Molds of regular tournaments

Víctor Neumann-Lara*, Mika Olsen†

Instituto de Matemáticas,
Universidad Nacional Autónoma de México,
México D. F., México

Abstract

We introduce the concept of molds, which together with an appropriate weight function, gives all the information of a regular tournament. We use the molds to give a shorter proof of the characterization of domination graphs than the one given in [4, 5]. We also use the molds to give a lower and an upper bound of the dichromatic number for all regular tournaments with the same mold.

Keywords: regular tournaments, domination digraph, dichromatic number, molds.

MSC 2000: 05C20, 05C15

1 Introduction

The domination graph of a tournament was introduced by Fisher et. al. in [9]. Two vertices u, v form a *dominant pair* of a tournament T if for every vertex $w \in V(T) \setminus \{u, v\}$ at least one of the arcs uw or vw belongs to T. Let the *domination graph* of a tournament T, denoted by *dom* (T), be the graph on the vertex set $V(T)$ with edges between dominant pairs of T. The *domination digraph* $\mathfrak{D}(T)$ of a tournament T is the domination graph *dom* (D) with the orientation induced by T. It was introduced and studied by Fisher et. al. in [8]. The domination graphs of tournaments have been characterized in a series of papers [4, 5, 6, 7, 8, 9].

Two vertices u, v form a dominant pair of a regular tournament T (see [5]) if

$$N^{-}(u, T \setminus \{u, v\}) = N^{+}(v, T \setminus \{u, v\}),$$

*Passed away February 2004

†Current address: Departamento de Matemáticas Aplicadas y Sistemas, Universidad Autónoma Metropolitana, Unidad Cuajimalpa, México D.F., México

where standard notation follows from [1, 2, 14].

We introduce the concept of molds. A *mold* is a regular tournament M such that all the paths of the domination digraph $\mathfrak{D}(M)$ are of order at most 2. Let the *cyclic* tournament $\overrightarrow{C}_{2m+1}\langle\emptyset\rangle$ be the tournament on the vertex set $\mathbb{Z}_{2m+1}$ and $uv \in A\left(\overrightarrow{C}_{2m+1}\langle\emptyset\rangle\right)$ if $v-u \in \mathbb{Z}_m$. We assign a mold M^T and associate a weight function φ_T to every regular non cyclic tournament T and prove that every regular noncyclic tournament can be reconstructed from its mold and the associated weight function (Proposition 6). Consequently, the study of molds is an important tool in the study of regular tournaments. Using molds, we give a shorter proof of the characterization of domination graphs of regular tournaments than the one given by Cho et. al. in [4, 5], with and respectively, without isolated vertices. Also we use molds to give an upper and a lower bound for the dichromatic number of every regular noncyclic tournament (the definition of the dichromatic number of a digraph is given in section 4).

Molds have also been used to prove that a large class of regular tournaments are tight, that is, every vertex coloring with exactly 3 colors induces a cyclic triangle with the 3 colors [10].

2 Domination digraphs

In this section, we review some basic facts about domination digraphs of regular tournaments. Proposition 5 will be the key for the definition of molds.

Note that $(\mathfrak{D}(T))^{op} = \mathfrak{D}(T^{op})$. Also it is clear that the automorphism group, $Aut(T)$, of a regular tournament T acts on the domination digraph $\mathfrak{D}(T)$ of T, so it fixes the set of $\mathfrak{D}$-arcs. We will need the following results.

Lemma 1 *[9] If T' is a subtournament of T, then $\mathfrak{D}(T)\mid_{T'}$ is a subdigraph of $\mathfrak{D}(T')$.*

As a consequence of Theorem 2.7 of [5], if T is a regular tournament, then $\mathfrak{D}(T)$ is a directed cycle or it has at least 2 components that are directed paths. Do note that in this case the indegree $d^-(u)$ and the outdegree $d^+(u)$ of any vertex $u \in V(\mathfrak{D}(T))$ are at most one.

In section 3 we use the domination structure of a regular tournament T to define a (regular) subtournament (the mold of T), that incode the acyclic and cyclic structure of T. We define the mold as the subtournament induced by the vertex set with the two first vertices of every even ordered maximal $\mathfrak{D}$-path and the first vertex of every odd ordered maximal $\mathfrak{D}$-path. We then use molds to determine which unions of n odd paths and m even paths are possible in $\mathfrak{D}(T)$ for T a regular tournament.

Let T be the regular tournament in Figure 1 on the vertex set

$$V(T) = \{w_0, w_1, w_2, u_0, u_1, v_0, v_1, x_0, x_1, x_2, x_3\}.$$

The fat arcs are the arcs of $\mathfrak{D}(\mathfrak{T})$. Note that w_0 and w_2 have exactly the same adjacency in T except with respect to the vertex w_1 (they are concordant with respect to $T \setminus \{w_1\}$). The vertices x_0 and x_3 have exactly opposite adjacency in T except with respect to the vertex x_1 and x_2 (they are discordant with respect to $T \setminus \{x_1, x_2\}$).

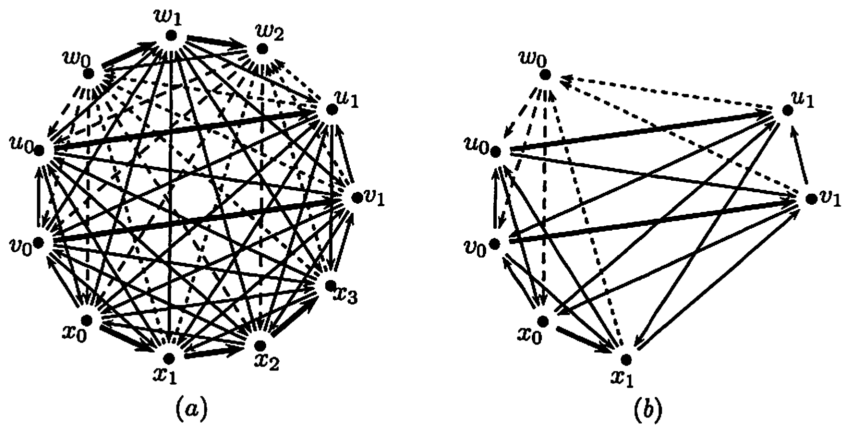


Figure 1: (a) A regular tournament T with domination paths of length tree, two, two and four respectively (b) the mold of T

We prove that the adjacency in a path of $\mathfrak{D}(T)$ of a regular tournament T is completely determined. Moreover, the behavior of one vertex of the path P with respect to another vertex $v \in V(T) \setminus P$, determines the behavior of all the vertices of the path P with respect to v. Let T be a regular tournament and let v be a vertex of the non trivial $\mathfrak{D}$-path P of $\mathfrak{D}(T)$. Note if we know the adjacency of the vertex v, then we know the adjacency of all vertices of the $\mathfrak{D}$-path P in the tournament T.

Let T be a tournament, $u, v \in V(T)$ and $S \subset V(T)$. We say that u, v are *concordant* modulo S, denoted by $u \equiv v \pmod{S}$, if

$$\begin{aligned} N^-(u, S \setminus \{u, v\}) &= N^-(v, S \setminus \{u, v\}) \\ N^+(u, S \setminus \{u, v\}) &= N^+(v, S \setminus \{u, v\}). \end{aligned}$$

We say that u, v are *discordant* modulo S, denoted by $u \mid v \pmod{S}$, if

$$\begin{aligned} N^+(u, S \setminus \{u, v\}) = N^-(v, S \setminus \{u, v\}) \\ N^-(u, S \setminus \{u, v\}) = N^+(v, S \setminus \{u, v\}). \end{aligned}$$

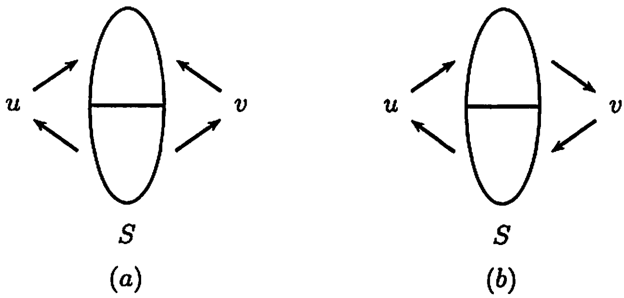


Figure 2: (a) u, v are *concordant* modulo S, (b) u, v are *discordant* modulo S

In Figure 1 $w_0 \equiv w_2 \pmod{T \setminus \{w_1\}}$ and $x_0 \mid x_3 \pmod{T \setminus \{x_1, x_2\}}$.

Remark 1 *If T is a regular tournament, then the following are equivalent*

- (i) $uv \in dom\,(T)$
- (ii) $u \mid v \pmod{T}$
- (iii) $N^+(u, T \setminus \{u, v\}) = N^-(v, T \setminus \{u, v\})$
- (iv) $N^-(u, T \setminus \{u, v\}) = N^+(v, T \setminus \{u, v\})$.

Lemma 2 *Let $P = (u_0, u_1, \ldots, u_k)$ be a path of $\mathfrak{D}(T)$, where T is a regular tournament, $P_{i,j} = (u_i, u_{i+1}, \ldots, u_j)$ an induced subpath of P with $0 \le i < j \le k$, and $int\,(P_{i,j}) = P_{i,j} \setminus \{u_i, u_j\}$.*

- (i) *If $j - i$ is even, then $u_i \equiv u_j \pmod{T \setminus P_{i,j}}$, $u_j u_i \in A(T)$ and $u_i \mid u_j \pmod{P_{i,j}}$.*
- (ii) *If $j - i$ is odd, then $u_i \mid u_j \pmod{T \setminus P_{i,j}}$, $u_i u_j \in A(T)$ and $u_i \equiv u_j \pmod{P_{i,j}}$.*

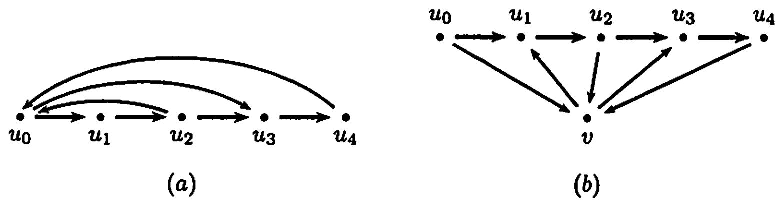


Figure 3: (a) the adjacency between u_0 and the vertices on P and (b) the adjacency between P and $v \in V(T \setminus P)$.

Proof

(i) Let $j - i = 2n$ for some $n \in \mathbb{N}$. Since $u_i u_{i+1}, u_{i+1} u_{i+2} \in A(\mathfrak{D}(T))$, then $u_i \equiv u_{i+2} \pmod{T \setminus \{u_{i+1}\}}$. Analogously $u_i \equiv u_{i+2n} \pmod{T \setminus \{u_{i+1}, u_{i+2,\dots,} u_{i+2n-1}\}}$. Since $int(P_{i,j}) = \{u_{i+1}, u_{i+2,\dots,} u_{i+2n-1}\}$ and $u_{i+2n} = u_j$, then it follows that $u_i \equiv u_j \pmod{T \setminus P_{i,j}}$.

So, we have that $u_{i+1} \equiv u_{j-1} \pmod{T \setminus \{u_{i+2}, u_{i+3}, \dots, u_{j-2}\}}$. Since $u_i u_{i+1}$, then $u_i u_{j-1} \in A(T)$, and since $u_{j-1} u_j \in A(\mathfrak{D}(T))$, then we conclude that $u_j u_i \in A(T)$.

Let $u_l \in int(P_{ij})$. If $j - l = 2m < 2n$, then $l - i = 2(n - m)$ and $u_j u_l, u_l u_i \in A(T)$. If $j - (l + 1) = 2m < 2n$, then $(l + 1) - i = 2(n - m)$ and $u_j u_{l+1}, u_{l+1} u_i \in A(T)$. Since $u_l u_{l+1} \in \mathfrak{D}(T)$, then $u_i u_l, u_l u_j \in A(T)$. It follows that $u_i \mid u_j \pmod{P_{i,j}}$.

(ii) Let $j - i = 2n + 1$ for some $n \in \mathbb{N}$. Then $(j - 1) - i = 2n$. Since $u_{j-1} u_j \in \mathfrak{D}(T)$, then by (i),

$$u_{j-1} u_i \in A(T) \quad \Rightarrow \quad u_i u_j \in A(T), \ (see\ Figure\ 3a)$$
$$u_i \mid u_{j-1} \pmod{P_{i,j-1}} \quad \Rightarrow \quad u_i \equiv u_j \pmod{P_{i,j}}, \ (Figure\ 3a)$$
$$u_i \equiv u_{j-1} \pmod{T \setminus P_{i,j-1}} \quad \Rightarrow \quad u_i \mid u_j \pmod{T \setminus P_{i,j}},$$
$$(see\ Figure\ 3b).$$

□

Remark 2 *Let $P = (u_0, u_1, \dots, u_k)$ be a path of $\mathfrak{D}(T)$, where T is a regular tournament. Let $F_0 = (u_0, u_2, \dots, u_{2l})$ and $F_1 = (u_1, u_3, \dots, u_{2l+1})$, where $2l, 2l + 1 \le k$, then $T[F_i]$ is transitive, $i = 0, 1$.*

If P is a path of $\mathfrak{D}(T)$, then $T[V(P)]$ is completely determined by Lemma 2.

The Figure 4 shows the adjacency between two $\mathfrak{D}$-paths.

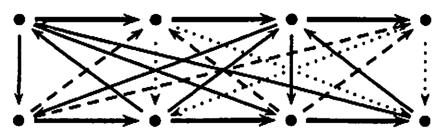


Figure 4: The adjacency between two $\mathfrak{D}$-paths

Let $\mathbb{Z}_{2m+1}$ be the cyclic group of integers modulo $2m + 1$ $(m \ge 1)$ and let J be subset of $\mathbb{Z}_{2m+1} \setminus \{0\}$ of order m, such that $|\{-i, i\} \cap J| = 1$ for every $i \in J$. A *circulant* (or *rotational) tournament* $\overrightarrow{C}_{2m+1}(J)$ is defined by $V(\overrightarrow{C}_{2m+1}(J)) = \mathbb{Z}_{2m+1}$ and

$$A(\overrightarrow{C}_{2m+1}(J)) = \{(i,j) : i,j \in \mathbb{Z}_{2m+1} \text{ and } j - i \in J\}.$$

Recall that the circulant tournaments are regular and their automorphism groups are vertex transitive. We call a tournament *vertex-transitive* if its automorphism group is vertex transitive.

The following proposition gives two characterizations of a cyclic tournament

Proposition 1 *A circulant tournament T is cyclic if and only if*

(*i*) *[5, 9] dom* $(T) \cong C_n$, *and*

$$\mathfrak{D}\left(\overrightarrow{C}_{2m+1}\langle\emptyset\rangle\right) = (0, m, 2m, m-1, \ldots, 1, m+1, 0).$$

(*ii*) *[13]* $T[N^+(v)]$ *is transitive for all* $v \in V(T)$.

Proposition 2 *[9] Let T be a vertex-transitive tournament. Then either T is a cyclic tournament, or dom (T) is edgeless.*

We will often use the following construction:

Remark 3 *Let uv be a $\mathfrak{D}$-arc of a regular tournament T of order $2m+3$ $(m \geq 1)$ and denote by $T' = T \setminus \{u, v\}$ the residual tournament of uv. By the definition of $\mathfrak{D}(T)$, there exists a natural partition of $V(T')$ into the sets $V^- = N^+(u; T')$ and $V^+ = N^+(v; T')$. Moreover, $V^- = N^-(v, T')$ and $V^+ = N^-(u, T')$. Since T is regular, $|V^-| = m$ and $|V^+| = m + 1$. See Figure 5.*

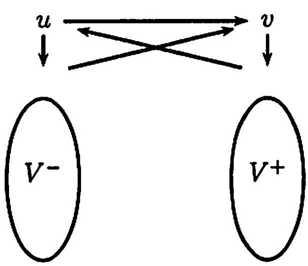


Figure 5: The partition of $T \setminus \{u, v\}$ into V^- and V^+

Proposition 3 *If T is a regular tournament and $u, v \in V(T)$, then the residual tournament $T' = T \setminus \{u, v\}$ is regular if and only if $uv \in \mathfrak{D}(T)$.*

Proof By regularity of T and $T \setminus \{u, v\}$, and Remark 3 (see Figure 5) the result follows. □

Corollary 1 *If T is a regular tournament and $P \subset \mathfrak{D}(T)$ a path of even order, then $T \setminus P$ is regular.*

Let $P_{0,k} = (u_0, u_1, \ldots, u_k)$ be a directed maximal path in the domination digraph of a regular tournament T. Note that if $|V(T)| \neq k+1$, then by Lemma 2, u_1 is the only vertex of the path $P_{0,k}$ which is discordant to u_0 with respect to T. Analogously u_{k-1} is the only vertex from the path $P_{0,k}$ discordant to u_k with respect to T.

Proposition 4 *$P_{0,2k+1} \subset \mathfrak{D}(T)$ is maximal in $\mathfrak{D}(T)$ if and only if there does not exist a source or a sink in T induced by $N^+(u_{2k+1}, T \setminus P_{0,2k+1})$.*

Proof Let $P_{0,2k+1} \subset \mathfrak{D}(T)$, and $T' = T \setminus P_{0,2k+1}$. Then $T' \neq \emptyset$, and T' is regular by Corollary 1. Note that $u_0u_{2k+1} \in \mathfrak{D}(T')$ by Lemma 2 (ii).

There exists $w \in N^+(u_{2k+1}, T')$, a source (sink) of $N^+(u_{2k+1}, T')$ if and only if

$$zw \in A(T) \text{ for all } z \in N^+(u_0, T') \cap N^-(u_{2k+1}, T') \text{ and } wu_0 \in A(\mathfrak{D}(T))$$

$$(wz \in A(T) \text{ for all } z \in N^+(u_0, T') \cap N^-(u_{2k+1}, T'),\ u_{2k+1}w \in A(\mathfrak{D}(T))).$$

Then $(w, u_0, u_1, \ldots, u_{2k+1})$ (resp. $(u_0, u_1, \ldots, u_{2k+1}, w)$) is a path in $\mathfrak{D}(T)$. And so $P_{0,2k+1}$ is not maximal in $\mathfrak{D}(T)$. □

Note that in Lemma 1, $\mathfrak{D}(T)|_{T'}$ is not necessarily induced in $\mathfrak{D}(T)$. When $\mathfrak{D}(T)|_{T'}$ is induced in $\mathfrak{D}(T)$, we say that T' is a *faithful* subtournament of T.

Proposition 5 *Let $P_{u_0u_r} = (u_0, u_1, \ldots, u_r)$ be a maximal path in $\mathfrak{D}(T)$ with $r \geq 2$. Then $T \setminus \{u_0, u_1\}$ and $T \setminus \{u_{r-1}, u_r\}$ are regular faithful subtournaments of T.*

Proof Let $T' = T \setminus \{u_0, u_1\}$. By Corollary 1, T' is regular. Let $vw \in A(\mathfrak{D}(T'))$. We assume that $u_2 \in N^+(v; T') \cap N^-(w; T')$. By Lemma 2 (i),

$$u_0 \equiv u_2 \pmod{T \setminus \{u_0, u_1, u_2\}} \text{ and } u_0u_1 \in \mathfrak{D}(T),$$

then $u_0 \in N^+(v) \cap N^-(w)$ and $u_1 \in N^-(v) \cap N^+(w)$. And $v \mid w \pmod{\{u_0, u_1\}}$ and $vw \in \mathfrak{D}(T)$. The proof for $T \setminus \{u_{r-1}, u_r\}$ is analogous. □

Corollary 2 *Let $P_{u_0u_r} = (u_0, u_1, \ldots, u_r)$ be a maximal path in $\mathfrak{D}(T)$ with $r \geq 2$ and $P = (u_{r-2k-1}, u_{r-2k}, \ldots, u_{r-1}, u_r)$ with $r - 2k - 1 > 0$. Then $T \setminus P$ is a regular faithful subtournament of T.*

3 Molds

A *mold* is a regular tournament M such that all the paths of the domination digraph $\mathfrak{D}(M)$ are of order at most 2. We assign a mold M^T and associate a weight function φ_T, to every regular non cyclic tournament T as follows:

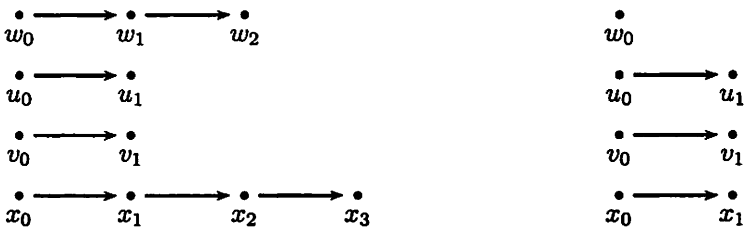
(i) Let $\{P_i^1\}$ be the set of maximal paths of odd order of $\mathfrak{D}(T)$, and let $\{P_i^2\}$ be the set of maximal paths of even order of $\mathfrak{D}(T)$. Denote $P_i^1 = (u_{i,0}, u_{i,1}, \ldots, u_{i,2k_i})$ and $P_i^2 = (v_{i,0}, v_{i,1}, \ldots, v_{i,2k_i+1})$. We define the mold M^T of T on the following vertex set

$$\{u_{i,0} \in V(T) : u_{i,0} \in P_i^1\} \cup \{v_{i,0}, v_{i,1} \in V(T) : (v_{i,0}, v_{i,1}) \in P_i^2\},$$

and $M^T = T[V(M^T)]$.

(ii) The weight function φ_T assigns to each component of $\mathfrak{D}(M^T)$ the order of the corresponding path in $\mathfrak{D}(T)$.

For example, let T be the regular tournament in Figure 1, (a), for the mold M^T of T see Figure 1 (b), and if the $\mathfrak{D}$-paths of $\mathfrak{D}(M^T)$ are $P_0^1 = \{w_0\}$, $P_0^2 = (u_0, u_1)$, $P_1^2 = (v_0, v_1)$, $P_2^2 = (x_0, x_1)$, and the weight function φ_T of T is $\varphi_T(P_0^1) = 3$, $\varphi_T(P_0^2) = 2$, $\varphi_T(P_1^2) = 2$, $\varphi_T(P_2^2) = 4$.



(a) (b)

Figure 6: (a) The digraph $\mathfrak{D}(T)$, (b) the digraph $\mathfrak{D}(M^T)$.

Clearly M^T is unique and the weight function φ_T is well-defined by Proposition 6. Note that if T is a mold, then $M^T = T$.

Proposition 6 *Let M^T be the mold of a regular non cyclic tournament T. Then*

(i) *M^T is a regular, faithful subtournament of T.*

(ii) *T can be reconstructed from its mold M^T and the weight function φ_T.*

Proof (i) follows by Corollaries 1 and 2.

(ii) By the definition of the weight function φ_T, we know the length of every path in $\mathfrak{D}(T)$. The adjacency between every pair of vertices of $V(T)$ is completely determined by the mold M^T and by Lemma 2. □

If T and T' are regular non cyclic tournaments such that $M^T \ncong M^{T'}$, then $T \ncong T'$.

Cho et al. characterized the domination graphs of regular tournaments in [4, 5]. Using molds we give a simpler proof of this characterization:

Definition 1 *Let T be a regular tournament and uv a $\mathfrak{D}$-arc. We say that the arc uv is $\mathfrak{C}$-residual ($\mathfrak{C} - arc$) if $T \setminus \{u, v\}$ is a cyclic tournament.*

Let $\mathbb{Z}_{2m+1}$ be the cyclic group of integers modulo $2m+1$ $(m \geq 1)$ and K a subset of $\mathbb{Z}_m \setminus \{0\}$. A *circulant* (or *rotational) tournament* $\overrightarrow{C}_{2m+1}\langle K\rangle$ is defined by $V(\overrightarrow{C}_{2m+1}\langle K\rangle) = \mathbb{Z}_{2m+1}$ and the set of arcs of $\overrightarrow{C}_{2m+1}\langle K\rangle$ is the set

$$\{(i,j) : i,j \in \mathbb{Z}_{2m+1} \text{ and } j-i \in (\{1,2,\ldots m\} \setminus K) \cup (-K)\}.$$

If T is non cyclic circulant tournament, then by Proposition 2 $\mathfrak{D}(T)$ is the arcless digraph.

Lemma 3 *Let $2m+1 \geq 9$. Then there exists a mold M, such that $\mathfrak{D}(M)$ has exactly one $\mathfrak{D}$-arc.*

Proof Let $t = 2t'+1 \geq 7$. Note that $\mathfrak{D}(\overrightarrow{C}_{2t'+1}\langle t'\rangle)$ is arcless.

Let T be the following regular tournament:

$V(T) = V\left(\overrightarrow{C}_{2t'+1}\langle t'\rangle\right) \cup \{w^-, w^+\}, T\left[V\left(\overrightarrow{C}_{2t'+1}\langle t'\rangle\right)\right] \cong \overrightarrow{C}_{2t'+1}\langle t'\rangle$, $w^-w^+ \in \mathfrak{D}(T)$, with $N^+(w^+) = \{0,1,2,\ldots,t'\}$. Since $N^+(w^+)$ has no source nor a sink in T and $\mathfrak{D}\left(\overrightarrow{C}_{2t'+1}\langle t'\rangle\right)$ is arcless, then by Lemma 1, the only $\mathfrak{D}$-arc of T is w^-w^+. □

Lemma 4 *Let $2m+1 \geq 9$, then there exists a mold M, such that $\mathfrak{D}(M)$ has exactly two $\mathfrak{D}$-arcs.*

Proof Let $t = 2t'+1 \geq 5$, note that $\mathfrak{D}(\overrightarrow{C}_{2t'+1}\langle\varnothing\rangle)$ is the cycle $\overrightarrow{C}_{2t'+1}$.

Let T' be the following regular tournament:

$$V(T') = V\left(\overrightarrow{C}_{2t'+1}\langle\varnothing\rangle\right) \cup \{w_0^-, w_0^+\}, T\left[V\left(\overrightarrow{C}_{2t'+1}\langle\varnothing\rangle\right)\right] \cong \overrightarrow{C}_{2t'+1}\langle\varnothing\rangle,$$

$w_0^- w_0^+ \in \mathfrak{D}(T)$ and $N^+\left(w_0^+\right) = V_0^+$, with

$$V_0^+ = \{(t'+1)\,t', (t'+2)\,t', \ldots, (2t')\,t', (2t'+1)\,t' = 0\}.$$

The induced subtournament $T\left[V_0^+\right]$ has no source nor a sink, then by Lemma 1, $A\left(\mathfrak{D}\left(T'\right)\right) = \left\{\left(w_0^-, w_0^+\right), (0, t'), ((t')\,t', (t'+1)\,t')\right\}$, T' is a mold, and $\mathfrak{D}(T')$ has exactly three $\mathfrak{D}$-arcs.

Let T be the following regular tournament: $V(T) = V(T') \cup \left\{w_1^-, w_1^+\right\}$, $T[T'] \cong T'$; $w_1^- w_1^+ \in \mathfrak{D}(T)$ and $N^+\left(w_1^+\right) = V_1^+$, with

$$V_1^+ = \left\{w_0^+, (t')\,t', (t'+1)\,t', \ldots, (2t')\,t'\right\}.$$

Since $T[V^+]$ has no source nor a sink, $A(\mathfrak{D}(T)) = \left\{\left(w_0^-, w_0^+\right), \left(w_1^-, w_1^+\right)\right\}$, T is a mold, and $\mathfrak{D}(T)$ has exactly two $\mathfrak{D}$-arcs. □

Let us construct infinite families of molds, first with an odd number of $\mathfrak{D}$-arc and then with an even number of $\mathfrak{D}$-arcs.

Proposition 7 *Let m, s be two integers, such that $2s+3 \geq 9$ and $0 < m \leq s$. Then there exists a mold M such that $\mathfrak{D}(M)$ has exactly $2m+1$ $\mathfrak{D}$-arcs.*

Proof Let (w^-, w^+) be the $\mathfrak{C}$-arc of M and $M \setminus \{w^-, w^+\} \cong \vec{C}_{2s+1}\langle\varnothing\rangle$. Let $N^+(w^+)$ be the following set

$$\{0, s-1, 2s-1, s-3, \ldots, s-(2m-3), 2s-(2m-3)\} \cup$$
$$\{(s-2m+3)\,s-2\,(m-1), (s-2m+4)\,s-2\,(m-1), \ldots, -s\}.$$

By Proposition 1, the set of arcs of $\mathfrak{D}(M)$ is

$$\{(0,s), (2s, s-1), (2s-1, s-2), \ldots, (2s-(2m-3), s-(2m-1))\} \cup$$
$$\{((s-2m+2)\,s-2\,(m-1), (s-2m+3)\,s-2\,(m-1)), (w^-, w^+)\}.$$

□

Let $D[U \leftrightarrow V]$ be the subdigraph of D on the vertex set $U \cup V$ with

$$A(D[U \leftrightarrow V]) = \{uv \in A(D) : u \in U, v \in V\} \cup \{vu \in A(D) : u \in U, v \in V\}$$

Lemma 5 *Let $P = (u_0, u_1, \ldots, u_r)$, $r \geq 0$, be a maximal path of $\mathfrak{D}(T)$ and let $V^- = N^+(u_0; T \setminus P)$ and $V^+ = N^-(u_0; T \setminus P)$. Then $\mathfrak{D}(T) \setminus P$ is a subdigraph of $T[V^- \leftrightarrow V^+]$. Moreover $\mathfrak{D}(T) \setminus P = \mathfrak{D}(T)[V^- \leftrightarrow V^+]$.*

Proof By Lemma 1, $\mathfrak{D}(T)[V^- \leftrightarrow V^+] \subset \mathfrak{D}(T) \setminus P$. Let vw be an arc of $\mathfrak{D}(T) \setminus P$, then $v \mid w \pmod{u_0}$ and $vw \in \mathfrak{D}(T)[V^- \leftrightarrow V^+]$. □

Proposition 8 *Let T be a regular non cyclic tournament with a $\mathfrak{C}$-arc (w^-, w^+), then M^T has an odd number of $\mathfrak{D}$-arcs.*

Proof Let T be a non cyclic regular tournament, then $T' = T \setminus \{w^-, w^+\}$ is a cyclic tournament. If $T[N^+(w^+)]$ is vertex transitive, then $T[N^+(w^+)]$ has a source and a sink, and by Proposition 4, (w^-, w^+) is not a component of $\mathfrak{D}(T)$. It follows that $\mathfrak{D}(T)$ is a cycle and by Proposition 1 (i), T is a cyclic tournament. Then $T[N^+(w^+)]$ is not a transitive subtournament of T'. Since $T[N^+(w^+)]$ is not transitive, then it has neither a source nor sink, by Proposition 1 (ii). By Proposition 4, the arc (w^-, w^+) is a component of $\mathfrak{D}(T)$. Let $V^- = N^+(w^-, T')$ and $V^+ = N^+(w^+, T')$. By Lemma 5, $\mathfrak{D}(T')[V^- \leftrightarrow V^+]$ has an even number of paths of even order in $\mathfrak{D}(T)$. Then M^T has an odd number of $\mathfrak{D}$-arcs. □

In [12], the regular tournament W_0 was defined as:

$$\begin{aligned} V(W_0) &= \{w_1^-, w_2^-, w_3^-, w_1^+, w_2^+, w_3^+, w_0\} \\ A(W_0) &= \{(w_i^-, w_0)\} \cup \{(w_0, w_i^+)\} \cup \{(w_i^-, w_i^+)\} \cup \{(w_i^+, w_{i-1}^-)\} \cup \\ &\quad \{(w_i^+, w_{i+1}^-)\}, \text{with } i-1, i, i+1 \in \mathbb{Z}_3. \end{aligned}$$

Note that $\mathfrak{D}(W_0) = \{(w_2^+, w_1^-), (w_3^+, w_2^-), (w_1^+, w_3^-), \{w_0\}\}$, the $\mathfrak{D}$-arcs of W_0 are all $\mathfrak{C}$-arcs and W_0 is transitive in $\mathfrak{C}$-arcs.

Remark 4 *Up to isomorphism there are* 3 *regular tournaments of order* 7, *the cyclic, the Paley tournament and* W_0. *So* W_0 *is the only mold of order* 7 *with* $\mathfrak{D}$*-arcs.*

Proposition 9 *There is no mold* M *of order* 9, *such that* $\mathfrak{D}(M)$ *has four* $\mathfrak{D}$*-arcs.*

Proof Suppose that M is a mold of order 9 with 4 $\mathfrak{D}$-arcs. Let u_0u_1 be a component of $\mathfrak{D}(M)$ and $T = M \setminus \{u_0, u_1\}$, then T is regular tournament of order 7. Since M has an even number of $\mathfrak{D}$-arcs, then by Proposition 8, u_0u_1 is not a $\mathfrak{C}$-arc, moreover T has at least 3 $\mathfrak{D}$-arcs so $T \cong W_0$, by Remark 4.

Since the $\mathfrak{D}$-arcs of W_0 form an orbit of $Aut(W_0)$, then there are 4 ways to construct a mold with 4 $\mathfrak{D}$-arcs: it is easy to verify that in each case $N^+(u_1)$ has a source or a sink, contradicting that u_0u_1 is a component of $\mathfrak{D}(M)$. If $N^+(u_1) = \{w_1^-, w_2^-, w_3^-, w_0\}$ ($N^+(u_1) = \{w_1^-, w_1^+, w_3^+, w_0\}$), then w_0 (w_1^+) is a sink of $N^+(u_1)$. If $N^+(u_1) = \{w_1^-, w_2^-, w_1^+, w_0\}$ ($N^+(u_1) = \{w_1^+, w_2^+, w_3^+, w_0\}$), then w_1^- (w_0) is a source of $N^+(u_1)$. □

We can now construct infinite families of molds with an even number of $\mathfrak{D}$-arcs.

Proposition 10 *Let* $m \geq 4$ *be an even integer and let* $2s+1 \geq 11$, $s \geq m$. *Then there exist a mold* M *of order* $2s+1$ *with* m $\mathfrak{D}$*-arcs.*

Proof Note that if $s = 4$, then by Proposition 9, $n > 1$, where n is the number of isolated vertices in $\mathfrak{D}(M)$.

By Proposition 7, there exists a mold T' with $m-1$ $\mathfrak{D}$-arcs. Let $\left(w_0^-, w_0^+\right)$ be a $\mathfrak{C}$-arc of T' and $V_0^- = N^+\left(w_0^-\right) \setminus \left\{w_0^+\right\}$, $V_0^+ = N^+\left(w_0^+\right)$. The residue of the $\mathfrak{C}$-arc $\left(w_0^-, w_0^+\right)$ is $\vec{C}_{2(s-2)+1}\langle\varnothing\rangle$. Since $s-2 \geq 2$, then T' has another $\mathfrak{D}$-arc. We assume that $(s-1, 0)$ is a $\mathfrak{D}$-arc of T', with $0 \in V_0^-$ and $s-1 \in V_0^+$.

Let T be the following regular tournament: $V(T) = V(T') \cup \left\{w_1^-, w_1^+\right\}$,

$$\begin{aligned} A(T) &= A(T') \cup \left\{w_1^- v : v \in V_1^-\right\} \cup \left\{u w_1^- : u \in V_1^+\right\} \cup \\ &\quad \left\{w_1^+ v : v \in V_1^+\right\} \cup \left\{u w_1^+ : u \in V_1^-\right\} \cup \left\{\left(w_1^-, w_1^+\right)\right\}, \end{aligned}$$

with $V_1^- = \left\{w_0^-\right\} \cup V_0^- \setminus \{0\} \cup \{s-1\}$, $V_1^+ = \left\{w_0^+\right\} \cup V_0^+ \setminus \{s-1\} \cup \{0\}$. Note that T' is faithful in T. Since $\left(0, w_0^+\right)$, $\left(w_0^+, v\right)$ with $v \in V_0^+ \setminus \{s-1\}$, then w_0^+ is neither a source nor a sink of V_1^+. If w is a source or a sink of V_1^+, then it would be a source or a sink of $V_1^+ \setminus w_0^+ \subset \vec{C}_{2(s-2)+1}\langle\varnothing\rangle$ and $T\left[V_1^+ \setminus w_0^+\right] \cong TT_{s-1}$. Since $s \geq 5$ $(s-1 \geq 4)$, then there exists $i, i+1 \in V_1^+ \cap V_0^+$ such that $i+s-1 \in V_0^-$ and $(i+1, i+s-1), (i+s-1, i) \in A(\mathfrak{D}(T'))$ contradicting that T' is a mold. □

After proving several properties of molds we are now able to give a shorter proof of Theorems 3 and 3.14 by Cho et. al. ([4, 5] respectively).

Theorem 1 *Let D be a disjoint union of m arcs and n vertices. Then D is the domination digraph of a mold of a regular tournament if and only if n is odd and either*

(*i*) $m = 3$ *or* $m \geq 5$, *or*

(*ii*) $m = 0, 1, 2, 4$ *and* $m + n \geq 7$.

Proof For necessity, let D be the domination digraph of the mold M, then by Proposition 3 n is odd. By Remark 4, if D has order 7, then $\mathfrak{D}(M)$ is arcless and then $m = 0$, $n = 7$ and $n + m = 7$, or $M \cong W_0$ and $m = 3$. If D has order 9, and $m = 0, 1, 2, 4$, then $n + m \geq 7$ by Proposition 9. Clearly if D has order at least 11, and $m = 0, 1, 2, 4$, then $n + m \geq 7$.

For sufficiency, let m, n be non negative integers, n odd. If $m = 0$ (resp. $m = 1$) and $n \geq 7$, then there exists a mold by Proposition 2 (resp. by Lemma 3) with these parameters. If $m = 2$ and $n \geq 5$ (resp. $m = 4$ and $n \geq 3$), then there exists a mold by Lemma 4 (resp. Propositions 9 and 10) with these parameters. If m is odd (resp. even) and $m \geq 3$, then there exists a mold by Proposition 7 (resp. Proposition 10) with these parameters. □

4 Colorings

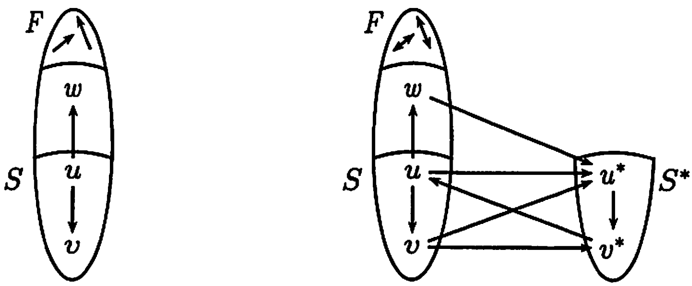
We give lower and upper bounds to the dichromatic number of a regular non cyclic tournament in terms of the dichromatic number of its mold. We also bound the dichromatic number of the tournaments that belong to a family with a fixed mold, in terms of the dichromatic number of the *quotient* of a regular *ample* tournament of this family (to be defined).

First we construct some digraphs that will be useful.

Let D be a digraph and $S \subset V(D)$. Let $S^* = \{u^* : u \in S\}$ be a copy of S disjoint to $V(D)$ and $D[S]^* = (S^*, A^*)$ be the digraph isomorphic to $D[S]$ with the bijection $\varphi : U \to U^*$ defined by $\varphi(u) = u^*$. Let F be a set (possibly empty) of disjoint arcs of $A(D)$ such that $V(F) \cap S = \varnothing$. We define the following digraphs $D\langle S, F; \bullet\rangle$, $D\langle S, F; \to\rangle$, $D\langle S, F; \leftrightarrow\rangle$.

$$\begin{aligned} D\langle S, F; \bullet\rangle &= D \cup D[S]^* \{v^*u : uv \in A(D) \text{ and } v \in S\} + \\ &\quad \{vu : uv \in F\} + \{vu^* : uv \in A(D) \text{ and } u \in S\} \\ D\langle S, F; \to\rangle &= D\langle S, F; \bullet\rangle + \{uu^* : u \in S\} \ (\textit{see Figure 7}). \\ D\langle S, F; \leftrightarrow\rangle &= D\langle S, F; \bullet\rangle + \{uu^* : u \in S\} + \{u^*u : u \in S\} \end{aligned}$$

Note that the arcs of F are symmetric and u, u^* $(u \in S)$ is a dominant pair in $D\langle S, F; *\rangle$, $* \in \{\bullet, \to, \leftrightarrow\}$. When $F = \varnothing$, we simply write $D\langle S; \to\rangle$ and $D\langle S; \leftrightarrow\rangle$.



(a) (b)

Figure 7: (a) The digraph D, (b) the digraph $D\langle S, F; \to\rangle$.

An acyclic coloring of a digraph D is a coloring of the vertices of D such that no directed monochromatic cycles are formed. The *dichromatic number* $dc(D)$ of a digraph D was defined by Neumann-Lara [11] as the smallest number of colors needed to color the vertices of D such that no directed monochromatic cycles are formed.

Lemma 6 *Let D be a digraph and $S \subset V(D)$. Then $dc(D\langle S, \leftrightarrow\rangle) \leq 2dc(D)$.*

Proof Let $\varphi : D \to C$ be an optimal coloring of D, where C is the set of colors used by φ. Let $C' = \{c' \mid c \in C\}$ be a set of colors disjoint to C with $|C'| = |C|$. We define a coloring $\gamma : D\langle S, \leftrightarrow\rangle \to C \cup C'$ such that:

(i) If $u \in D$, then $\gamma(u) = \varphi(u)$.

(ii) If $u \in S^*$ and $\varphi(u) = c_i$, then $\gamma(u) = c_i'$.

Clearly γ is an acyclic coloring of $D\langle S, \leftrightarrow\rangle$, and γ uses at most $2|C|$ colors.
□

If M is a mold, let $\mathcal{F}(M)$ be the family of regular tournaments such that if $T \in \mathcal{F}(M)$, then M is the mold of T.

If T is a regular tournament and all the paths of $\mathfrak{D}(T)$ are of order at least 3, then we say that T is an *ample* tournament.

We say that the vertices $u, v \in V(T)$ are *equivalent* in T, $u \sim v \pmod{T}$, if for some path P in $\mathfrak{D}(T)$, $u, v \in V(P)$ and $u \equiv v \pmod{T \setminus P}$. The equivalence relation $\sim$ induces a partition $\mathcal{P}$ of $V(T)$. Let $T/\mathcal{P}$ be the quotient of T induced by $\mathcal{P}$. Note that $M^T \hookrightarrow T/\mathcal{P}$ and that $T/\mathcal{P}$ is a tournament only when T is a mold.

Remark 5 *Let $T, T' \in \mathcal{F}(M)$ with T' ample, then $T/\mathcal{P}$ can be naturally identified with a subtournament of $T'/\mathcal{P}$, where the mold M remains fixed.*

Remark 6 *Let M^T be the mold of the regular tournament T. Let S_0 be the set of trivial components of $\mathfrak{D}(M^T)$ that are not components of $\mathfrak{D}(T)$, S_1 be the set of arcs of $A(\mathfrak{D}(M^T))$ belonging to some component of order at least 3 of $\mathfrak{D}(T)$. Then $M^T\langle S_0, S_1; \leftrightarrow\rangle \cong T/\mathcal{P}$.*

Theorem 2 *Let T be a regular non cyclic tournament, then $dc(T) = dc(T/\mathcal{P})$.*

Proof If $\varphi : D \to D'$ is a epimorphism with acyclic preimages $\varphi^{-1}(D)$, then $dc(D) \leq dc(D')$. By Remark 2, $dc(T) \leq dc(T/\mathcal{P})$. Let φ be an optimal acyclic coloring of T and $\mathbf{P}$ the set of maximal paths in $\mathfrak{D}(T)$ of order at least 3. In each path $P \in \mathbf{P}$ there are at least 2 colors by Lemma 2 (i). Let u_p,v_p be consecutive vertices in P with different color. We define a new coloring γ of T as follows:

(i) If $x \in V(P)$ for some path $P \in \mathbf{P}$, then

$$\gamma(x) = \begin{cases} \varphi(u_p) & \text{if } x \equiv u_p \pmod{T \setminus P} \\ \varphi(v_p) & \text{if } x \equiv v_p \pmod{T \setminus P} \end{cases}$$

(ii) Otherwise, $\gamma(x) = \varphi(x)$.

Suppose that γ is not acyclic. Let C be a directed monochromatic cycle of minimum order of T. Then C is a directed triangle. Clearly, C has some vertex u whose color was changed, and $u \in P$ for some path $P \in \mathbf{P}$. Since γ induces an acyclic coloring in P, then C has some vertex $v \in V(T) \setminus P$. Suppose that C has the color $\varphi(u_p)$. If the third vertex $z \in C$ was in P, then $u \equiv z \pmod{v}$, which contradicts that C is a cycle. So $z \in V(T) \setminus P$. But $u_p \equiv u \pmod{\{v, z\}}$, and then T has a triangle by φ. Therefore γ is an acyclic coloring of T. Moreover γ induces an acyclic coloring of $T/\mathcal{P}$ and $dc(T) \geq dc(T/\mathcal{P})$. □

As consequence of this Theorem, we can prove that all ample tournaments of a family $\mathcal{F}(M)$ have the same dichromatic number.

Theorem 3 *Let T be a regular non cyclic tournament, then*

$$dc(M^T) \leq dc(T) \leq 2dc(M^T).$$

Proof Since $M^T \subseteq T$, the first inequality is valid. The second one follows by Theorem 2 and Lemma 6. □

By Remark 5, we have

Corollary 3 *Let $T, T' \in \mathcal{F}(M)$ with T' ample, then*

$$dc(M) \leq dc(T) \leq dc(T').$$

Acknowledgment

The authors would like to thank the referee for the valuable suggestions.

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