Roman domination number of Generalized Petersen Graphs $P(n, 2)$ *

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Abstract

A *Roman domination function* on a graph $G = (V, E)$ is a function $f : V(G) \rightarrow \{0, 1, 2\}$ satisfying the condition that every vertex u with $f(u) = 0$ is adjacent to at least one vertex v with $f(v) = 2$. The *weight* of a Roman domination function f is the value $f(V(G)) = \sum_{u \in V(G)} f(u)$. The minimum weight of a Roman dominating function on a graph G is called the *Roman domination number* of G, denoted by $\gamma_R(G)$. In this paper, we study the *Roman domination number* of generalized Petersen graphs $P(n, 2)$ and prove that $\gamma_R(P(n, 2)) = \lceil \frac{8n}{7} \rceil (n \geq 5)$.

Keywords: *Roman domination number; Generalized Petersen Graph;*

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1 Introduction

Let $G = (V, E)$ be a simple graph, i.e., loopless and without multiple edges, with vertex set $V(G)$ and edge set $E(G)$. The open neighborhood, $N(v)$, and the closed neighborhood, $N[v]$, of a vertex $v \in V$ are denoted by $N(v) = \{u \in V(G) : vu \in E(G)\}$ and $N[v] = N(v) \cup \{v\}$, respectively. For a vertex set $S \subseteq V(G)$, $N(S) = \underset{v \in S}{\cup} N(v)$ and $N[S] = \underset{v \in S}{\cup} N[v]$. The maximum degree of any vertex in $V(G)$ is denoted by $\Delta(G)$.

A set $S \subseteq V(G)$ is a dominating set if for each $v \in V(G)$ either $v \in S$ or v is adjacent to some $w \in S$. That is, S is a dominating set if and only if $N[S] = V(G)$. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set of G, and a dominating set S of minimum cardinality is called a *γ-set* of G.

For a graph G, let $f : V \to \{0, 1, 2\}$, and let $(V_0; V_1; V_2)$ be the order partition of V induced by f, where $V_i = \{v \in V(G) | f(v) = i\}$ and $|V_i| = n_i$, for $i = 0, 1, 2$. Note that there exists a 1-1 correspondence between the functions $f : V(G) \to \{0, 1, 2\}$ and the ordered partitions $(V_0; V_1; V_2)$ of $V(G)$. So we denote $f = (V_0; V_1; V_2)$.

A function $f : V(G) \to \{0, 1, 2\}$ is a *Roman domination function* (*RDF*) if V_2 dominates V_0, i.e. $V_0 \subseteq N[V_2]$. The *weight* of f is $f(V(G)) = \sum_{v \in V(G)} f(v) = 2n_2 + n_1$. The minimum weight of an RDF of G is called the *Roman domination number* of G, denoted by $\gamma_R(G)$. We say that a function $f = (V_0; V_1; V_2)$ is a *γ_R-function* if it is an RDF and $f(V) = \gamma_R(G)$.

In 2004, Cockayne et al[?] studied the graph theoretic properties of this variant of the domination number of a graph and proved

Proposition 1.1. For any graph G of order n, $\frac{2n}{\Delta(G)+1} \leq \gamma_R(G)$.

Proposition 1.2. For any graph G of order n, $\gamma(G) \leq \gamma_R(G) \leq 2\gamma(G)$.

For more references and other Roman dominating problems, we can refer to [? ? ? ? ? ?].

The generalized Petersen graph $P(n,k)$ is defined to be a graph on $2n$ vertices with $V(P(n,k)) = \{v_i, u_i : 0 \leq i \leq n-1\}$ and $E(P(n,k)) = \{v_i v_{i+1}, v_i u_i, u_i u_{i+k} : 0 \leq i \leq n-1$, subscripts taken modulo $n\}$.

In 2007, Yang Yuansheng et al [? ?] studied the domination number of generalized Petersen graphs $P(n,2)$ and $P(n,3)$. They proved

Theorem 1.3. $\gamma(P(n,2)) = n - \lfloor \frac{n}{5} \rfloor - \lfloor \frac{n+2}{5} \rfloor$.

Theorem 1.4. $\gamma(\mathrm{P}(n,3)) = n - 2\lfloor \frac{n}{4} \rfloor (n \neq 11)$.

In this paper, we study the *Roman* domination in the generalized Petersen graphs $P(n,2)$ and prove $\gamma_R(P(n,2)) = \lceil \frac{8n}{7} \rceil (n \geq 5)$.

2 Roman domination number of $P(n,2)$

Let $m = \lfloor \frac{n}{7} \rfloor$, $t = n (\mathrm{mod}\ 7)$, then $n = 7m + t$.

Lemma 2.1. $\gamma_R(P(n,2)) \leq \lceil \frac{8n}{7} \rceil (n \geq 5)$.

Proof. In order to prove that for $n \geq 5$, $\gamma_R(P(n,2)) \leq \lceil \frac{8n}{7} \rceil$, it suffices to give an RDF f of $P(n,2)$ with $f(V(P(n,2))) = \lceil \frac{8n}{7} \rceil$. For $n = 5$, let

$V_2 = \{v_0, u_2, u_3\}$, $V_1 = \emptyset$, $V_0 = N(V_2)$.

For $n = 6$, let

$$V_2 = \{v_0, u_3, u_4\},\ V_1 = \{v_2\},\ V_0 = N(V_2).$$

For $n \geq 7$, let

$$V_2 = \begin{cases} \{v_{7i}, u_{7i+3}, u_{7i+4} \ :\ 0 \leq i \leq m-1\}, & \text{if } t = 0; \\ \{v_{7i}, u_{7i+3}, u_{7i+4} \ :\ 0 \leq i \leq m-1\}, & \text{if } t = 1; \\ \{v_{7i}, u_{7i+3}, u_{7i+4} \ :\ 0 \leq i \leq m-1\} \cup \{v_{7m}\}, & \text{if } t = 2; \\ \{v_{7i}, u_{7i+3}, u_{7i+4} \ :\ 0 \leq i \leq m-1\} \cup \{v_{7m}\}, & \text{if } t = 3; \\ \{v_{7i}, u_{7i+3}, u_{7i+4} \ :\ 0 \leq i \leq m-1\} \cup \{v_{7m-1}, u_{7m+1}, u_{7m+2}\}, & \text{if } t = 4; \\ \{v_{7i}, u_{7i+3}, u_{7i+4} \ :\ 0 \leq i \leq m-1\} \cup \{v_{7m}, u_{7m+2}, u_{7m+3}\}, & \text{if } t = 5; \\ \{v_{7i}, u_{7i+3}, u_{7i+4} \ :\ 0 \leq i \leq m\}, & \text{if } t = 6. \end{cases}$$

$$V_1 = \begin{cases} \{v_{7i+2}, v_{7i+5} \ :\ 0 \leq i \leq m-1\}, & \text{if } t = 0; \\ \{v_{7i+2}, v_{7i+5} \ :\ 0 \leq i \leq m-1\} \cup \{v_{7m-1}, u_{7m}\}, & \text{if } t = 1; \\ \{v_{7i+2}, v_{7i+5} \ :\ 0 \leq i \leq m-1\} \cup \{u_{7m+1}\}, & \text{if } t = 2; \\ \{v_{7i+2}, v_{7i+5} \ :\ 0 \leq i \leq m-1\} \cup \{u_{7m+1}, u_{7m+2}\}, & \text{if } t = 3; \\ \{v_{7i+2}, v_{7i+5} \ :\ 0 \leq i \leq m-1\} \setminus \{v_{7m-2}\}, & \text{if } t = 4; \\ \{v_{7i+2}, v_{7i+5} \ :\ 0 \leq i \leq m-1\}, & \text{if } t = 5; \\ \{v_{7i+2}, v_{7i+5} \ :\ 0 \leq i \leq m-1\} \cup \{v_{7m+2}\}, & \text{if } t = 6. \end{cases}$$

$$V_0 = N(V_2).$$

Note that V_0, V_1 and V_2 are pairwise disjoint, and $V(P(n,2)) = V_1 \cup$

$V_2 \cup V_0 = V_1 \cup N[V_2]$. Hence $f = (V_0; V_1; V_2)$ is an RDF of $P(n,2)$ with

$$f(V(P(n,2))) = \begin{cases} 2\times 3m + 2m = 8m = \lceil \frac{8n}{7} \rceil, & \text{if } t = 0; \\ 2\times 3m + 2m + 2 = 8m + 2 = \lceil \frac{8n}{7} \rceil, & \text{if } t = 1; \\ 2\times (3m+1) + 2m + 1 = 8m + 3 = \lceil \frac{8n}{7} \rceil, & \text{if } t = 2; \\ 2\times (3m+1) + 2m + 2 = 8m + 4 = \lceil \frac{8n}{7} \rceil, & \text{if } t = 3; \\ 2\times (3m+3) + 2m - 1 = 8m + 5 = \lceil \frac{8n}{7} \rceil, & \text{if } t = 4; \\ 2\times (3m+3) + 2m = 8m + 6 = \lceil \frac{8n}{7} \rceil, & \text{if } t = 5; \\ 2\times (3m+3) + 2m + 1 = 8m + 7 = \lceil \frac{8n}{7} \rceil, & \text{if } t = 6. \end{cases}$$

□

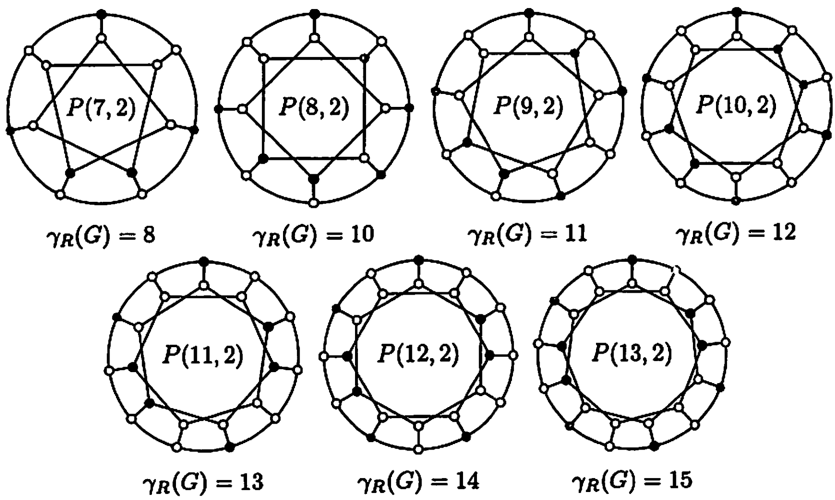

Figure 2.1: The RDFs of $P(n,2)$ for $7 \le n \le 13$

In Figure 2.1, we give the RDFs of $P(n,2)$ for $7 \le n \le 13$, where the vertices of V_2 are in dark, the vertices of V_1 are in grey, and the vertices of V_0 are in white.

Let $f = (V_0; V_1; V_2)$ be an arbitrary γ_R-function of $P(n,2)$. Then we have

Lemma 2.2. For any vertex $w_1 \in V_2$, if $w_2 \in N(w_1)$, then $w_2 \notin V_1$.

Proof. Suppose to the contrary that $w_2 \in V_1$. Let $f'(w_2) = 0$ and $f'(w) = f(w)$ for every vertex $w \in V(P(n,2)) \setminus w_2$. Then f' is an RDF of $P(n,2)$ with $f'(V(P(n,2))) = \gamma_R(P(n,2)) - 1$, a contradiction. □

Let $f_m = (V_0; V_1; V_2)$ be an arbitrary γ_R-function of $P(n,2)$ with minimum cardinality of V_2, i.e. $|V_2| \leq |V_2'|$ for any γ_R-function $f' = (V_0'; V_1'; V_2')$ of $P(n,2)$. Then we have

Lemma 2.3. For any vertex $w_1 \in V_2$, if $w_2 \in N(w_1)$, then $w_2 \notin V_2$.

Proof. Suppose to the contrary that $w_2 \in V(P(n,2))$ such that $w_1, w_2 \in V_2$. Let $N(w_1) = \{w_2, w_3, w_4\}$ where $f_m(w_3) \geq f_m(w_4)$. There are two cases depending on w_4:

Case 1. $w_4 \in V_1 \cup V_2$. Let $f'(w_1) = 0$ and $f'(w) = f_m(w)$ for every vertex $w \in V(P(n,2)) \setminus \{w_1\}$. Then f is an RDF of $P(n,2)$ with $f'(V(P(n,2))) = \gamma_R(P(n,2)) - 2$, a contradiction.

Case 2. $w_4 \in V_0$. For $w_3 \in V_1 \cup V_2$, let $f'(w_1) = 0$, $f'(w_4) = 1$ and $f'(w) = f_m(w)$ for every vertex $w \in V(P(n,2)) \setminus \{w_1, w_4\}$. We have that f is an RDF of $P(n,2)$ with $f'(V(P(n,2))) = \gamma_R(P(n,2)) - 1$, a contradiction. For $w_3 \in V_0$, let $f'(w_1) = 0$, $f'(w_3) = f'(w_4) = 1$ and $f'(w) = f_m(w)$ for every vertex $w \in V(P(n,2)) \setminus \{w_1, w_3, w_4\}$. We have that $f' = (V_0'; V_1'; V_2')$ is a γ_R-function of $P(n,2)$ with $|V_2| > |V_2'|$, a contradiction. □

For an arbitrary γ_R-function $f = (V_0; V_1; V_2)$ of $P(n,2)$, we define a function g_f as follows. Let

$$g_f(w) = \begin{cases} 0.5, & \text{if } w \in V_2; \\ 1, & \text{if } w \in V_1; \\ 0.5|N(w) \cap V_2|, & \text{if } w \in V_0. \end{cases}$$

Then $g_f(w) \geq 0.5$ for every vertex $w \in V(P(n,2))$.

Lemma 2.4. $g_f(V(P(n,2))) = \sum_{v \in V(P(n,2))} g_f(v) = \gamma_R(P(n,2))$.

Proof. By Lemmas 2.2-2.3, we have that $N(w) \subseteq V_0$ for any vertex $w \in V_2$. It follows that $\gamma_R(P(n,2)) = |V_1| + 2|V_2| = |V_1| + 0.5|V_2| + 0.5\sum_{w \in V_2} |N(w) \bigcap V_0| = g_f(V_1)+g_f(V_2)+0.5\sum_{w \in V_0} |N(w) \bigcap V_2| = g_f(V_1) + g_f(V_2) + g_f(V_0) = g_f(V(P(n,2)))$. □

For every vertex $w \in V(P(n,2))$, let $r_f(w) = g_f(w) - 0.5$. For every subset $S \subseteq V(P(n,2))$, let $r_f(S) = \sum_{w \in S}(r_f(w))$. Let $V'(i,t) = \{v_j, u_j : i \leq j \leq i+t-1\}$. Then we have

Lemma 2.5. If $r_f(V'(i,7)) \leq 0.5$, then $v_{i+3} \notin V_2$.

Proof. Suppose to the contrary that $v_{i+3} \in V_2$. Then $v_{i+2}, v_{i+4}, u_{i+3} \in V_0$. If $\{v_{i+5}, u_{i+5}\} \cap (V_1 \cup V_2) \neq \emptyset$, then $rf(V'(i,7)) = 0.5$. It follows that $v_{i+1}, u_{i+1} \in V_0$, $v_i \in V_2$, $u_i, u_{i+2} \in V_0$ and $u_{i+4} \in V_2$, which implies that $r_f(V'(i,7)) \geq 1$(see Figure 2.2(1) for $v_{i+5} \in V_1$), a contradiction. Hence, $v_{i+5}, u_{i+5} \in V_0$. It follows that $v_{i+6} \in V_2$. There are three cases depending on u_{i+4}:

Case 1. $u_{i+4} \in V_2$. Then $r_f(V'(i,7)) \geq 1$(see Figure 2.2(2)), a contradiction.

Case 2. $u_{i+4} \in V_1$. Then $r_f(V'(i,7)) \geq 0.5$. It follows $u_{i+2} \in V_0$, $u_i \in V_2$ and at least one vertex of $\{v_{i+1}, u_{i+1}\}$ belongs to $V_1 \cup V_2$, which implies that $r_f(V'(i,7)) \geq 1$(see Figure 2.2(3)), a contradiction.

Case 3. $u_{i+4} \in V_0$. Then $u_{i+2} \in V_2$ and $r_f(V'(i,7)) \geq 0.5$. Since at least one vertex of $\{v_i, v_{i+1}, u_{i+1}\}$ belongs to $V_1 \cup V_2$, we have $r_f(V'(i,7)) \geq 1$(see Figure 2.2(4)), a contradiction.

From the above discussion, the lemma follows. □

Lemma 2.6. If $r_f(V'(i,7)) \leq 0.5$, then $v_{i+2}, v_{i+4} \notin V_2$.

Proof. By symmetry, it suffices to prove that $v_{i+4} \notin V_2$. Suppose to the

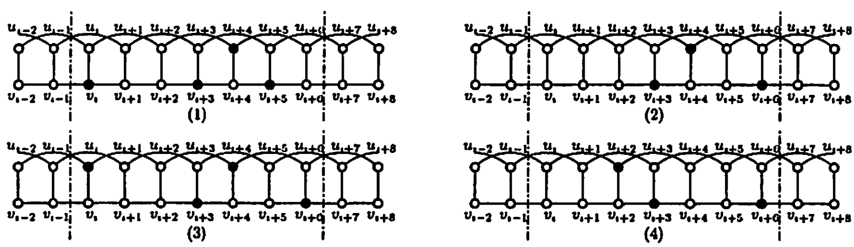

Figure 2.2: Some cases for $v_{i+3} \in V_2$

contrary that $v_{i+4} \in V_2$. Then $v_{i+3}, v_{i+5}, u_{i+4} \in V_0$. If $\{u_{i+5}, v_{i+6}, u_{i+6}\} \cap (V_1 \cup V_2) \neq \emptyset$, then $rf(V'(i,7)) = 0.5$. It follows that $v_{i+2}, u_{i+2} \in V_0$, $v_{i+1} \in V_2$, $u_{i+2} \in V_0$ and $u_i \in V_2$, which implies that $r_f(V'(i,7)) \geq 1$(see Figure 2.3(1) for $u_{i+5} \in V_1$), a contradiction. Hence, $u_{i+5}, v_{i+6}, u_{i+6} \in V_0$. Then $v_{i+7} \in V_2$, $u_{i+3} \in V_2$ and $r_f(V'(i,7)) \geq 0.5$. It forces $u_{i+1}, v_{i+1}, v_{i+2} \in V_0$, $v_i \in V_2$ and $N[u_{i+2}] \subseteq V_0$(see Figure 2.3(2)), a contradiction. □

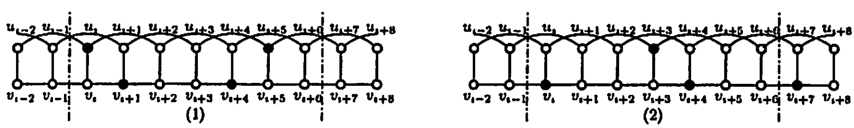

Figure 2.3: Some cases for $v_{i+4} \in V_2$

Lemma 2.7. If $r_f(V'(i,7)) \leq 0.5$, then $u_{i+3} \notin V_2$.

Proof. Suppose to the contrary that $u_{i+3} \in V_2$. Then $u_{i+1}, v_{i+3}, u_{i+5} \in V_0$. If $\{v_{i+4}, v_{i+5}\} \cap (V_1 \cup V_2) \neq \emptyset$, then $r_f(V'(i+4,3)) = 0.5$. If $v_{i+4}, v_{i+5} \in V_0$, then $u_{i+4}, v_{i+6} \in V_2$, we also have $r_f(V'(i+4,3)) = 0.5$. It follows that $v_{i+1} \in V_0$ and $v_{i+2} \in V_1$, which implies that $r_f(V'(i+4,3)) \geq 1$(see Figure 2.4), a contradiction. □

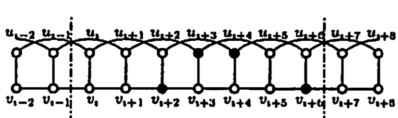

Figure 2.4: Case for $u_{i+3} \in V_2$

Lemma 2.8. If $r_f(V'(i,7)) \le 0.5$, then

(1) $r_f(V'(i,7))$ is at least 0.5,

(2) $V'(i,7) \cap V_1 = \{v_{i+3}\}$,

(3) $V'(i,7) \cap V_2 = \{u_{i+1}, u_{i+2}, v_{i+5}\}$ and $V'(i,7) \cap V_0 = V'(i,7) \setminus \{v_{i+3}, u_{i+1}, u_{i+2}, v_{i+5}\}$, or $V'(i,7) \cap V_2 = \{u_{i+4}, u_{i+5}, v_{i+1}\}$ and $V'(i,7) \cap V_0 = V'(i,7) \setminus \{v_{i+3}, u_{i+4}, u_{i+5}, v_{i+1}\}$.

Proof. By Lemma 2.5-2.7, we have that $r_f(V'(i,7))$ is at least 0.5 and $v_{i+3} \in V_1$. It follows that $u_{i+3}, v_{i+2}, v_{i+4} \in V_0$ and one vertex of $\{u_{i+1}, u_{i+5}\}$ belongs to V_2(see Figure 2.5(1)). If $u_{i+1} \in V_2$, then $u_{i+2}, v_{i+5} \in V_2$. If $u_{i+5} \in V_2$, then $u_{i+4}, v_{i+1} \in V_2$(see Figure 2.5(2)). □

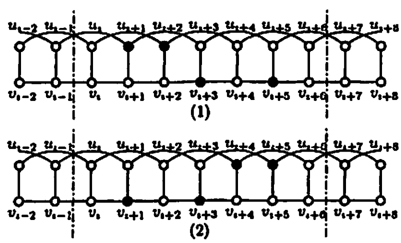

Figure 2.5: Case for $v_{i+3} \in V_1$

Lemma 2.9. If $r_f(V'(i,7)) = 0.5$, then $r_f(V'(i-7,7)) \ge 1$ and $r_f(V'(i+7,7)) \ge 1.5$, or $r_f(V'(i-7,7)) \ge 1.5$ and $r_f(V'(i+7,7)) \ge 1$.

Proof. By symmetry, we only need to consider the case shown in Figure 2.5(1). Since $N(v_i) \cap V_2 \ne \emptyset$ and $N(u_{i+6}) \cap V_2 \ne \emptyset$, we have $v_{i-1}, u_{i+8} \in V_2$. By Lemma 2.5 and Lemma 2.7, we have $r_f(V'(i-4,7)) \ge 1$ and $r_f(V'(i+5,7)) \ge 1$. We see that $r_f(V'(i,3)) = 0$ and $r_f(V'(i+5,2)) = 0$. It follows that $r_f(V'(i-4,4)) = r_f(V'(i-4,7)) - r_f(V'(i,3)) \ge 1$ and $r_f(V'(i+7,5)) = r_f(V'(i+5,7)) - r_f(V'(i+5,2)) \ge 1$. Therefore, $r_f(V'(i-7,7)) \ge 1$ and $r_f(V'(i+7,7)) \ge 1$.

Now, we prove that $r_f(V'(i-7,7)) \ge 1.5$. Suppose to the contrary that

$r_f(V'(i-7,7)) = 1$. Since $r_f(V'(i,7)) = 0.5$, we have $u_{i-2} \notin V_2$. If $u_{i-2} \in V_1$, then $r_f(V'(i-7,7)) \geq 1$ and $u_{i-3}, v_{i-3} \in V_0$. It follows that $u_{i-5}, v_{i-4} \in V_2$ and $r_f(V'(i-7,7)) \geq 1.5$(see Figure 2.5(1)), a contradiction. Hence, $u_{i-2} \in V_0$. It follows that $u_{i-4} \in V_2$ and $v_{i-4} \in V_0$. There are three cases depending on u_{i-3}:

Case 1. $u_{i-3} \in V_2$. Then $r_f(V'(i-7,7)) \geq 1$ and $u_{i-5} \in V_0$. It follows that at least one vertex of $\{v_{i-6}, v_{i-5}\}$ belongs to $V_1 \cup V_2$, which implies that $r_f(V'(i-7,7)) \geq 1.5$(see Figure 2.5(2)), a contradiction.

Case 2. $u_{i-3} \in V_1$. Then $v_{i-3} \in V_1$, which implies that $r_f(V'(i-7,7)) \geq 1.5$(see Figure 2.5(3)), a contradiction.

Case 3. $u_{i-3} \in V_0$. Then $v_{i-3} \in V_1$ and $u_{i-5} \in V_2$. It follows that at least one vertex of $\{v_{i-7}, v_{i-6}\}$ belongs to $V_1 \cup V_2$, which implies that $r_f(V'(i-7,7)) \geq 1.5$(see Figure 2.5(4)), a contradiction.

From the above discussion, the lemma follows. □

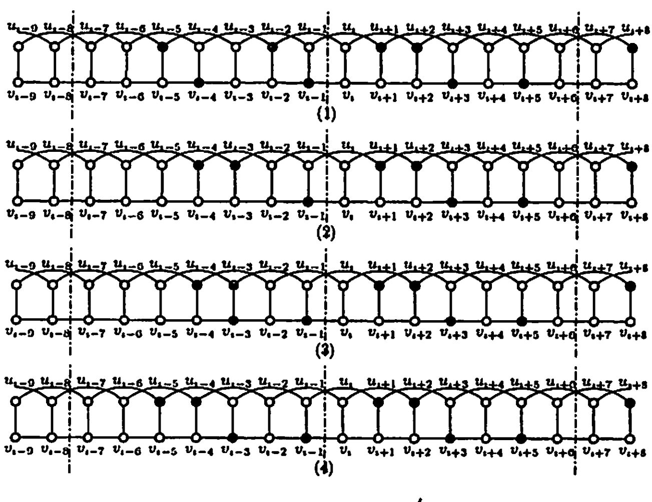

Figure 2.5: Some cases for $r_f(V'(i-7,7)) = 1$

Lemma 2.10. If $r_f(V'(i-7,7)) = 0.5$ and $r_f(V'(i+7,7)) = 0.5$, then

$r_f(V'(i,7)) \geq 2$.

Proof. By Lemma 2.8, there are three cases.

Case 1. $v_{i-2}, u_{i+8} \in V_2$. Then $u_{i+1}, v_{i+6} \in V_2$. It follows that $v_{i+1}, u_{i+3}, v_{i+5}, u_{i+6} \in V_0$ and $r_f(V'(i,7)) \geq 0.5$. Since $N[u_{i+5}] \cap (V_1 \cup V_2) \neq \emptyset$, we have $u_{i+5} \in V_1 \cup V_2$, which implies that $r_f(V'(i,7)) \geq 1$. Since $r_f(V'(i-7,7)) = 0.5$, we have $v_i, u_i \notin V_2$. Since $N[v_i] \cap (V_1 \cup V_2) \neq \emptyset$, we have $v_i \in V_1$. Then $r_f(V'(i,7)) \geq 1.5$. Since $N[v_{i+3}] \cap (V_1 \cup V_2) \neq \emptyset$, we have $\{v_{i+2}, v_{i+3}, v_{i+4}\} \cap (V_1 \cup V_2) \neq \emptyset$, which implies that $r_f(V'(i,7)) \geq 2$ (see Figure 2.6).

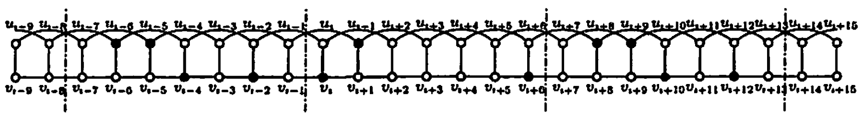

Figure 2.6: Case for $v_{i-2}, u_{i+8} \in V_2$

Case 2. $v_{i-2}, v_{i+8} \in V_2$. Then $u_{i+1}, u_{i+5} \in V_2$. It follows that $v_{i+1}, u_{i+3}, v_{i+5} \in V_0$ and $r_f(V'(i,7)) \geq 0.5$. Since $N[v_i] \cap (V_1 \cup V_2) \neq \emptyset$, we have $v_i \in V_1$. Since $N[v_{i+6}] \cap (V_1 \cup V_2) \neq \emptyset$, we have $v_{i+6} \in V_1$. It follows that $r_f(V'(i,7)) \geq 1.5$. Since $N[v_{i+3}] \cap (V_1 \cup V_2) \neq \emptyset$, we have $\{v_{i+2}, v_{i+3}, v_{i+4}\} \cap (V_1 \cup V_2) \neq \emptyset$, which implies $r_f(V'(i,7)) \geq 2$ (see Figure 2.7).

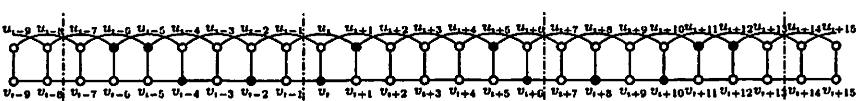

Figure 2.7: Case for $v_{i-2}, v_{i+8} \in V_2$

Case 3. $u_{i-2}, u_{i+8} \in V_2$. Then $v_i, v_{i+6} \in V_2$. It follows that $u_i, v_{i+1}, v_{i+5}, u_{i+6} \in V_0$ and $r_f(V'(i,7)) \geq 1$. Since $r_f(V'(i-7,7)) = 0.5$ and $r_f(V'(i+7,7)) = 0.5$, we have $u_{i+1}, u_{i+5} \notin V_2$.

If $u_{i+1} \in V_1$ and $u_{i+5} \in V_1$, then $r_f(V'(i+7,7)) \geq 2$ (see Figure 2.8(1)).

If $u_{i+1} \in V_0$ or $u_{i+5} \in V_0$, then $u_{i+3} \in V_2$. It follows that $u_{i+1}, v_{i+3}, u_{i+5} \in V_0$. Since $N[v_{i+2}] \cap (V_1 \cup V_2) \neq \emptyset$ and $N[v_{i+4}] \cap (V_1 \cup V_2) \neq \emptyset$, we have that $\{v_{i+2}, u_{i+2}\} \cap (V_1 \cup V_2) \neq \emptyset$ and $\{v_{i+4}, u_{i+4}\} \cap (V_1 \cup V_2) \neq \emptyset$, which implies that $r_f(V'(i+7,7)) \geq 2$ (see Figure 2.8(2)).

This completes the proof. □

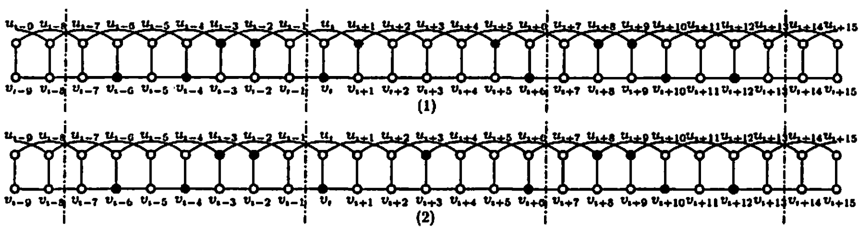

Figure 2.8: Case for $u_{i-2}, u_{i+8} \in V_2$

Lemma 2.11. $\gamma_R(P(n,2)) \geq \lceil \frac{8n}{7} \rceil (n \geq 5)$.

Proof. Let

$$\begin{aligned}
S_1 &= \{i : 0 \leq i \leq n-1, r_f(V'(7i,7)) = 0.5\},\\
S_2 &= \{i : 0 \leq i \leq n-1, r_f(V'(7i,7)) = 1\},\\
S_{31} &= \{i : 0 \leq i \leq n-1, r_f(V'(7i,7)) \geq 1.5, |\{i-1, i+1\} \cap S_1| \leq 1\},\\
S_{32} &= \{i : 0 \leq i \leq n-1, r_f(V'(7i,7)) \geq 1.5, |\{i-1, i+1\} \cap S_1| = 2\}.
\end{aligned}$$

By Lemma 2.8, $r_f(V'(7i,7)) \geq 0.5$, hence we have $\{0, 1, \ldots, n-1\} = S_1 \cup S_2 \cup S_{31} \cup S_{32}$. By Lemma 2.9, we have $|S_1| \leq |S_{31}| + 2|S_{32}|$. By Lemma 2.10, we have that $r_f(V'(7i,7)) \geq 2$ for any integer $i \in S_{32}$. By

Lemma 2.4, we have

$$\begin{aligned}
& 7\times\gamma_R(P(n,2)) \\
= & 7\times\textstyle\sum_{v\in V(P(n,2))} g_f(v) \\
= & 7\times\textstyle\sum_{v\in V(P(n,2))}(r_f(v)+0.5) \\
= & 7\times\textstyle\sum_{v\in V(P(n,2))} r_f(v)+7n \\
= & \textstyle\sum_{0\le i\le n-1} r_f(V'(7i,7))+7n \\
= & \textstyle\sum_{i\in S_1} r_f(V'(7i,7))+\sum_{i\in S_2} r_f(V'(7i,7))+\sum_{i\in S_{31}} r_f(V'(7i,7)) \\
& +\textstyle\sum_{i\in S_{32}} r_f(V'(i,7))+7n \\
\ge & 0.5|S_1|+|S_2|+1.5|S_{31}|+2|S_{32}|+7n \\
= & 0.5|S_1|+|S_2|+|S_{31}|+|S_{32}|+0.5(|S_{31}|+2|S_{32}|)+7n \\
\ge & |S_1|+|S_2|+|S_{31}|+|S_{32}|+7n \\
= & 8n,
\end{aligned}$$

which implies that $\gamma_R(P(n,2))\ge\lceil\frac{8n}{7}\rceil$. □

From Lemma 2.1 and Lemma 2.11, we have the following

Theorem 2.12. $\gamma_R(P(n,2))=\lceil\frac{8n}{7}\rceil(n\ge 5)$. □

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