# INDEPENDENCE OF COUNTABLE SETS OF FORMULAS OF THE PROPOSITIONAL LOGIC

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ABSTRACT. In this paper, we prove that every countable set of formulas of the propositional logic has at least one equivalent independent subset. We illustrate the situation by considering axioms for Boolean algebras; the proof of independence we give uses model forming.

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## 1. INTRODUCTION

As it was stated in [11], independence is no more or no less important than consistency and decidability. In reality, independence was on of the four pillars of Hilbert's project to study formal reasoning (the three others were consistency, completeness and decidability). The reason why independence was neglected comes from the fact that it is less substantial than three other keystones for a logic; in particular, completeness is the preeminent task for a logic. But one should take into account that independence, which is much as relevant only to elegance, is germane to - among other things - problems of consistency.

In fact, independent axiomatization of any set of formulas of classical (propositional or predicate) logic was difficult to settle and many logicians tried to find a solution but they failed. The first partial success for countable sets was obtained by Tarski [10] in 1930 and from then on the topic remained insignificant until Kreisler [6] and [7] reconsidered it. He pointed out that one can prove independent axiomatizability in classical propositional logic, by use of $\Rightarrow$ only, of any set of formulas of cardinality $\aleph_2$. None the less, an example given by Reznikoff [8] showed the failure at $\aleph_2$.

The works mentioned above, being too much sophisticated in notation as well as in contents, prompted us to simplify things and to give a proof both clear and comprehensible for sets of formulas of cardinality at most $\aleph_0$. Moreover, we avoided to copy the main facts available in [8] and [11] for the care of not increasing the page number of the paper.

## 2. Preliminaries and Examples

Denote by $P$ the set of propositional variables or atoms, i.e. let $P := \{p_n : n \in \mathbb{N}\}$. A set $S$ of formulas constructed from $P$ is called *independent* if for every formula $\varphi \in S$, $\varphi$ is not a logical consequence of $S - \{\varphi\}$, in symbols, $S - \{\varphi\} \not\models \varphi$.

Given a finite set of formulas $\{\varphi_1, \varphi_2, \ldots, \varphi_n\}$, to show that it is independent, it suffices to find, for each $1 \leq i \leq n$, an assignment of truth values that satisfies all the $\varphi_k$, where $k \neq i$, and that does not satisfy $\varphi_i$; on the contrary, to demonstrate that it is not independent, one shows that one of the $\varphi_i$ is a consequence of the others.

In cases where the set of formulas under consideration is not independent, we appeal to the following observation when we desire to find an *equivalent independent* subset, namely, if $S$ is a set of formulas and if the formula $\varphi$ is a consequence of $S - \{\varphi\}$, then the sets $S$ and $S - \{\varphi\}$ are equivalent.

**Examples** (See [1]) Let us see whether the following sets of formulas built from $P$ are independent.

$S_1 = \{p_1 \Rightarrow p_2, p_2 \Rightarrow p_3, p_3 \Rightarrow p_1\}$,

$S_2 = \{p_1 \Rightarrow p_2, p_2 \Rightarrow p_3, p_1 \Rightarrow p_3\}$,

$S_3 = \{(p_1 \Rightarrow p_2) \Rightarrow p_3, p_1 \Rightarrow p_3, p_2 \Rightarrow p_3, (p_3 \Rightarrow p_2) \Rightarrow p_1, (p_1 \Rightarrow p_2) \Rightarrow (p_1 \Leftrightarrow p_2)\}$.

The set $S_1$ is independent. To see this, consider the assignment of truth values $v(p_1) = T$, $v(p_2) = T$, and $v(p_3) = T$, where $T$ and $F$ are propositional constants denoting *true* and *false*, respectively. Note by the way that we could take the Boolean constants 1 and 0 instead of $T$ and $F$, respectively. Then $v(p_1 \Rightarrow p_2) = F$, $v(p_2 \Rightarrow p_3) = T$, and $v(p_3 \Rightarrow p_1) = T$, so the first formula is falsified by this assignment while the other two are satisfied. It can be checked that in fact, any assignment of truth values that makes one of these formulas false will necessarily satisfy the other two. Hence $S_1$ is an independent set.

The set $S_2$ is obviously not independent because $p_1 \Rightarrow p_3$ is a consequence of $p_1 \Rightarrow p_2$ and $p_2 \Rightarrow p_3$ by the classical syllogism law:

$\{p_1 \Rightarrow p_2, p_2 \Rightarrow p_3\} \vdash p_1 \Rightarrow p_3$.

However, the subset $\{p_1 \Rightarrow p_2, p_2 \Rightarrow p_3\}$ is independent and equivalent to $S_2$.

The set $S_3$ is not independent. First, observe that $(p_1 \Rightarrow p_2) \Rightarrow (p_1 \Leftrightarrow p_2)$ and $p_2 \Rightarrow p_1$ are logically equivalent, so we have two independent equivalent subsets of $S_3$ as follows:

$S_{31} = \{(p_1 \Rightarrow p_2) \Rightarrow p_3, p_1 \Rightarrow p_3, p_3 \Rightarrow (p_2 \Rightarrow p_1)\}$

and

$S_{32} = \{(p_1 \Rightarrow p_2) \Rightarrow p_3, p_1 \Rightarrow p_3, (p_1 \Rightarrow p_2) \Rightarrow (p_1 \Leftrightarrow p_2)\}$.

That the following sets of formulas are not independent can be checked by the reader:

$T_1 = \{p_1 \vee p_2, p_1 \Rightarrow p_3, p_2 \Rightarrow p_3, \neg p_1 \Rightarrow (p_1 \vee p_3)\}$,

$T_2 = \{p_1, p_2, p_1 \Rightarrow p_3, p_3 \Rightarrow p_2\}$,

$T_3 = \{p_1 \Rightarrow (p_2 \vee p_3), p_3 \Rightarrow (\neg p_2), p_2 \Rightarrow (p_2 \vee p_3), (p_2 \wedge p_3) \Leftrightarrow p_2, (p_1 \Rightarrow p_3) \Rightarrow (p_1 \Leftrightarrow p_2)\}$.

## 3. Independence of Sets of Formulas

The aim of this paper is to prove that every countably infinite set of formulas of the propositional logic has at least one equivalent independent subset. To show this, we start with the empty set, set with a single element, singleton, and sets with finite elements and then move to the infinite countable cases. At some point, we shall need Completeness Theorem, which is stated under the following version:

'For any set $\Sigma$ of formulas of the propositional logic and any propositional formula $\psi$, $\psi$ is a consequence of $\Sigma$ if and only if $\psi$ is a consequence of at least one *finite* subset of $\sum$'.

We would like to lay emphasis on the fact that for the proofs that follow, we inspired from [1] to a large extent.

**Propositon 1**

(a) *The empty set (of formulas) is independent.*

(b) *A set consisting of a single element is independent if and only if the formula is not a tautology.*

**Proof**

(a) Assume, to the contrary, that the empty set $\emptyset$ is not independent. Then it contains a formula $\varphi$ such that $\emptyset - \{\varphi\} \models \varphi$; that is clearly impossible, so the empty set is independent.

(b) Let $S = \{\varphi\}$. Then $S - \{\varphi\} \models \varphi$ is equivalent to $\emptyset \models \varphi$, which means that $\varphi$ is a tautology. Consequently, for a singleton to be independent it is necessary and sufficient that the formula is not a tautology.■

**Propositon 2***Every finite set of formulas of the propositional logic has at least one independent equivalent subset.*

**Proof**We establish this property by induction on the number of formulas in the set.

*Base step*: When this number is 0 or 1, the property holds by virtue of Proposition 1.

*Induction hypothesis*: Suppose that every set containing $n$ formulas has at least one equivalent independent subset; i.e. the property is true for any set of $n$ formulas.

*Induction step*: Show that the property is also true for a set $\Sigma$ of $n+1$ formulas.

If $\Sigma$ is independent, it is itself an independent subset equivalent to $\Sigma$. If not, then we can find in $\Sigma$ a formula $\psi$ that is a consequence of $\Gamma = \Sigma - \{\psi\}$. $\Gamma$, which contains $n$ formulas, has, by the Induction hypothesis, an independent subset $\Delta$ that is equivalent to $\Gamma$. But $\Gamma$ is equivalent to $\Sigma$, so $\Delta$ is a subset of $\Sigma$ that is independent and equivalent to $\Sigma$.■

**Remark** It is high time to correct two major errors concerning the proof that we give for Proposition 2; these two mistakes are, in fact, related.

The first consists in believing that if $S$ is an independent set of formulas and if $\varphi$ is a formula that is not a consequence of $S$, then $S \cup \{\varphi\}$ is independent.

The second reposes on thinking that a maximal independent subset of a set of formulas is necessarily equivalent to this set.

These two ideas are not correct. To give an example, consider $S = \{p_1\}$, $\varphi = p_1 \wedge q_1$, and $\Gamma = S \cup \{\varphi\}$. Obviously, $S$ is independent and $\varphi$ is not a consequence of $S$, and $S \cup \{\varphi\}$ is not independent and $S$ is a maximal independent subset of $\Gamma$ but is not equivalent to $\Gamma$. Indeed, $S$ is independent by Proposition 1 (b) and $\varphi$ is not a consequence of $S$. Now $S \cup \{\varphi\} = \{p_1, p_1 \wedge p_2\}$ is not independent since $p_1$ is a consequence of $p_1 \wedge p_2$ and $S$ is a maximal independent subset of $\Gamma$ but not equivalent to $\Gamma$.

**Theorem 3***For a set of formulas of the propositional logic to be independent, it is necessary and sufficient that all its finite subsets be independent.*
**Proof** Let $\Sigma$ be a set of formulas.

*Condition is necessary*: If $\Sigma$ is independent and if $\Gamma$ is a subset of $\Sigma$ (finite or not), then $\Gamma$ is independent.

*Condition is sufficient*: Assume, to the contrary, that $\Sigma$ is a set of formulas that is not independent. Then there is at least one formula $\psi$ in $\Sigma$ such that $\Sigma - \{\psi\} \models \psi$. Now, according to Completeness Theorem, there is some finite subset $\Gamma$ of $\Sigma - \{\psi\}$ such that $\Gamma \vdash \psi$. Set $\Delta = \Gamma \cup \{\psi\}$; we then have $\Delta - \{\psi\} \vdash \psi$, which proves that $\Delta$ is a finite subset of $\Sigma$ that is not independent; this is a contradiction with the hypothesis that all finite subsets of $\Sigma$ are independent. Thus, for a set of formulas to be independent, it is necessary and sufficient that all its subsets are independent.■

**Theorem 4***The infinite countable set of formulas of the propositional logic* $\Sigma = \{p_1, p_1 \wedge p_2, p_1 \wedge p_2 \wedge p_3, \ldots, p_1 \wedge p_2 \wedge \ldots \wedge p_n, \ldots\}$ *has no independent subset that are equivalent to* $\Sigma$.
**Proof** For each integer $n \geq 1$, set $\varphi_n = p_1 \wedge p_2 \wedge \ldots \wedge p_n$ and let $\Sigma$ denote the set $\{\varphi_n : n \in \mathbb{Z}^+\}$. Clearly, for $n \leq m$, $\varphi_n$ is a consequence of $\varphi_m$; thus, the only subset of $\Sigma$ that are independent are composed of a single element. It is also immediate that for every $n$, $\varphi_{n+1}$ is not a consequence

of $\varphi_n$; to see this, it suffices to take an assignment of truth values that satisfies $p_1, p_2, \dots, p_n$ and not $p_{n+1}$. Thus, it follows that no independent subset of $\Sigma$ can be equivalent to $\Sigma$. However, there exist independent sets that are equivalent to $\Sigma$, for example, the set $P = \{p_1, p_2, \dots, p_n, \dots\}$ is one of them.■

The next - and the last - theorem is the main theorem of our work.

**Theorem 5***For any infinite countable set of formulas of the propositional logic, there exists at least one equivalent independent set.*
**Proof** Let $\Sigma = \{\varphi_n : n \in \mathbb{N}\}$ be our set of formulas. We are searching a set of formulas equivalent to $\Sigma$. First, we obtain a set that is equivalent to $\Sigma$ by removing those $\varphi_n$ that are a consequence of $\{\varphi_0, \varphi_1, \dots, \varphi_{n-1}\}$. In other words, we may assume that for every $n$, the formula $\varphi_n$ is not a consequence of $\{\varphi_0, \varphi_1, \dots, \varphi_{n-1}\}$ and, in particular, that $\varphi_0$ is not a tautology. We now consider the following set

$\Gamma = \{\varphi_0, \varphi_0 \Rightarrow \varphi_1, (\varphi_0 \wedge \varphi_1) \Rightarrow \varphi_2, \dots, (\varphi_0 \wedge \varphi_1 \wedge \dots \wedge \varphi_n) \Rightarrow \varphi_{n+1}, \dots\}$.

It is clear that if an assignment of truth values satisfies all the formulas $\varphi_n$, then it satisfies $\Gamma$; conversely, if it satisfies all the formulas in $\Gamma$, then by induction on $n$ we see that it satisfies all the formulas $\varphi_n$. The sets $\Sigma$ and $\Gamma$ are thus equivalent. We shall show that $\Gamma$ is independent by exhibiting, for every formula $\gamma$ in $\Gamma$, an assignment of truth values that does not satisfies $\gamma$ but that satisfies all other formulas of $\Gamma$.

If $\gamma = \varphi_0$, we take an assignment of truth values that makes $\varphi_0$ false (such a $\gamma$ exists since $\varphi_0$ is not a tautology). The remaining formulas of $\Gamma$ are then all satisfied.

If $\gamma = (\varphi_0 \wedge \varphi_1 \wedge \dots \wedge \varphi_n) \Rightarrow \varphi_{n+1}$, we choose an assignment of truth values that satisfies $\varphi_0, \varphi_1, \dots, \varphi_n$ and that makes $\varphi_{n+1}$ false (such a $\gamma$ exists since $\varphi_{n+1}$ is not a consequence of $\{\varphi_0, \varphi_1, \dots, \varphi_{n-1}\}$; that such an assignment has the required properties is easy to verify.■

## 4. An Example of the Independence of Axioms

We illustrate the independence of axioms for Boolean algebras.

Recall that a Boolean algebra is a mathematical structure $\mathfrak{B} =< B, \vee, \wedge, ', 0, 1 >$, where $B$ is a nonempty set, $\vee$, $\wedge$ and $'$ are two binary and unary operations on $B$, respectively, and 0, 1 are the distinguished elements of $B$ with the following axioms:

| | |
|---|---|
| (1) $x \wedge 1 = x$ | (1)' $x \vee 0 = x$ |
| (2) $x \wedge x' = 0$ | (2)' $x \vee x' = 1$ |
| (3) $x \wedge y = y \wedge x$ | (3)' $x \vee y = y \vee x$ |

(4) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ (4)′ $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$
for all $x, y, z \in B$.

This set of axioms is logically equivalent to various sets, four of which are discussed in [4]. The axioms above are due to Huntington and can be found in [5]. The statement "Huntington's axioms for Boolean algebras are independent" occurs in [9] as an exercise. This was already established in [5] but only for two axioms, the remaining six left to be proved by the reader. A similar treatment appears in [2]. For more knowledge and details on Boolean algebras, see [3].

The sketching proof given in the mentioned sources are neither convincing nor clear enough. We show, here, the independence not using truth tables but forming two-elements models and by interpreting Boolean operations and distinguished elements property.

Note that a member of a set of axioms is said to be independent of the remaining axioms if it is not derivable from them. Our approach will be based on forming a model in which that axiom is falsified while the remaining ones are valid.

Let $\Sigma = \{(i); (i)' : i = 1, 2, 3, 4\}$. We will show the following:

1. $\Sigma - \{(4)'\} \nvdash (4)'$,
2. $\Sigma - \{(4)\} \nvdash (4)$,
3. $\Sigma - \{(2)\} \nvdash (2)$,
4. $\Sigma - \{(2)'\} \nvdash (2)'$.

**Proofs**

1. Consider the two-element Boolean ring $\mathit{2} = < \{0,1\}, \vee, \wedge, ', 0, 1 >$; this is our model.

Interpret join $\vee$ as ring addition; meet $\wedge$ as ring multiplication; complement $'$ as the unary operation that interchanges 0 and 1; zero and one as 0 and 1, respectively.

Then under these interpretations, the axioms become

(1) $x \cdot 1 = x$, (1)′ $x + 0 = x$,
(2) $x \cdot x' = 0$, (2)′ $x + x' = 1$,
(3) $x \cdot y = y \cdot x$, (3)′ $x + y = y + x$,
(4) $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$, (4)′ $x + (y \cdot z) = (x + y) \cdot (x + z)$.

Now for $x \in \{0, 1\}$, we have

$0 \cdot 1 = 0$ and $0 + 0 = 0$,
$1 \cdot 1 = 1$ and $1 + 0 = 1$,

so $\mathit{2} \models (1)$ and $\mathit{2} \models (1)'$.

As

$0 \cdot 0' = 0 \cdot 1 = 0$ and $0 + 0' = 0 + 1 = 1$,
$1 \cdot 1' = 1 \cdot 0 = 0$ and $1 + 1' = 1 + 0 = 1$,

we have $\mathit{2} \models (2)$ and $\mathit{2} \models (2)'$.

The axioms (3) and (3)′ are clearly valid in $\mathit{2}$.

Show that $2 \models (4)$. If $x = 0$, then as $0 \cdot (y+z) = 0$ and $(0 \cdot y)+(0 \cdot z) = 0$, regardless of the value of $y$ and $z$ we see that $2 \models (4)$. When $x = 1$, then $1 \cdot (y+z) = y+z$ and $(1 \cdot y)+(1 \cdot z) = y+z$, so again we have $2 \models (4)$. The seven axioms are satisfied by $2$. Let's show that $2 \not\models (4)'$. To see this, it sufficient to consider a triple $(x, y, z)$ for which $x+(y+z) \neq (x+y) \cdot (x+z)$. Now let $x = 1, y = 1$, and $z = 0$. Then

$$x + (y \cdot z) = 1 + (1 \cdot 0) = 1 + 0 = 1$$

and

$$(x + y) \cdot (x + z) = (1 + 1) \cdot (1 + 0) = 0 \cdot 1 = 0$$

so that $x + (y \cdot z) \neq (x + y) \cdot (x + z)$, and hence $2 \not\models (4)'$.

2. The model is the same model as in 1.

Interpret join $\vee$ as ring multiplication; meet $\wedge$ as ring addition; the Boolean zero as 1 and the Boolean unity as 0.

Then under these interpretations, the axioms become

| | |
|---|---|
| (1) $x + 0 = x$, | (1)′ $x \cdot 1 = x$, |
| (2) $x + x' = 1$, | (2)′ $x \cdot x' = 0$, |
| (3) $x + y = y + x$, | (3)′ $x \cdot y = y \cdot x$, |
| (4) $x + (y \cdot z) = (x + y) \cdot (x + z)$, | (4)′ $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$. |

Similar computation as in 1. will show that all the seven axioms are valid in $2$ except (4).

3. Our model is the two-element Boolean algebra, with the complement redefined. To establish the independence of (2), define complement to be the unary operation whose value is constantly 1. The validity of axioms $(1), (1)', (3), (3)'$ and $(4), (4)'$ remains unaffected, since the complement is not affected in these axioms. (2)′ continues to hold because the join of any element with 1 is 1. However, (2) fails, since

$$x \wedge x' = x \wedge 1 = x$$

for $x = 0, 1$. In particular, $1 \wedge 1' = 1 \neq 0$.

Finally, to show the independent of (2)′, define complement to be the unary operation whose values in constantly 0. Then argue as above.■

Note that (3) and (3)′ can be deduced from the remaining axioms plus $x \vee 1 = 1$ and $x \wedge 0 = 0$.

**Conclusion** In this work, we prove that every countable set of formulas of the propositional logic has at least one equivalent independent subset. The results we give are almost well-known except that our proofs are accessible, clear and comprehensible to anyone who possesses basic concepts of the classical logic.

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