SUPER MEAN LABELING OF GRAPHS

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Abstract

In this paper, we introduce a new type of graph labeling known as super mean labeling. We investigate the super mean labeling for the Complete graph K_n, the Star $K_{1,n}$, the Cycle C_{2n+1}, and the graph $G_1 \cup G_2$ where G_1 and G_2 are super mean graphs and some standard graphs.

Key words: Super mean labeling, super mean graph.
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1. Introduction:

By a graph we mean a finite, simple and undirected one. The vertex set and edge set of a graph G are denoted by V(G) and E(G) respectively. The disjoint union of m copies of the graph G is denoted by mG. The union of two graphs G_1 and G_2 is the graph $G_1 \cup G_2$ with $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$.

A vertex of degree one is called a pendant vertex. The graph obtained by joining a single pendant edge to each vertex of a path is called a comb. The square G^2 of a graph G has $V(G^2) = V(G)$ with u and v adjacent in G^2 whenever $d(u, v) \leq 2$ in G. A chain of n-cycles denoted by $C_n(p_1, p_2, \ldots, p_n)$ is a graph obtained from a path $v_1v_2\ldots v_n$ by joining v_i and v_{i+1} by a path of length p_i-1 for $1 \leq i \leq n - 1$. That is, every edge of a path is replaced by a cycle of length p_i. The corona $G_1 \odot G_2$ of two graphs G_1 and G_2 is obtained by taking one copy of G_1(with p vertices) and p copies of G_2 and then joining the i^{th} vertex of G_1 to every vertex in the i^{th} copy of G_2. Terms and notations not defined here are used in the sense of Harary [1].

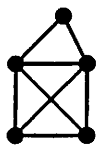
S.Somasundaram and R.Ponraj[2] have introduced the concept of mean labeling. Analogous to the mean labeling, we introduce a new labeling known as super mean labeling of graphs.

2. Super Mean Labeling

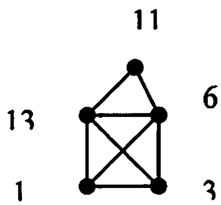
Definition 2.1: Let G be a (p,q) graph and f: $V(G) \rightarrow \{1,2,3,\ldots,p+q\}$ be an injection. For each edge e = uv, let $f^*(e) = (f(u) + f(v)) / 2$ if $f(u) + f(v)$ is even and $f^*(e) = (f(u) + f(v) + 1) / 2$ if $f(u) + f(v)$ is odd. Then f is called a super mean labelingif $f(V) \cup \{f^*(e): e \in E(G)\} = \{1, 2, 3,\ldots, p+q\}$. A graph that admits a super mean labeling is called a super mean graph.

Remark 2.2:

(i) 1 and p + q must be vertex labels of a super mean graph. In a super mean graph, x and x + 1 cannot be labels of adjacent vertices.

(ii) The graph G  is not a mean graph [3]. But G is a super mean graph

The super mean labeling of G is given below:



(iii) The cycle C_4 is a mean graph [2]. But C_4 is not a super mean graph. For, if 1 and 8 are labels of non adjacent vertices then either 3 or 6 acts as both vertex and edge labels; if 1 and 8 are labels of adjacent vertices then 5 acts as both vertex and edge labels, it is not allowed.

3. Super mean labeling of trees:

In this section, we investigate super mean labelings of some special trees.

Theorem 3.1: Any path is a super mean graph.

Proof: Let P_n be the path $u_1u_2u_3\ldots u_n$. Define f: $V(P_n) \rightarrow \{1, 2, 3,\ldots,p+q = 2n-1\}$ by $f(u_i) = 2i-1$, $1 \leq i \leq n$. Now $f(V) = \{1, 3, 5, \ldots, 2n-1\}$ and $\{f^*(e): e \in E(P_n)\} = \{2, 4, 6,\ldots,2n-2\}$. Hence P_n is a super mean graph.

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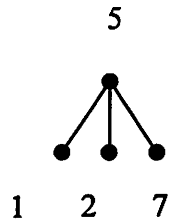
The Star $K_{1,1}$ is P_2 and $K_{1,2}$ is P_3. Hence $K_{1,1}$ and $K_{1,2}$ are super mean graphs by Theorem.3.1 .The super mean labeling of $K_{1,3}$ is



Now we have

Theorem 3.2: If $n > 3$, $K_{1,n}$ is not a super mean graph.

Proof: Suppose $K_{1,n}$ is a super mean graph with a super mean labeling f. Let (V_1, V_2) be the bipartition of $K_{1,n}$ with $V_1 = \{u\}$ and $V_2 = \{u_1, u_2, u_3, \ldots, u_n\}$. Here $p + q = 2n + 1$. First we show that $p + q$ cannot be a label for any vertex.
Suppose $f(u) = p + q$. Then the minimum number, which can be an edge label, is $n + 1$. This implies, $\{f(u_i): 1 \le i \le n\} = \{1, 2, 3, \ldots, n\}$. If $f(u_i) = 2$ and $f(u_j) = 3$, then uu_i and uu_j get the same label. This is a contradiction.
Suppose $f(u_i) = p + q$ for some i.

Case (i): $p + q - 1$ is a vertex label.

By Remark 2.2 (i), $p + q - 1$ cannot be a label for u. Therefore $f(u_j) = p + q - 1$ for some j. Since $p + q$ is odd, $f(u) \neq m$ for all odd m. Therefore $f(u_r) = 1$ for some r. Also u is adjacent to u_r, $f(u) \neq 2$. Therefore $f(u_k) = 2$ for some k. Hence $f(u) \neq m$ for all even m. This is a contradiction.

Case (ii): $p + q - 1$ is an edge label.

Then $f(u) \in \{p + q - 2, p + q - 3\}$. Since $n > 3$, $p + q - 3 \ge 6$. Therefore 1, 2, 3 must be labels of u_i, u_j, u_r for some i, j, r. If $f(u) = p + q - 2$, then the labels of the edges uu_r, uu_j are $(p + q + 1)/2$; if $f(u) = p + q - 3$, the edges uu_i and uu_j get the same label $(p + q - 1)/2$ which is a contradiction.

Hence $p + q$ is cannot be a vertex label. Therefore, $K_{1,n}$, $n > 3$ is not a super mean graph.

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Theorems 3.3: Combs are mean graphs.

Proof: Let G be a comb obtained from the path P_n: $v_1 v_2 \ldots v_n$ by joining a vertex u_i to v_i $(1 \le i \le n)$. Define $f: V(G) \rightarrow \{1, 2, \ldots, p + q = 4n - 1\}$ by $f(v_i) = 4i - 1 (1 \le i \le n)$,
$f(u_1) = 1$ and $f(u_i) = 4(i - 1)$, $(1 \le i \le 2n)$. Then f is a super mean labeling of G.

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Theorem 3.4: The bistar $B_{m,n}$ is a super mean graph for $m = n$ or $n + 1$.

Proof: Let $V(K_2) = \{u, v\}$ and u_i, v_j be the vertices adjacent to u and v respectively
$(1 \le i \le m, 1 \le j \le n)$.

Case (i) m = n

Define f: $V(B_{m,n}) \rightarrow \{1,2,3\ldots,p + q = 4n + 3\}$ by $f(u) = 3$, $f(u_1) = 1$, $f(u_i) = f(u_{i-1}) + 4$ for $1 \leq i \leq n$; $f(v_1) = p + q$; $f(v) = p + q-2$ and $f(v_i) = f(v_{i-1}) - 4$ for $2 \leq i \leq n$ .Then f is a super mean labeling.

Case (ii) m = n + 1

Define f: $V(B_{m,n}) \rightarrow \{1, 2, 3,\ldots,p + q = 4n +5\}$ by $f(u) = 3$, $f(u_1)=1$, $f(u_i) = f(u_{i-1}) + 4(2 \leq i \leq n +1)$, $f(v) = p + q$, $f(v_1) = p + q-2$ and $f(v_i) = f(v_{i-1}) - 4$ for $2 \leq i \leq n$ Then f is a super mean labeling.

Hence by Case (i) and Case (ii), $B_{m,n}$ is a super mean graph for m = n or n +1.

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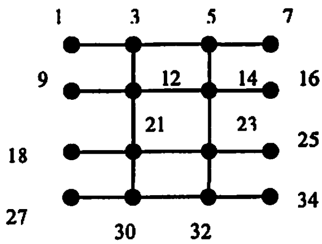
Theorem 3.5: The graph $L_n \odot K_1$ is a super mean graph

Proof: Let $V(L_n) = \{a_i,b_i : 1 \leq i \leq n\}$ and $E(L_n) = \{a_ib_i : 1 \leq i \leq n\} \cup \{a_ia_{i+1}, b_ib_{i+1}: 1\leq i \leq n-1\}$. Let c_i and d_i be the pendant vertices adjacent to a_i and b_i respectively. Define f: $V(L_n \odot K_1) \rightarrow \{1, 2, 3,\ldots,p + q = 9n - 2\}$ by $f(c_1) = 1$; $f(a_1) = 3$;

$f(b_1) = 5$; $f(d_1) = 7$; $f(c_i) = 9(i-1)$ for $2 \leq i \leq n$; $f(a_i) = f(a_{i-1}) + 9$ for $2 \leq i \leq n$; $f(b_i) = f(b_{i-1}) + 9$ for $2 \leq i \leq n$; $f(d_i) = f(d_{i-1}) + 9$ for $2 \leq i \leq n$; Then f is a super mean labeling of G.

Example 3.6: A super mean labeling of $L_4 \odot K_1$ is given below:



4. Cycle Related graphs

In this section, we investigate super mean labeling for some graphs, which contain cycles.

Theorem 4.1: Any cycle of odd length is a super mean graph.

Proof: Let C_{2n+1} be the cycle $u_1u_2u_3\ldots u_{2n+1}u_1$.

Define $f:V(C_{2n+1}) \rightarrow \{1, 2, 3,\ldots,p + q = 4n +2\}$ by $f(u_i) = 2i-1$ for $1 \leq i \leq n+1$; $f(u_{n+1+j}) = 2(n + 1 + j)$ forl $\leq j \leq n$. Since $f(V) = \{1,3,5,\ldots,2n+1,2n+4,2n+6,\ldots,4n+2\}$ and $\{f^*(e): e \in E(C_{2n+1})\} = \{2,4,.6,\ldots2n,2n+3,\ldots,4n+1,2n+2\}$, f is a super mean labeling and hence C_{2n+1} is a super mean graph.

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Next we investigate super mean labeling of the complete graph K_n. K_1, K_2 are super mean graphs by Theorem 3.1, K_3 is a super mean graph by Theorem 4.1.

Theorem 4.2: If n >3, K_n is not a super mean graph.
Proof: Let $V(K_n) = \{u_i: 1 \leq i \leq n\}$. Suppose K_n, n > 3 is a super mean graph with a super mean labeling f. By remark 2.2 (i), p+q, 1 must be vertex labels so let $f(u_1) = p + q$,
$f(u_2) = 1$.Then $p + q-1$ and 2 cannot be vertex labels. To get the edge label 2, 3 must be a vertex label so let $f(u_3) = 3$.To get the edge label $p + q-1$, either $p + q-2$ or $p + q-3$ is a vertex label. We consider the following two cases.
Case (i): p + q -2 is a vertex label. Let $f(u_4) = p + q-2$. Hence the edges u_1u_2 and u_3u_4 get the same label. This is a contradiction.
Case (ii): p + q −3 is a vertex label. Let $f(u_4) = p + q - 3$. Then p + q −2 cannot be a vertex label. To get the edge label p + q −2 , the only possibility is that p + q −5 must be a vertex label. If p + q −5 is the label of u_5 then edges u_2u_4 and u_3u_5 get the same label. This is a contradiction. Hence K_n , n >3, is not a super mean graph.

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Theorem 4.3: The graph P_n^2 is a super mean graph.
Let P_n be the path $u_1u_2u_3...u_n$. Clearly P_n^2 has n vertices and 2n-3 edges.
Case (i): n is even.
Define f: $V(P_n^2) \rightarrow \{1, 2, 3,...,p + q = 3n - 3\}$ by $f(u_1) = 1$, $f(u_2) = 3$, $f(u_{2i+1}) = f(u_{2i-1}) + 6, 1 \leq i \leq (n-2)/2$; $f(u_{2i+2}) = f(u_{2i}) + 6$, $1 \leq i \leq (n-2)/2$.
Case (ii): n is odd
Define f: $V(P_n^{\ 2}) \rightarrow \{1,2,3,..., p + q = 3n-3\}$ by $f(u_1) = 1$, $f(u_2) = 3$, $f(u_3) = 6$, $f(u_4) = 10$, $f(u_{2i+1}) = f(u_{2i-1}) + 6$ for $2 \leq i \leq (n-1)/2$ $f(u_{2i+2}) = f(u_{2i}) + 6$ for $2 \leq i \leq (n-3)/2$. Clearly
f is a super mean labeling. Therefore P_n^2 is a super mean graph.

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5. Union and Identification of graphs

Theorem 5.1: If G_1 and G_2 are two super mean graphs, then $G_1 \cup G_2$ is also a super mean graph.
Proof: Let $V(G_1) = \{ u_i : 1 \leq i \leq p_1\}$, $E(G_1) = \{e_i : 1 \leq i \leq q_1\}$, $V(G_2) = \{v_i : 1 \leq i \leq p_2\}$,
$E(G_2) = \{e_i' : 1 \leq i \leq q_2\}$. Let f and g be the super mean labeling of G_1 and G_2 respectively. Define h: $V(G_1 \cup G_2) \rightarrow \{1, 2, 3,..., p_1 + p_2 + q_1 + q_2\}$ by $h(u_i) = f(u_i)$ and $h(v_i) = p_1 + q_1 + g(v_i)$ for $1 \leq i \leq p_2$.

We show that h is an injection. For, $h(u_i) = h(u_j)$ implies $f(u_i) = f(u_j)$. Since f is an injective function, $u_i = u_j$. $h(v_i) = h(v_j)$ implies $p_1 + q_1 + g(v_i) = p_1 + q_1 + g(v_j)$. Since g is an injective function, $v_i = v_j$ .Therefore h is an injection. Suppose $h(v_i) = h^*(e_j')$, then $p_1 + q_1 + g(v_i) = p_1 + q_1 + g^*(e_j')$, a contradiction to g is a super mean labeling. Hence h is a super mean labeling of $G_1 \cup G_2$.

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Corollary 5.2: mP_n , $m \geq 1$ is a super mean graph.
Proof: It follows from Theorem 3.1 and Theorem 5.1.

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Theorem 5.3: $C_m \cup P_n$ is a super mean graph for all $m \geq 3$ and $n \geq 2$.
Proof: **Case (i)** m is odd.
It follows from Theorem 3.1, Theorem 4.1 and Theorem 5.1.
Case (ii) m is even.
Let $m = 2k$ and C_m be the cycle $u_1u_2\ldots u_m$ and $v_1v_2\ldots v_n$ be the path P_n.
Define $f: V(C_m \cup P_n) \rightarrow \{1, 2, 3,\ldots, p + q = 2m + 2n - 1\}$ by $f(u_i) = 2i - 1$ for $1 \leq i \leq k + 1$, $f(u_{k+2}) = 2k + 4$, $f(u_{k+2+j}) = 2k + 4 + 2j$ for $1 \leq j \leq m-k-3$, $f(u_m) = 2m + 2$, $f(v_1) = 2m-1$, $f(v_2) = 2m + 3$, $f(v_j) = 2m + 2j - 1$ for $3 \leq j \leq n$ .Clearly f is a super mean labeling and hence $C_m \cup P_n$ is a super mean graph.

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Remark 5.4: If m and n are odd then $C_m \cup C_n$ is a super mean graph.

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Theorem 5.5: $C_3 \cup C_m$ is a super mean graph for $m \geq 3$.
Proof: If m is odd then by Remark 5.4 $C_3 \cup C_m$ is a super mean graph.
If m is even, then $m = 2n$ for some n. Let $V(C_3)=\{u_1,u_2,u_3\}$ and $V(C_m) = \{v_1,v_2,v_3,\ldots,v_m= v_{2n}\}$. Define $f: V(C_3 \cup C_m) \rightarrow \{1,2,3,\ldots,2m + 6 = 4n + 6\}$
By $f(u_1) = 1; f(u_2) = 3; f(u_3) = 7; f(v_1) = 6; f(v_{2n}) = 11; f(v_i) = 4i + 2$ for $2 \leq i \leq n+1$; $f(v_{2n-i}) = 4i + 11$ for $1 \leq i \leq n - 2$.Then f is a super mean labeling of $C_3 \cup C_m$.
Hence $C_3 \cup C_m$ is a super mean graph.

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Theorem 5.6: Let $G_1 = (p_1, q_1)$ and $G_2 = (p_2, q_2)$ be two super mean graphs with super mean labeling f and g respectively. Let $f(u) = p_1 + q_1$ and $g(v) = 1$. Then the graph $(G_1)_f * (G_2)_g$ obtained from G_1 and G_2 by identifying the vertices u and v is also a super mean graph.
Proof: Let $V(G_1) =\{u, u_i : 1 \leq i \leq p_1 - 1\}$, $V(G_2) = \{v, v_i : 1 \leq i \leq p_2 - 1\}$
Define $h: V((G_1)_f * (G_2)_g) \rightarrow \{1, 2, 3, \ldots, p_1 + p_2 + q_1 + q_2\}$ by $h(u_i) = f(u_i)$, $h(u)= f(u)$ and $h(v_i)= p_1 + q_1 + g(v_i) - 1$. It is easy to verify that h is a super mean labeling. Therefore $(G_1)_f * (G_2)_g$ is a super mean graph.

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Theorem 5.7: A chain of n-cycles $C_n(p_1, p_2,\ldots, p_n)$ is a super mean graph for $p_1, p_2,\ldots, p_n$ are odd.

Proof: Let $G = C_n(p_1, p_2,\ldots, p_n)$ and $V(G) = \bigcup_{i=1}^{n}\{v_{ij} : 1 \leq j \leq p_i\}$ and we assume that
$v_{ip_i} = v_{(i+1)1}$ for $1 \leq i \leq n - 1$. Take $p_i = 2n_i + 1$ for $1 \leq i \leq n$.
Define $f: V(G) \rightarrow \{1, 2,\ldots, p + q = 2(p_1 + p_2 +\ldots+ p_n) - n + 1\}$ by
$f(v_{1j}) = 2j-1$ for $1 \leq j \leq n_1 + 1$, $f(v_{1j}) = 2j$ for $n_1 + 2 \leq j \leq 2n_1 + 1$.
For $2 \leq i \leq n$, $f(v_{ij}) = 2j-1 + 2(p_1 + p_2 +\ldots+ p_{i-1}) - i + 1$ for $1 \leq j \leq n_i + 1$ and

$f(v_{ij}) = 2j + 2(p_1 + p_2 + \ldots + p_{i-1}) - i + 1 \quad$ for $n_i + 2 \leq j \leq 2n_i + 1$

Clearly f is a super mean labeling and hence $C_n(p_1, p_2, \ldots, p_n)$ is a super mean graph for $p_1, p_2, \ldots, p_n$ are odd.

■

In a graph G, d(u, v) denotes the length of a shortest path joining u and v and $d_H(u, v)$ denotes the length of a shortest path joining u and v in a sub graph H of G. Now we find the super mean labeling of C_n together with a chord uv such that $d_{C_n}(u, v) = 2$ and $d_{C_n}(u, v) = 3$.

Theorem 5. 8: Let C_n be a cycle of length $n \geq 4$ and let G be a graph obtained from C_n by taking $V(G) = V(C_n)$ and $E(G) = E(C_n) \cup \{uv\}$ such that $d_{C_n}(u, v) = 2$ where $u, v \in V(C_n)$. Then G is a super mean graph.

Proof: Let C_n be a cycle $u_1u_2u_3\ldots u_nu_1$ and take $u = u_2$ and $v = u_n$. Then $d_{C_n}(u,v) = 2$.

Case (i) suppose n is even, $n \geq 4$

Define f: $V(G) \rightarrow \{1, 2, 3, \ldots, p + q = 2n + 1\}$ by $f(u_1) = 2n + 1$; $f(u_2) = 2n - 2$; $f(u_3) = 1$;

$f(u_i) = f(u_{i-1}) + 2$ for $4 \leq i \leq \frac{n}{2} + 2$; $f(u_{\frac{n}{2}+2+j}) = f(u_{\frac{n}{2}+2}) + 2j + 1$ for $1 \leq j \leq \frac{n}{2} - 2$.

Then the induced edge labels are $\{2, 4, 6, \ldots, n, n + 1, n + 3, \ldots, 2n - 1, 2n\}$. Clearly

$f(V) \cup \{f^*(e) : e \in E(G)\} = \{1, 2, \ldots, 2n + 1\}$. Therefore f is a super mean labeling of G. Hence G is a super mean graph.

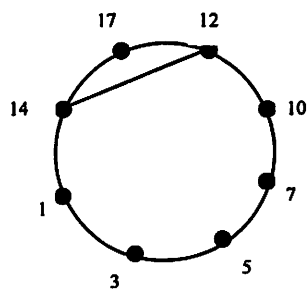
Case (ii) suppose n is odd.

Define f: $V(G) \rightarrow \{1, 2, 3, \ldots, p + q = 2n + 1\}$ by $f(u_1) = 2n - 2$; $f(u_2) = 2n + 1$;

$f(u_3) = 1$; $f(u_i) = f(u_{i-1}) + 2$ for $4 \leq i \leq \frac{n+1}{2} + 2$; $f(u_{\frac{n+1}{2}+2+j}) = f(u_{\frac{n+1}{2}+2}) + 2j + 1$,

for $1 \leq j \leq \frac{n-1}{2} - 2$. Then $\{f^*(e) : e \in E(G)\} = \{2, 4, 6, \ldots, n-1, n + 1, n+2, n + 4, \ldots,$
$2n - 1, 2n\}$. Clearly f is a super mean labeling of G. Hence G is a super mean graph. ■

Example 5.9: A super mean graph obtained from C_8 is given below:



Theorem 5.10: Let C_n be a cycle of length $n \geq 5$ and G be a graph obtained from C_n by taking $V(G) = V(C_n)$ and $E(G) = E(C_n) \cup \{uv\}$ such that $d_{C_n}(u, v) = 3$ where $u, v \in V(C_n)$. Then G is a super mean graph.

Proof: Let C_n be a cycle $u_1u_2u_3 \ldots u_nu_1$ and take $u = u_3$ and $v = u_n$. Then $d_{C_n}(u,v) = 3$.

Case (i) suppose n is even

Define $f: V(G) \to \{1, 2, 3,\ldots,p + q = 2n +1\}$ by $f(u_1) = 1$; $f(u_i) = f(u_{i-1}) +2$ for $2 \leq i \leq \frac{n}{2}$; $f(u_{\frac{n}{2}+j}) = f(u_{\frac{n}{2}})+2(j +1)$ for $1 \leq j \leq \frac{n}{2}-2$; $f(u_{n-1}) = 2n +1$; $f(u_n) =2n-2$.

Then the induced edge labels are $\{2, 4, 6,\ldots,n, n +1, n +2, n + 4, n + 6,\ldots,2n-4, 2n - 1,2n\}$.Clearly

$f(V) \cup \{ f^*(e) : e \in E(G)\} = \{1, 2, 3,\ldots,p + q = 2n + 1\}$.Therefore f is a super mean labeling of G. Hence G is a super mean graph.

Case (ii) suppose n is odd.

Define $f: V(G))\to\{1, 2, 3,\ldots,p + q = 2n +1\}$ by $f(u_i) = 2i - 1$ for $1 \leq i \leq \frac{n+1}{2}$;

$f(u_{\frac{n+1}{2}+j}) = f(u_{\frac{n+1}{2}}) + 2(j +1)$ for $1 \leq j \leq \frac{n-1}{2}$; Then $\{ f^*(e) : e \in E(G)\}=\{2, 4, 6,\ldots,$
$n -1, n +1, n +2, n + 4,\ldots, 2n - 2, 2n\}$. Clearly f is a super mean labeling of G. Hence G is a super mean graph.

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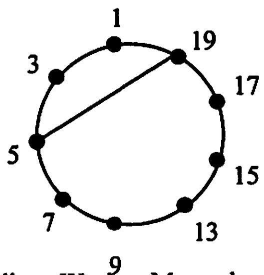
Example 5.11: A super mean graph obtained from C_9 is given below:



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