Reversing arcs in transitive tournaments to obtain maximum number of spanning cycles

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Abstract

Consider the following problem: Given a transitive tournament T of order $n \geq 3$ and an integer k with $1 \leq k \leq \binom{n}{2}$, which k arcs in T should be reversed so that the resulting tournament has the largest number of spanning cycles? In this note, we solve the problem when n is sufficiently large compared to k.

1 Introduction

A *tournament* T is a digraph in which every two vertices are joined by one and only one arc. A tournament T is *transitive* if for any three vertices u, v, w in T, u dominates w whenever u dominates v and v dominates w. Given a transitive tournament T of order $n \geq 3$ and an integer k with $1 \leq k \leq \binom{n}{2}$, choose any k arcs and reverse their directions in T. Which k arcs in T should be chosen so that the resulting tournament has the largest number of spanning cycles? In this note, we study the problem and solve it when $n \geq k^3$.

Let T_n be the transitive tournament of order $n \geq 3$. For convenience, we denote its vertex set by $\{0, 1, ..., n-1\}$ so that 'i' dominates 'j' if and only if $i < j$. Let $T_n(u_1w_1, ..., u_kw_k)$ denote the tournament obtained from T_n by reversing the k arcs $w_1u_1, ..., w_ku_k$ in T_n, with $u_i > w_i$ for each $i = 1, 2, ..., k$. For the basic terminology on digraphs not defined here, the reader is referred to the book [1].

Our basic method of counting spanning cycles is illustrated in the rudimentary case when $k = 2$ in the next section. Let T^* denote the resulting tournament $T_n(u_1w_1, ..., u_kw_k)$ which has the maximum number of spanning cycles. In Section 3 we estimate the number of spanning cycles in a family of tournaments to which(as shown in Section 4) T^* belongs. The proof of our main result is completed in Section 5.

2 The Case $k = 2$

The transitive T_n has no spanning cycles. When $k = 1$, $T_n(u_1w_1)$ has a spanning cycle if and only if $u_1 = n-1$ and $w_1 = 0$ for $n \geq 3$. We now consider the case when $k = 2$. The following proof illustrates our method of counting the number of spanning cycles used throughout this paper.

Proposition 2.1. *For $n \geq 5$, $T_n(u_1w_1, u_2w_2)$ has the maximum number of spanning cycles when $u_1 = n-1, w_1 = 0, u_2 = n-2$ and $w_2 = 1$.*

Proof. We determine the number of spanning cycles obtainable by reversing two arcs of the transitive T_n. For the resulting tournament to be strong, we must have the reversed arcs $x0$ and $(n-1)y$ for some $0 \leq x, y \leq n-1$. There are three cases.

Case 1: $n-1$ dominates 0
Without loss of generality, let $u_1w_1 = (n-1)0$. There is another reversed arc ab with $a > b$. If $a = n-1$, then $b \neq 0$ and $(n-1)b$ cannot be included in any spanning cycle, else $(n-1)0$ must be excluded from the cycle and there is no way to reach vertex 0. Thus the resulting tournament has only the unique spanning cycle $\{0, 1, 2, ..., n-2, n-1, 0\}$. Similarly, there is only one spanning cycle if $b = 0$.

If $a \neq n-1$ and $b \neq 0$, note that any spanning cycle must go over at least one reversed arc. If only one reversed arc is used, it must be $(n-1)0$, with only the spanning cycle $\{0, 1, 2, ..., n-2, n-1, 0\}$. If both reversed arcs are used, then in any spanning cycle $\{v_1, v_2, ..., v_n, v_1\}$ going over both reversed arcs, $n-1$ must be followed by 0, and a must be followed by b. The number of ways to permute these two arcs in a cycle is $(2-1)! = 1$. So each spanning cycle takes the form below:

$$\{n-1, 0, \square, a, b, \square, n-1\}.$$

It remains to determine the number of ways to place the remaining vertices into the two boxes. Within each box, the vertices must be arranged in ascending order, since no other reversed arc is used for the cycle. Now each vertex z such that $0 < z < b$ must go into the box following 0, as z dominates b and so cannot go into the box following b. Similarly, each vertex z such that $a < z < n-1$ can only go into the box preceding $n-1$, as a dominates z and so z cannot go into the box preceding a. For z such that $b < z < a$, since $\{0, b\}$ dominates z and z dominates $\{a, n-1\}$, z can go into either box. The number of such vertices z is $a - b - 1$. So the number of spanning cycles using both reversed arcs is 2^{a-b-1}, and the total number of spanning cycles is $2^{a-b-1} + 1$.

Case 2: 0 dominates $n-1$ and $x \leq y$
If $x < y$, then the resulting tournament is not strong and hence has no spanning cycles. If $x = y$, both $(n-1)x$ and $x0$ must be included in any spanning cycle, and the only one is $\{0, 1, 2, ..., x-1, x+1, x+2, ..., n-1, x, 0\}$.

Case 3: 0 dominates $n-1$ and $x > y$
Note that both reversed arcs must be present in any spanning cycle. Arguing similarly as Case 1 above, each spanning cycle is of the form below:

$$\{n-1, y, \Box, x, 0, \Box, n-1\}.$$

Similar to Case 1, the vertices $z, 0 < z < y$ can only go into the box following 0, the vertices $z, x < z < n-1$ can only go into the box preceding $n-1$, while the vertices $z, y < z < x$ can go into either box. Hence the total number of spanning cycles is 2^{x-y-1}.

Note that for $n \geq 5$, the maximum number of spanning cycles is obtained when the reversed arcs are $(n-1)0$ and $(n-2)1$ in Case 1, with $2^{n-4} + 1$ spanning cycles. □

How about the general case when $k \geq 3$? By a similar case-by-case verification, it can be shown that for large n, $T_n(u_1w_1, u_2w_2, u_3w_3)$ has the maximum number of spanning cycles when $u_1 = n-1, w_1 = 0, u_2 = n-2, w_2 = 1, u_3 = n-3$ and $w_3 = 2$. (Such a tournament has $2 \cdot 3^{n-6} + 2^{n-4} + 2^{n-6} - 1$ cycles.) However, this method of brute force is certainly impractical when k is large. In what follows, we shall present a more elegant approach, and show that for $k \geq 3$ and for sufficiently large n compared to k, $T_n(u_1w_1, ..., u_kw_k)$ has the maximum number of spanning cycles when $u_i = n - i$ and $w_i = i - 1$ for $1 \leq i \leq k$.

3 Counting

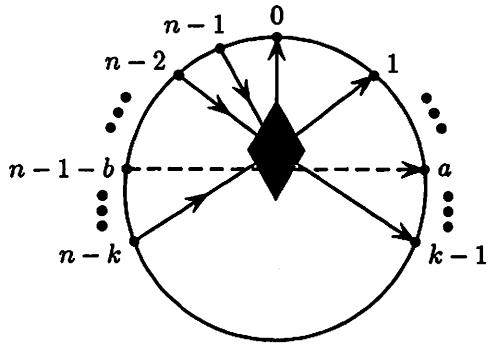

Fig. 1

Let $\mathcal{F}_{n,k}$ denote the family of tournaments of form $T_n(u_1w_1, ..., u_kw_k)$ with $k(k \leq \frac{n}{2})$ nonadjacent reversed arcs, such that $u_i \in \{n-1, n-2, ..., n-k\}$ and $w_j \in \{0, 1, ..., k-1\}$ for all $1 \leq i, j \leq k$. Such a tournament is depicted in Fig. 1 above, where the bold diamond matches incoming half-arcs from $\{n-1, n-2, ..., n-k\}$ one-to-one with outgoing half-arcs to $\{0, 1, ..., k-1\}$. The number of spanning cycles of a tournament in $\mathcal{F}_{n,k}$ satisfying certain properties is calculated.

Proposition 3.1. *For each $T \in \mathcal{F}_{n,k}$, the number of spanning cycles that go over all k reversed arcs is $(k-1)!k^{n-2k}$.*

Proof. Let $u_1w_1, ..., u_kw_k$ denote the k reversed arcs. In any spanning cycle going over all k arcs, u_i must be followed by w_i for all $1 \leq i \leq k$. The number of ways to permute these k arcs in a cycle is $(k-1)!$. Consider one such permutation:

$$\{u_1, w_1, \square, u_2, w_2, \square, ..., u_k, w_k, \square, u_1\}.$$

Now any number of vertices from $\{k, k+1, ..., n-k-1\}$ can go into each of the k boxes above, a total of $n-2k$ vertices. Inside each box, the vertices must be arranged in ascending order, since no other reversed arc is used for the cycle. Hence there are k^{n-2k} ways to arrange $\{k, k+1, ..., n-k-1\}$ into the specific permuation $\{u_1, w_1, \square, u_2, w_2, \square, ..., u_k, w_k, \square, u_1\}$. Combining, the total number of spanning cycles using all k arcs is $(k-1)!k^{n-2k}$. □

Proposition 3.2. *For each $T \in \mathcal{F}_{n,k}$, the number of spanning cycles using $k-1$ of the reversed arcs, but avoiding the reversed arc $(n-1-b)a$ where*

$a, b \in \{0, 1, ..., k-1\}$, is bounded above by $ab(k-2)!(k-1)^{n-2k}$, and below by $ab(k-2)!(k-1)^{n-2k} - (k-1)!(k-2)^{n-2k}$.

Proof. Similar arguments as presented in the proof of Proposition 3.1 give $(k-2)!$ permutations of $k-1$ reversed arcs and $(k-1)^{n-2k}$ ways to arrange $\{k, k+1, ..., n-k-1\}$ into $k-1$ boxes for each permutation.

Now vertex a can go into the box following w_j if and only if w_j dominates a, i.e. if and only if $w_j < a$. So vertex a can only go into the boxes following $\{0, 1, ..., a-1\}$, a total of a choices. Similarly, vertex $n-1-b$ can only go into the boxes preceding $\{n-b, n-b+1, ..., n-1\}$, a total of b choices. Altogether, the total number of spanning cycles is at most $ab(k-2)!(k-1)^{n-2k}$.

Note however that a and $n-1-b$ cannot be consecutive in any cycle, since this contradicts $n-1-b$ dominating a. Such a situation occurs if and only if a and $n-1-b$ are the only vertices in any of the $k-1$ boxes, since the vertices inside each box are required to be arranged in increasing order. (Although in practice, a and b can both be present in at most $min(a, b)$ boxes.) If this happens, the $n-2k$ vertices in $\{k, k+1, ..., n-k-1\}$ can only go into the remaining $k-2$ boxes. So such problem cases number at most $(k-2)!(k-2)^{n-2k}(k-1) = (k-1)!(k-2)^{n-2k}$. Subtracting this offset term from the main term above gives the required lower bound. □

4 Some Properties of T^*

We shall see that if $n \geq k^3$ and $T_n(u_1w_1, ..., u_kw_k)$ has the maximum number of spanning cycles, then the k reversed arcs are nonadjacent and go from $\{n-1, n-2, ..., n-k\}$ to $\{0, 1, ..., k-1\}$; that is, it belongs to $\mathcal{F}_{n,k}$.

Proposition 4.1. *Let $k \geq 3$ and $n \geq k^3$. If $T_n(u_1w_1, ..., u_kw_k)$ has the maximum number of spanning cycles, then the k reversed arcs $u_1w_1, ..., u_kw_k$ are nonadjacent; that is, all the u_i and w_j are distinct for $1 \leq i, j \leq k$.*

Proof. Let $0 \leq a, b, c \leq n-1$ be three vertices of graph $T_n(u_1w_1, ..., u_kw_k)$. There are two cases to consider.

Case 1: ab and bc for some $a > b > c$

We first obtain an upper bound for the number of spanning cycles that use all k reversed arcs. Since ab and bc are used, vertices $\{a, b, c\}$ must appear in sequence in any spanning cycle. If each of the other $k-2$ reversed arcs $u_1w_1, ..., u_{k-2}w_{k-2}$ is vertex-disjoint(nonadjacent) with one another as well as with $\{a, b, c\}$, the number of ways to arrange the $k-1$ entities $u_1w_1, ..., u_{k-2}w_{k-2}, abc$ in a spanning cycle is $(k-2)!$. Clearly, if any of the $k-2$ reversed arcs are adjacent to each other or to $\{a, b, c\}$, the number of entities to permute, and hence the number of permutations, will be fewer.

Now the maximum number of vertices not incident to any reversed arc is obviously bounded by n, and these can go into at most $k-1$ boxes. Altogether, the maximum number of spanning cycles using all k reversed arcs is thus bounded above by $(k-2)!(k-1)^n$.

Claim: In any $T_n(u_1w_1, ..., u_kw_k)$, the total number of spanning cycles, using up to $k-1$ of the reversed arcs and avoiding at least one reversed arc, is bounded above by $2^k(k-2)!(k-1)^n$.
To see this, consider the number of spanning cycles using m arcs, $1 \leq m \leq k-1$. Clearly, the total number of subsets of m arcs is less than 2^k. The number of ways to permute m arcs in a spanning cycle is $(m-1)!$, and $(m-1)! \leq (k-2)!$. The number of vertices not incident to any of the m arcs is obviously bounded by n, and these can go into at most $m \leq k-1$ boxes. So the grand total of spanning cycles using up to $k-1$ of the reversed arcs is bounded above by $2^k(k-2)!(k-1)^n$, and the claim holds.

Since a spanning cycle must use anything from 1 to k reversed arcs, the total number of spanning cycles is bounded above by

$$2^k(k-2)!(k-1)^n + (k-2)!(k-1)^n = (2^k+1)(k-2)!(k-1)^n.$$

Comparing with Proposition 3.1, we need to show that

$$(k-1)!k^{n-2k} > (2^k+1)(k-2)!(k-1)^n \tag{4.1}$$

whenever $k \geq 3$ and $n \geq k^3$. Cancelling $(k-1)!$, we need to show $k^{n-2k} > (2^k+1)(k-1)^{n-1}$. By Lemma 1 in the Appendix,

$$\begin{aligned} k^{n-2k} &= k(k^{n-2k-1}) \\ &> k(2^k(k-1)^{n-1}) \\ &= 2^k(k-1)^{n-1} + 2^k(k-1)^n \\ &> 2^k(k-1)^{n-1} + (k-1)^{n-1} \\ &= (2^k+1)(k-1)^{n-1}. \end{aligned}$$

Hence (4.1) holds whenever $k \geq 3$ and $n \geq k^3$.

Case 2: $\{ab$ and $ac\}$ or $\{ac$ and $bc\}$ for some $a > b > c$
Consider $\{ab$ and $ac\}$(the other case is similar). Note that no spanning cycle uses all k reversed arcs, since ab and ac cannot both be present in the same spanning cycle. Using the claim in Case 1 above, the total number of spanning cycles(using up to $k-1$ reversed arcs) is thus bounded above by $2^k(k-2)!(k-1)^n$. Again, comparing with Proposition 3.1 we need to show that

$$(k-1)!k^{n-2k} > 2^k(k-2)!(k-1)^n$$

whenever $k \geq 3$ and $n \geq k^3$. But from (4.1) above,

$$(k-1)!k^{n-2k} > (2^k+1)(k-2)!(k-1)^n > 2^k(k-2)!(k-1)^n.$$

Hence the required inequality holds. □

Proposition 4.2. *Let $k \geq 3$ and $n \geq k^3$. If $T_n(u_1w_1, ..., u_kw_k)$ has the maximum number of spanning cycles, then the k nonadjacent reversed arcs $u_1w_1, ..., u_kw_k$ are such that $u_i, w_j \in \{0, 1, ..., k-1\} \cup \{n-1, n-2, ..., n-k\}$ for all $1 \leq i, j \leq k$.*

Proof. Suppose some vertex a, $0 \leq a \leq k-1$ is not incident to any of the reversed arcs(the case $n-1-c$, $0 \leq c \leq k-1$ is similar). Consider the number of spanning cycles using all k reversed arcs. Again, the number of ways to permute the k arcs in a spanning cycle is $(k-1)!$. Excluding all u_i, w_j and vertex a, there are a total of $n-2k-1$ vertices left, which can go into at most k boxes. Now a can go into the box following w_j if and only if $w_j < a$. Since $a \leq k-1$, there are at most $k-1$ such w_j(from 0 to $k-2$). So vertex a has at most $k-1$ choices. Combining, the number of spanning cycles using all k reversed arcs is bounded above by $(k-1)!k^{n-2k-1}(k-1)$.

Using the claim in Proposition 4.1, the total number of spanning cycles is at most

$$(k-1)!k^{n-2k-1}(k-1) + 2^k(k-2)!(k-1)^n.$$

Comparing with Proposition 3.1, we require

$$(k-1)!k^{n-2k} > (k-1)!k^{n-2k-1}(k-1) + 2^k(k-2)!(k-1)^n$$

whenever $k \geq 3$ and $n \geq k^3$. Cancelling $(k-1)!$ throughout and rearranging terms, this is equivalent to $k^{n-2k-1} > 2^k(k-1)^{n-1}$, which is exactly the inequality that is handled by Lemma 1 in the Appendix. □

Proposition 4.3. *Let $k \geq 3$ and $n \geq k^3$. If $T_n(u_1w_1, ..., u_kw_k)$ has the maximum number of spanning cycles, then the k nonadjacent reversed arcs $u_1w_1, ..., u_kw_k$ are such that $u_i \in \{n-1, n-2, ..., n-k\}$ and $w_j \in \{0, 1, ..., k-1\}$ for all $1 \leq i, j \leq k$.*

Proof. Let $u_1w_1, ..., u_kw_k$ denote the k reversed arcs. Comparing with Proposition 4.2, we need to show that there is no $1 \leq i \leq k$ such that either $u_i, w_i \in \{0, 1, ..., k-1\}$ or $u_i, w_i \in \{n-1, n-2, ..., n-k\}$. (In fact, it can be shown that $\exists u_i, w_i \in \{0, 1, ..., k-1\} \Leftrightarrow \exists u_j, w_j \in \{n-1, n-2, ..., n-k\}$.) So suppose $u_i, w_i \in \{0, 1, ..., k-1\}$(the other case is similar). Consider the number of spanning cycles using all k reversed arcs. Again, the number of ways to permute the k arcs in a spanning cycle is $(k-1)!$. Now none of the vertices from $\{k, k+1, ..., n-k-1\}$ can go into the box preceding u_i, since $u_i \in \{0, 1, ..., k-1\}$ implies that u_i dominates each of $\{k, k+1, ..., n-k-1\}$.

So these $n-2k$ vertices can go into at most $k-1$ boxes, and the number of spanning cycles using all k arcs is bounded above by $(k-1)!(k-1)^{n-2k}$.

Using the claim in Proposition 4.1, the total number of spanning cycles is at most

$$(k-1)!(k-1)^{n-2k}+2^k(k-2)!(k-1)^n.$$

Comparing with Proposition 3.1, we require

$$(k-1)!k^{n-2k}>(k-1)!(k-1)^{n-2k}+2^k(k-2)!(k-1)^n$$

whenever $k\geq 3$ and $n\geq k^3$. Cancelling $(k-1)!$ throughout, we need to show $k^{n-2k}>(k-1)^{n-2k}+2^k(k-1)^{n-1}$. By Lemma 1 in the Appendix,

$$k^{n-2k}=k(k^{n-2k-1})=(k-1)k^{n-2k-1}+k^{n-2k-1}>(k-1)^{n-2k}+2^k(k-1)^{n-1}.$$

Hence the required inequality holds. □

5 The Main Result

We are now in a position to establish our main result in this note.

Theroem 5.1. *Let $k\geq 3$ and $n\geq k^3$. Then $T_n((n-1)0,(n-2)1,...,(n-k)(k-1))$ is the unique tournament with the greatest number of spanning cycles.*

Proof. By Proposition 4.3, it suffices to consider the case where the k reversed arcs are nonadjacent and go from $\{n-1,n-2,...,n-k\}$ to $\{0,1,...,k-1\}$. Let $u_1w_1,...,u_kw_k$ denote the k reversed arcs. We will show that if there are some $1\leq i,j\leq k$ such that

$$u_iw_i=(n-1-(c+1))a, u_jw_j=(n-1-c)b, a<b,$$

then by setting

$$u_i'w_i'=u_jw_i=(n-1-c)a, u_j'w_j'=u_iw_j=(n-1-(c+1))b$$

and $u_m'w_m'=u_mw_m$ for all $m\neq i,j$, the resulting $T'=T_n(u_1'w_1',...,u_k'w_k')$ has more spanning cycles than the original $T=T_n(u_1w_1,...,u_kw_k)$. Informally, this breaks up 'crosses' in our diagrams, as depicted in Fig. 2 below. If the resulting T' also has a 'cross', i.e. there exists vertices $v_1<v_2<v_3<v_4$ such that v_3 dominates v_1 and v_4 dominates v_2, then we can apply the same procedure on T'. By repeatedly applying this procedure, we eventually get the required $T_n((n-1)0,(n-2)1,...,(n-k)(k-1))$ which cannot be further improved.

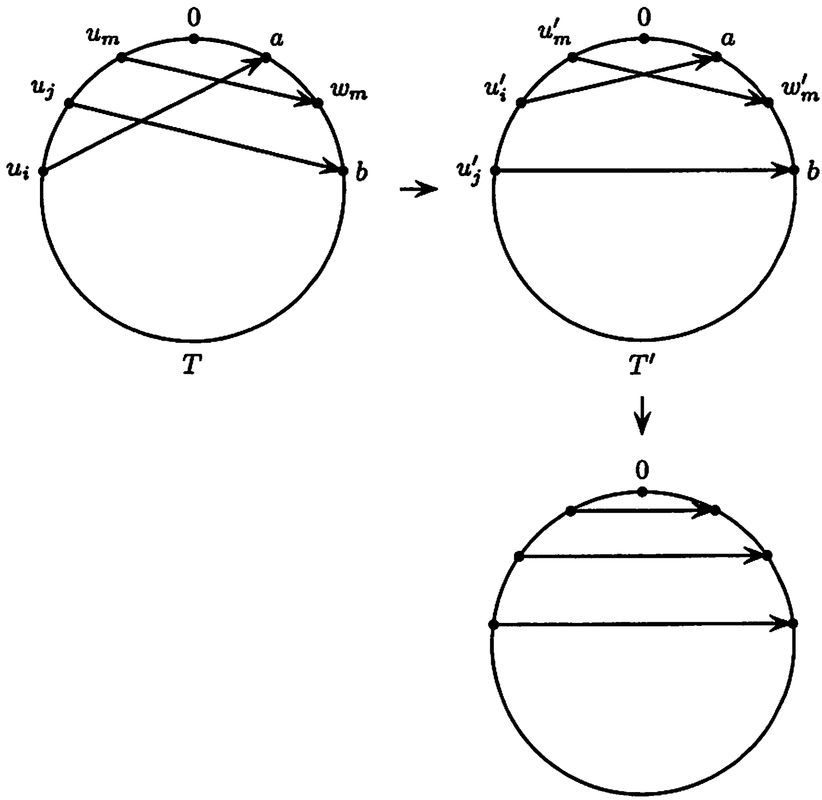

Fig. 2

By Proposition 3.1, T and T' have the same number (i.e. $(k-1)!k^{n-2k}$) of spanning cycles that go over all k reversed arcs. Compare the number of spanning cycles in T and T' that use $k-1$ reversed arcs.

Step 1: The unused arcs are $u_m w_m = u'_m w'_m$ where $m \neq i, j$
For each $m \neq i, j$, we claim that there is a bijection between spanning cycles in T using all k reversed arcs except $u_m w_m$, and spanning cycles in T' using all k reversed arcs except $u'_m w'_m$. This bijection is given by

$$f(\{v_1, v_2, ..., u_i, w_i, ..., u_j, w_j, ..., v_{n-1}, v_n, v_1\}) =$$
$$\{v_1, v_2, ..., u_j, w_i, ..., u_i, w_j, ..., v_{n-1}, v_n, v_1\}.$$

In other words, it suffices to interchange $u_i = n-1-(c+1)$ and $u_j = n-1-c$ in the cycle.

For simplicity, relabel vertices so that $i = 1, j = 2, m = k$. So the $k-1$ reversed arcs used in spanning cycles in T are $u_1w_1, u_2w_2, u_3w_3, ..., u_{k-1}w_{k-1}$, while those used in T' are $u_2w_1, u_1w_2, u_3w_3, ..., u_{k-1}w_{k-1}$. For each permutation of the entities $u_1w_1, u_2w_2, ..., u_{k-1}w_{k-1}$ in a spanning cycle in T, for example:

$$\{u_1, w_1, \square, u_2, w_2, \square, ..., u_{k-1}, w_{k-1}, \square, u_1\}, \tag{1}$$

there is a corresponding permutation of $u_2w_1, u_1w_2, ..., u_{k-1}w_{k-1}$ in T':

$$\{u_2, w_1, \square, u_1, w_2, \square, ..., u_{k-1}, w_{k-1}, \square, u_2\}, \tag{2}$$

that interchanges u_1 and u_2. As argued above, the vertices inside each box must be arranged in ascending order. Clearly the vertices from $\{k, k+1, ..., n-k-1\}$ can go into the boxes in both (1) and (2) without any restrictions.

Now w_k can go into the box following w_x if and only if w_x dominates w_k, i.e. if and only if $w_x < w_k$. Since only u_1 and u_2 were interchanged from (1) to (2), for all $1 \le x \le k-1$,

$$w_k \text{ can follow } w_x \text{ in } T \Leftrightarrow w'_x = w_x < w_k \Leftrightarrow w_k \text{ can follow } w'_x \text{ in } T'.$$

In other words, if w_k can go into a box in (1), then it can also go into the corresponding box in (2), and vice versa.

As for u_k, it can go into the box preceding u_y if and only if $u_k < u_y$. Since only u_1 and u_2 were interchanged, for $3 \le y \le k-1$,

$$u_k \text{ can precede } u_y \text{ in } T \Leftrightarrow u_k < u_y = u'_y \Leftrightarrow u_k \text{ can precede } u'_y \text{ in } T'.$$

Now we require that either u_k dominates $\{u_1, u_2\}$, or $\{u_1, u_2\}$ dominates u_k, so that u_k can either go into the boxes preceding u_1 and u_2 in both (1) and (2), or neither. But this follows immediately from the fact that $u_2 = u_1 + 1$ and u_1, u_2, u_k are all distinct, so either $u_k < u_1, u_2$ or $u_1, u_2 < u_k$.

It remains to show that (1) and (2) have the same number of 'problem cases' that were considered in Proposition 3.2, where w_k and u_k are consecutive in one of the boxes. But this follows immediately from the above discussion since each unused vertex can go into the exact same boxes in both (1) and (2), so any 'problem cases' are directly mirrored.

Step 2: The unused arcs are u_mw_m and $u'_mw'_m$ where $m = i, j$
Apply Proposition 3.2 to obtain an upper bound for the number of spanning cycles avoiding either u_iw_i or u_jw_j in T, and a lower bound for the number of spanning cycles avoiding either $u'_iw'_i = u_jw_i$ or $u'_jw'_j = u_iw_j$ in T'. For T, the number of spanning cycles avoiding $u_iw_i = (n-1-(c+1))a$ is bounded above by $a(c+1)(k-2)!(k-1)^{n-2k}$, while the number of spanning cycles avoiding $u_jw_j = (n-1-c)b$ is bounded above by $bc(k-2)!(k-1)^{n-2k}$. For

T', the number of spanning cycles avoiding $u_j w_i = (n-1-c)a$ is bounded below by $ac(k-2)!(k-1)^{n-2k} - (k-1)!(k-2)^{n-2k}$, while the number of spanning cycles avoiding $u_i w_j = (n-1-(c+1))b$ is bounded below by $b(c+1)(k-2)!(k-1)^{n-2k} - (k-1)!(k-2)^{n-2k}$. So the difference between the numbers of spanning cycles using $k-1$ arcs in T' and T exceeds

$$(k-2)!(k-1)^{n-2k}(ac + b(c+1) - a(c+1) - bc) - 2(k-1)!(k-2)^{n-2k}$$
$$= (k-2)!(k-1)^{n-2k}(b-a) - 2(k-1)!(k-2)^{n-2k}.$$

Note that $b - a \geq 1$ since $a < b$. By a similar argument to the claim in Proposition 4.1, the total number of spanning cycles using up to $k-2$ of the reversed arcs in T is bounded above by $2^k(k-3)!(k-2)^n$. Hence the number of spanning cycles in T' exceeds that in T when

$$(k-2)!(k-1)^{n-2k} - 2(k-1)!(k-2)^{n-2k} - 2^k(k-3)!(k-2)^n > 0.$$

Cancelling $(k-2)!$ throughout, we need to show that if $k \geq 3$ and $n \geq k^3$, then

$$(k-1)^{n-2k} - 2(k-1)(k-2)^{n-2k} - 2^k(k-2)^{n-1} > 0.$$

By Lemma 2 in the Appendix,

$$\begin{aligned}(k-1)^{n-2k} &= (k-2)(k-1)^{n-2k-1} + (k-1)^{n-2k-1} \\ &> (k-2)2^k(k-2)^{n-1} + 2^k(k-2)^{n-1} \\ &= 2 \cdot 2^{k-1}(k-2)^n + 2^k(k-2)^{n-1} \\ &> 2(k-1)(k-2)^{n-2k} + 2^k(k-2)^{n-1}\end{aligned}$$

since $2^{k-1} > k-1$. Hence the required inequality holds. □

Appendix

Lemma 1. *If $k \geq 3$ and $n \geq k^3$, then $k^{n-2k-1} > 2^k(k-1)^{n-1}$.*

Proof. Taking log and rearranging terms, this is equivalent to

$$n > \frac{k \log 2k^2}{\log \frac{k}{k-1}} + 1.$$

Since $n \geq k^3$, it suffices to show that

$$k^2 - \frac{\log 2k^2}{\log \frac{k}{k-1}} - \frac{1}{k} > 0$$

whenever $k \geq 3$. We verify that the inequality holds when $k = 3, 4, 5, 6$, and that the derivative wrt k is positive for $k \geq 6$. The derivative wrt k is

$$2k - \frac{2}{k \log \frac{k}{k-1}} - \frac{\log 2k^2}{k(k-1)(\log \frac{k}{k-1})^2} + \frac{1}{k^2}.$$

So it suffices to show that for $k \geq 6$,

$$2k - \frac{2}{k \log \frac{k}{k-1}} - \frac{\log 2k^2}{k(k-1)(\log \frac{k}{k-1})^2} > 0.$$

Now

$$\frac{1}{k \log \frac{k}{k-1}} = \frac{1}{\log (1 + \frac{1}{k-1})^k} < \frac{1}{\log(1 + \frac{k}{k-1})} < \frac{1}{\log 2}.$$

Also

$$\frac{1}{k(k-1)(\log \frac{k}{k-1})^2} < \frac{1}{(\log(1 + \frac{1}{k-1})^{k-1})^2} < \frac{1}{(\log(1 + \frac{k-1}{k-1}))^2} = \frac{1}{(\log 2)^2}.$$

So

$$2k - \frac{2}{k \log \frac{k}{k-1}} - \frac{\log 2k^2}{k(k-1)(\log \frac{k}{k-1})^2} > 2k - \frac{2}{\log 2} - \frac{\log 2k^2}{(\log 2)^2}.$$

Combining the terms on the RHS, we require

$$\frac{2k(\log 2)^2 - 2\log 2 - \log 2k^2}{(\log 2)^2} > 0.$$

It is not difficult to verify(e.g. by differentiating the numerator) that this inequality holds whenever $n \geq 6$. Hence the lemma holds. □

Lemma 2. *If $k \geq 3$ and $n \geq k^3$, then $(k-1)^{n-2k-1} > 2^k(k-2)^{n-1}$.*

Proof. Taking log and rearranging terms, this is equivalent to

$$n > \frac{k \log 2(k-1)^2}{\log \frac{k-1}{k-2}} + 1.$$

Since $n \geq k^3$, it suffices to show that

$$k^2 - \frac{\log 2(k-1)^2}{\log \frac{k-1}{k-2}} - \frac{1}{k} > 0$$

whenever $k \geq 3$. We verify that the inequality holds when $k = 3, 4$, and that the derivative wrt k is positive for $k \geq 4$. The derivative wrt k is

$$2k - \frac{2}{(k-1)\log\frac{k-1}{k-2}} - \frac{\log 2(k-1)^2}{(k-1)(k-2)(\log\frac{k-1}{k-2})^2} + \frac{1}{k^2}.$$

So it suffices to show that for $k \geq 4$,

$$2k - \frac{2}{(k-1)\log\frac{k-1}{k-2}} - \frac{\log 2(k-1)^2}{(k-1)(k-2)(\log\frac{k-1}{k-2})^2} > 0.$$

Now

$$\frac{1}{(k-1)\log\frac{k-1}{k-2}} = \frac{1}{\log\left(1+\frac{1}{k-2}\right)^{k-1}} < \frac{1}{\log(1+\frac{k-1}{k-2})} < \frac{1}{\log 2}.$$

Also

$$\frac{1}{(k-1)(k-2)(\log\frac{k-1}{k-2})^2} < \frac{1}{(\log(1+\frac{1}{k-2})^{k-2})^2} < \frac{1}{(\log(1+\frac{k-2}{k-2}))^2} = \frac{1}{(\log 2)^2}.$$

So

$$2k - \frac{2}{(k-1)\log\frac{k-1}{k-2}} - \frac{\log 2(k-1)^2}{(k-1)(k-2)(\log\frac{k-1}{k-2})^2} > 2k - \frac{2}{\log 2} - \frac{\log 2(k-1)^2}{(\log 2)^2}.$$

Combining the terms on the RHS, we require

$$\frac{2k(\log 2)^2 - 2\log 2 - \log 2(k-1)^2}{(\log 2)^2} > 0.$$

Again, it is not difficult to verify that this inequality holds whenever $n \geq 4$. Hence the lemma holds. □

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References

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