the $\frac{m-2}{2}$ pairwise edge-disjoint perfect matchings of G_0', and assign $f(e) = -1$ for the rest of edge in G_0'. The value of other edge in $K_{m,n}$ is

$$f(v_i u_{m+j}) = \begin{cases} 1 & i+j \equiv 0 \pmod 2, i = 1,2,\cdots,m; \\ & j = 1,2,\cdots,n-m \\ -1 & i+j \equiv 1 \pmod 2, i = 1,2,\cdots,m; \\ & j = 1,2,\cdots,n-m \end{cases}$$

Clearly, f is an SEDF of $K_{m,n}$ such that $f(K_{m,n}) = 2m$. Thus, $\gamma_s'(K_{m,n}) \leq 2m$.

Now let f' be an SEDF of $K_{m,n}$ such that $f'(K_{m,n}) = \gamma_s'(K_{m,n})$. Suppose that there exists a vertex u of V_2 such that $d_{f'}^*(u) = -2$. Then for every vertex v of V_1,

$$d_{f'}^*(v) \geq 2 \quad or \quad d_{f'}^*(v) \geq 4,$$

and the numbers of vertices in V_1 satisfying the former and the latter of the above inequalities are equal to $\frac{m+2}{2}$ and $\frac{m-2}{2}$, respectively. Therefore, $m \geq 4$ and

$$f'(K_{m,n}) \geq \frac{m+2}{2} \cdot 2 + \frac{m-2}{2} \cdot 4 = 3m - 2 > 2m,$$

which is a contradiction. Suppose that there exists a vertex u of V_2 such that $d_{f'}^*(u) \leq -4$. Then we can get a similar contradiction. Hence, for every vertex u of V_2, $d_{f'}^*(u) \geq 0$. Suppose that for every vertex u of V_2, $d_{f'}^*(u) \geq 2$. Then we have

$$f'(K_{m,n}) \geq 2n,$$

which is a contradiction. So there exists a vertex u of V_2 such that $d_{f'}^*(u) = 0$.

Let k be the number of vertices u of V_2 satisfying $d_{f'}^*(u) = 0$. Suppose that $k < n - m$. Then

$$f'(K_{m,n}) > 2(n - (n - m)) = 2m,$$

which is a contradiction. Hence, $k \geq n - m$.

If $k = n - m$, then

$$f'(K_{m,n}) \geq 2(n - (n - m)) = 2m.$$

It follows that $\gamma'_s(K_{m,n}) = 2m$.

If $k > n - m$, then for every vertex v of V_1, there exists a vertex u of V_2 satisfying $d^*_{f'}(u) = 0$ and $f'(vu) = 1$. Since f' is an SEDF of $K_{m,n}$, we have

$$d^*_{f'}(v) + d^*_{f'}(u) - f(vu) \geq 1.$$

So

$$d^*_{f'}(v) \geq 2.$$

Hence

$$f'(K_{m,n}) \geq 2m.$$

It follows that $\gamma'_s(K_{m,n}) = 2m$, completing the proof. □

Since the line graph of $K_{m,n}$ corresponds to $K_m \times K_n$, we can get the following corollary.

Corollary 1 *Let $K_m \times K_n$ be the cartesian product of K_m and K_n, where $m \leq n$. Then*

$$\gamma_s(K_m \times K_n) = \begin{cases} n, & m \equiv 1 \pmod 2,\ n \equiv 1 \pmod 2, \\ & n \leq 2m - 1; \\ 2m - 1, & m \equiv 1 \pmod 2,\ n \equiv 1 \pmod 2, \\ & n \geq 2m + 1; \\ n, & m \equiv 0 \pmod 2,\ n \equiv 0 \pmod 2, \\ & n \leq 2m - 1; \\ 2m, & m \equiv 0 \pmod 2,\ n \equiv 0 \pmod 2, \\ & n \geq 2m; \\ 2m, & m \equiv 1 \pmod 2,\ n \equiv 0 \pmod 2, \\ & n \leq 2m; \\ n, & m \equiv 1 \pmod 2,\ n \equiv 0 \pmod 2, \\ & 2m + 2 \leq n \leq 3m - 1; \\ 3m - 1, & m \equiv 1 \pmod 2,\ n \equiv 0 \pmod 2, \\ & n \geq 3m + 1; \\ 2m, & m \equiv 0 \pmod 2,\ n \equiv 1 \pmod 2, \\ & n \leq 2m; \\ n + 1, & m \equiv 0 \pmod 2,\ n \equiv 1 \pmod 2, \\ & 2m + 1 \leq n \leq 3m - 1; \\ 3m, & m \equiv 0 \pmod 2,\ n \equiv 1 \pmod 2, \\ & n \geq 3m + 1. \end{cases}$$

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