On extremal cacti with minimal degree distance

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Abstract. Let $Diag(G)$ and $D(G)$ be the degree-diagonal matrix and distance matrix of G, respectively. Define the multiplier $Diag(G)D(G)$ as degree distance matrix of G. The degree distance of G is defined as $D'(G) = \sum_{x\in V(G)} d_G(x)D_G(x)$, where $d_G(x)$ is the degree of vertex x, $D_G(x) = \sum_{u\in V(G)} d_G(u,x)$ and $d_G(u,x)$ the distance between u and x. Obviously, $D'(G)$ is also the sum of elements of degree distance matrix $Diag(G)D(G)$ of G. A connected graph G is a cactus if any two of its cycles have at most one common vertex. Let $\mathscr{G}(n,r)$ be the set of cacti of order n and with r cycles. In this paper, we give the sharp lower bound of the degree distance of cacti among $\mathscr{G}(n,r)$, and characterize the corresponding extremal cactus.

Keywords: degree distance; cactus; degree distance matrix

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1. Introduction

Let $G = (V(G), E(G))$ be a connected simple graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$. Set $N(v_i) = \{u|uv_i \in E(G)\}$, $N[v_i] = N(v_i)\cup\{v_i\}$. Let $d_G(v_i)(= |N(v_i)|)$ be the degree of vertex v_i of G. The number $\delta(G) = \min\{d_G(v)|v \in V(G)\}$ is the minimum degree of G, the number $\Delta(G) = \max\{d_G(v)|v \in V(G)\}$ its maximum degree. If $d_G(v) = k$, we name v as k-degree vertex. Denote by $Diag(G)$ the diagonal matrix of vertex degrees of G. For vertices $v_i, v_j \in V(G)$, the distance $d_G(v_i, v_j)$ is defined as the length of the shortest path between v_i and v_j in G. Let $D(G) = (d_{ij})_{v_i,v_j\in V(G)}$ be the distance matrix of G, where $d_{ij} = d_G(v_i, v_j)$. Define the multiplier $Diag(G)D(G)$ as degree distance matrix of G.

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The degree distance $D'(G)$ of G, which was introduced by Dobrynin and Kochetova [3] and Gutman [5], is defined as

$$D'(G) = \sum_{x \in V(G)} d_G(x) D_G(x) \tag{1.1}$$

where $d_G(x)$ is the degree of vertex x, $D_G(x) = \sum_{u \in V(G)} d_G(u, x)$ and $d_G(u, x)$ the distance between u and x. Obviously, $D'(G)$ is also the sum of elements of degree distance matrix $Diag(G)D(G)$ of G. Besides as a topological index itself, the degree distance is also the non-trivial part of the molecular topological index (MTI) (or Schultz index) [10], which may be expressed as $D'(G) + \sum_{u \in V(G)} d_G(u)^2$, for characterization of alkanes [5, 8, 9]. Some properties for the degree distance may be found, e.g., in [8, 9, 14] in the text of MTI.

The degree distance of graphs is well studied in the literature. In [12], Tomescu presented the graph with minimum degree distance among all connected graphs and disproved a conjecture posed in [3], and in [13] some properties of graphs having minimum degree distance in the class of connected graphs of order n and size $m \geq n - 1$ were deduced. In [2] the authors reported the minimum degree distance of graphs with given order and size. Dankelmann et al. [4] presented an asymptotically sharp upper bound of the degree distance of graphs with given order and diameter. Hou and Chang [6] obtained the maximum degree distance among unicyclic graphs on n vertices. In [11], Tomescu obtained the minimum degree distance of unicyclic and bicyclic graphs, and the authors in [7] characterized n-vertex unicyclic graphs with girth k, having minimum and maximum degree distance, and the maximum degree distance among bicyclic graphs, respectively.

In this paper, we will further study the degree distance of cacti. We call G a cactus if it is connected and all of blocks of G are either edges or cycles, i.e., any two of its cycles have at most one common vertex. Denote $\mathscr{G}(n, r)$ the set of cacti of order n and with r cycles. Specifically, $\mathscr{G}(n, 0)$ is the set of trees of order n and $\mathscr{G}(n, 1)$ is the set of unicyclic graphs of order n. In this paper, we will give the sharp lower bound of the degree distance of cacti among $\mathscr{G}(n, r)$, and characterize the corresponding extremal cactus.

In order to state our results, we introduce some notation and terminology. For other undefined notation we refer to Bollobás [1]. If $W \subset V(G)$, we denote by $G - W$ the subgraph of G obtained by deleting the vertices of W and the edges incident with them, and by $G[W]$ the induced subgraph of G. Similarly, if $E \subset E(G)$, we denote by $G - E$ the subgraph of G

obtained by deleting the edges of E. If $W = \{v\}$ and $E = \{xy\}$, we write $G - v$ and $G - xy$ instead of $G - \{v\}$ and $G - \{xy\}$, respectively.

Now we give some lemmas that will be used in the proof of our main results.

Lemma 1.1. *[13] If $T \in \mathscr{G}(n, 0)$, then $D'(T) \geq 3n^2 - 7n + 4$, the equality holds if and only if $T \cong S_n$.*

Denote by $C_k(1^{n-k})$ the graph obtained by attaching $n - k$ pendent edges to one vertex of C_k.

Lemma 1.2. *[13] If $G \in \mathscr{G}(n, 1)$, then $D'(G) \geq 3n^2 - 3n - 6$, the equality holds if and only if $G \cong C_3(1^{n-3})$.*

For a connected graph G with $u \in V(G)$, we define

$$D_G^*(u) = \sum_{x \in V(G)} d_G(x) d_G(x, u).$$

Lemma 1.3. *[7] Let G be a connected graph and v be a pendent vertex of G with $uv \in E(G)$. Then*

$$D'(G) = D'(G - v) + D_{G-v}(u) + D_G(v) + D^*(v).$$

Let G_1, G_2 be two connected graphs, $G_1 \cup G_2 = (V(G_1) \cup V(G_2), E(G_1) \cup E(G_2))$.

Lemma 1.4. *[7] Let G be a connected graph with a cut-vertex v such that G_1 and G_2 are two connected subgraphs of G having v as the only common vertex and $G = G_1 \cup G_2$. Let $n_i = |V(G_i)|$ and $m_i = |E(G_i)|$ for $i = 1, 2$. Then $D'(G) = D'(G_1) + D'(G_2) + 2m_1 D_{G_2}(v) + 2m_2 D_{G_1}(v) + (n_1 - 1) D_{G_2}^*(v) + (n_2 - 1) D_{G_1}^*(v)$.*

Let $G^0(n, r)$ be the graph as shown in Figure 1.

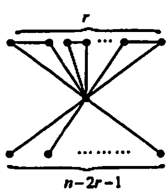

Figure 1: The graph $G^0(n, r)$

Lemma 1.5. *Let G be a graph in $\mathscr{G}(n,r)$ with a 1-degree vertices and $A = \{v \in V(G) : d_G(v) = 1\}$, then $n \geq 2r + a + 1$.*

Proof. It is easy to see that there are at least $2r+1$ vertices on the r cycles in G and $|A| = a$, then $n \geq a + 2r + 1$.

□

2. The minimum degree distance of cacti

In the section, we will study a sharp lower bound on the degree distance of cacti. First, we give some lemmas that will be used.

Lemma 2.1. *Let $G^0(n,r)$ be the graph shown in Figure 1, then*

$$D'(G^0(n,r)) = 3n^2 + 4nr - 7n - 10r + 4.$$

Proof. For simplicity, let $A = G^0(n,r)$. By (1.1), we have

$$D'(G^0(n,r)) = \sum_{v \in V(A)} d_A(v) D_A(v)$$

$$\begin{aligned} &= \sum_{v \in V(A), d_A(v)=1} D_A(v) + 2 \sum_{v \in V(A)}^{d_A(v)=2} D_A(v) + (n-1) \sum_{v \in V(A)}^{d_A(v)=n-1} D_A(v) \\ &= 3n^2 + 4nr - 7n - 10r + 4. \end{aligned}$$

□

Lemma 2.2. *Let $G \in \mathscr{G}(n,r)$ and v be a pendent vertex of G with $uv \in E(G)$, then $D^*(v) \geq 3n + 4r - 5$. The equality holds if and only if $G \cong G^0(n,r)$.*

Proof. By the definition of $D^*(v)$, we have

$$\begin{aligned} D^*(v) &= \sum_{x \in V(G)} d_G(x) d_G(x,v) \\ &= \sum_{x \in V(G)-\{u,v\}} d_G(x)[d_G(x,u) + 1] + d_G(u) \\ &= \sum_{x \in V(G)-\{u,v\}} d_G(x) d_G(x,u) + 2[(n-1) + r] - 1 \\ &\geq \sum_{x \in V(G)-\{u,v\}} d_G(x) + 2[(n-1) + r] - 1. \end{aligned}$$

We can assume that there are a 1-degree vertices in G. Then

$$\begin{aligned}\sum_{x\in V(G)-\{u,v\}} d_G(x) &= a-1+\sum_{x\in V(G)-\{u,v\},d_G(x)\geq 2} d_G(x)\\ &\geq a-1+2[n-1-a]=2n-a-3.\end{aligned}$$

Further, we have $D^*(v) \geq 4n+2r-a-6$. By Lemma 1.5, we have $-a \geq 2r+1-n$, then $D^*(v) \geq 3n+4r-5$. The equality holds if and only if $d_G(x)=2$ for any $x \in \{y \in V(G)-\{u,v\}, d_G(y)\geq 2\}$ and $d_G(x,u)=1$ for any $x \in V(G)-u$, that is, $G \cong G^0(n,r)$. □

Lemma 2.3. *Let $C_k = u_1u_2\dots u_ku_1$ ($k \geq 4$) and $Q = C_k - u_1u_2 + u_2u_k$, then*

(i) $\sum_{u\in V(C_k)} d_{C_k}(u,u_1) - \sum_{u\in V(Q)} d_Q(u,u_1) = \begin{cases} 1-l, & if \;\; k=2l, \\ -l, & if \;\; k=2l+1. \end{cases}$

(ii) $\sum_{u\in V(C_k)} d_{C_k}(u,u_k) - \sum_{u\in V(Q)} d_Q(u,u_k) = \lfloor \frac{k}{2} \rfloor - 1;$

(iii) $\sum_{u\in V(C_k)} d_{C_k}(u,u_1) - \sum_{u\in V(Q)} d_Q(u,u_k) = \lfloor \frac{k}{2} \rfloor - 1;$

(iv) *If $k=4$ or $k\geq 6$,* $\sum_{x\in V(C_k)-\{u_1,u_k\}} \sum_{u\in V(C_k)} d_{C_k}(u,x) - \sum_{x\in V(C_k)-\{u_1,u_k\}} \sum_{u\in V(Q)} d_Q(u,x) \geq 0.$

Proof. By direct calculation, it is easy to obtain the following results:

If $k=2l$ ($l \geq 2$), where l is an integer, then

$$\begin{aligned} \sum_{u\in V(C_k)} d_{C_k}(u,u_1) &= \sum_{u\in V(C_k)} d_{C_k}(u,u_k) \\ &= 2[1+2+\ldots+(l-1)]+l && (2.2)\\ \sum_{u\in V(Q)} d_Q(u,u_k) &= 2[1+2+\cdots+(l-1)]+1 && (2.3)\\ \sum_{u\in V(Q)} d_Q(u,u_1) &= 2[1+2+\cdots+(l-1)]+2l-1 && (2.4)\end{aligned}$$

If $k=2l+1$ ($l\geq 2$), where l is an integer, then

$$\begin{aligned} \sum_{u\in V(C_k)} d_{C_k}(u,u_1) &= \sum_{u\in V(C_k)} d_{C_k}(u,u_k) \\ &= 2(1+2+\cdots+l) && (2.5)\\ \sum_{u\in V(Q)} d_Q(u,u_k) &= 2[1+2+\cdots+(l-1)]+l+1 && (2.6)\\ \sum_{u\in V(Q)} d_Q(u,u_1) &= 2[1+2+\cdots+(l-1)]+l+2l && (2.7)\end{aligned}$$

(i) By (2.2) and (2.4), (2.5) and (2.7), we have

$$\sum_{u\in V(C_k)} d_{C_k}(u,u_1) - \sum_{u\in V(Q)} d_Q(u,u_1) = \begin{cases} 1-l, & if \;\; k=2l, \\ -l, & if \;\; k=2l+1. \end{cases}$$

(ii) By (2.2),(2.3), (2.5) and (2.6), it is easy to obtain that

$$\sum_{u\in V(C_k)} d_{C_k}(u,u_k) - \sum_{u\in V(Q)} d_Q(u,u_k) = l-1 = \lfloor\frac{k}{2}\rfloor - 1.$$

(iii) Similarly, by (2.2),(2.3), (2.5) and (2.6), it is easy to obtain that

$$\sum_{u\in V(C_k)} d_{C_k}(u,u_1) - \sum_{u\in V(Q)} d_Q(u,u_k) = \lfloor\frac{k}{2}\rfloor - 1.$$

(iv) Let $x \in V(C_k) - \{u_1, u_k\}$. If $k = 2l$ $(l \geq 2)$, for $x = u_2, \ldots, u_{2l-1}$, we have

$$\begin{aligned} \sum_{u\in V(Q)} d_Q(u,u_2) = \sum_{u\in V(Q)} d_Q(u,u_{2l-1}) &= 2[1+2+\cdots+(l-1)]+2, \\ &\vdots \\ \sum_{u\in V(Q)} d_Q(u,u_l) = \sum_{u\in V(Q)} d_Q(u,u_{l+1}) &= 2[1+2+\cdots+(l-1)]+l; \end{aligned}$$

and by symmetry, we have

$$\sum_{u\in V(C_k)} d_{C_k}(u,u_2) = \cdots = \sum_{u\in V(C_k)} d_{C_k}(u,u_k) = 2[1+2+\cdots+(l-1)]+l.$$

Then

$$\begin{aligned} &\sum_{x\in V(C_k)-\{u_1,u_k\}} \sum_{u\in V(C_k)} d_{C_k}(u,x) - \sum_{x\in V(C_k)-\{u_1,u_k\}} \sum_{u\in V(Q)} d_Q(u,x) \\ = \;\; &(l-1)(l-2) \geq 0. \end{aligned}$$

If $k = 2l+1$ $(l \geq 3)$, for $x = u_2, \ldots, u_{2l}$, we have

$$\begin{aligned} \sum_{u\in V(Q)} d_Q(u,u_2) = \sum_{u\in V(Q)} d_Q(u,u_{2l}) &= 2[1+2+\cdots+(l-1)]+l+2, \\ &\vdots \\ \sum_{u\in V(Q)} d_Q(u,u_l) = \sum_{u\in V(Q)} d_Q(u,u_{l+2}) &= 2[1+2+\cdots+(l-1)]+l+l, \\ \sum_{u\in V(Q)} d_Q(u,u_{l+1}) &= 2[1+\cdots+(l-1)]+2l+1; \end{aligned}$$

and by symmetry, we have

$$\sum_{u\in V(C_k)} d_{C_k}(u,u_2) = \cdots = \sum_{u\in V(C_k)} d_{C_k}(u,u_k) = 2(1+2+\cdots+l).$$

Then

$$\begin{aligned}&\sum_{x\in V(C_k)-\{u_1,u_k\}} \sum_{u\in V(C_k)} d_{C_k}(u,x) - \sum_{x\in V(C_k)-\{u_1,u_k\}} \sum_{u\in V(Q)} d_Q(u,x)\\ =\ & l^2-3l+1>0.\end{aligned}$$

This completes the proof. □

Let H be a connected graph and $C_k = u_1u_2\dots u_ku_1$ ($k=4$ or $k\geq 6$). Suppose that w is a vertex of H, let G_1 be the graph obtained from H and C_k by identifying w with u_k and $G_2 = G_1 - u_1u_2 + u_2u_k$.

Lemma 2.4. *Let G_1, G_2 be graphs as description above, then $D'(G_1) > D'(G_2)$.*

Proof. Let $V_1 = \{u_1,u_2,\dots,u_k\}$, $V_2 = V(G_1) - V_1$. It is obvious that we have the following results:

(i) $d_{G_1}(u_1) = 2$, $d_{G_2}(u_1) = 1$, $d_{G_2}(u_k) = d_{G_1}(u_k)+1$ and $d_{G_2}(x) = d_{G_1}(x) = 2$ for $x\in V(G_1)-\{u_1,u_k\}$;
(ii) $d_{G_1}(u,u_k) = d_{G_2}(u,u_k)$ for $u\in V_2$.

By (1.1), we have

$$\begin{aligned}D'(G_1) &= d_{G_1}(u_1)D_{G_1}(u_1) + d_{G_1}(u_k)D_{G_1}(u_k) + \sum_{x\in V_2} d_{G_1}(x)D_{G_1}(x) +\\ &\quad \sum_{x\in V_1-\{u_1,u_k\}} d_{G_1}(x)D_{G_1}(x),\\ D'(G_2) &= d_{G_2}(u_1)D_{G_2}(u_1) + d_{G_2}(u_k)D_{G_2}(u_k) + \sum_{x\in V_2} d_{G_2}(x)D_{G_2}(x) +\\ &\quad \sum_{x\in V_1-\{u_1,u_k\}} d_{G_2}(x)D_{G_2}(x).\end{aligned}$$

Note that

$$\begin{aligned}&A_1 = d_{G_1}(u_1)D_{G_1}(u_1) - d_{G_2}(u_1)D_{G_2}(u_1) = 2D_{G_1}(u_1) - D_{G_2}(u_1)\\ =\ & \{\sum_{u\in V_2}[d_{G_1}(u,u_k)+1] + \sum_{u\in V_1} d_{G_1}(u,u_1)\} + \{\sum_{u\in V_1} d_{G_1}(u,u_1) -\\ & \sum_{u\in V_1} d_{G_2}(u,u_1)\};\end{aligned}$$

$$
\begin{aligned}
& A_2 = d_{G_1}(u_k)D_{G_1}(u_k) - d_{G_2}(u_k)D_{G_2}(u_k) \\
= \; & d_{G_1}(u_k)D_{G_1}(u_k) - [d_{G_1}(u_k)+1]D_{G_2}(u_k) \\
= \; & d_{G_1}(u_k)[\sum_{u\in V_1} d_{G_1}(u,u_k) - \sum_{u\in V_1} d_{G_2}(u,u_k)] - [\sum_{u\in V_2} d_{G_2}(u,u_k) + \\
& \sum_{u\in V_1} d_{G_2}(u,u_k)].
\end{aligned}
$$

Then if $k=2l$, by Lemma 2.3 (i)-(iii), we have

$$
\begin{aligned}
A_1 + A_2 \; = \; & \{\sum_{u\in V_2}[d_{G_1}(u,u_k)+1] + \sum_{u\in V_1} d_{G_1}(u,u_1)\} + \{\sum_{u\in V_1} d_{G_1}(u,u_1) \\
& - \sum_{u\in V_1} d_{G_2}(u,u_1)\} + \{d_{G_1}(u_k)[\sum_{u\in V_1} d_{G_1}(u,u_k) - \\
& \sum_{u\in V_1} d_{G_2}(u,u_k)] - [\sum_{u\in V_2} d_{G_2}(u,u_k) + \sum_{u\in V_1} d_{G_2}(u,u_k)]\} \\
= \; & n - 2l + (l-1)d_{G_1}(u_k) > 0.
\end{aligned}
$$

Furthermore, by Lemma 2.3 (ii), we have

$$
\begin{aligned}
& A_3 = \sum_{x\in V_2} d_{G_1}(x)[D_{G_1}(x) - D_{G_2}(x)] \\
= \; & \sum_{x\in V_2} d_{G_1}(x)\{\sum_{u\in V_2} d_{G_1}(u,x) + \sum_{u\in V_1}[d_{G_1}(u,u_k) + d_{G_1}(u_k,x)] \\
& -[\sum_{u\in V_2} d_{G_2}(u,x) + \sum_{u\in V_1}(d_{G_2}(u,u_k) + d_{G_2}(u_k,x))]\} \\
= \; & \sum_{x\in V_2} d_{G_1}(x)[\sum_{u\in V_1} d_{G_1}(u,u_k) - \sum_{u\in V_1} d_{G_2}(u,u_k)] \\
= \; & (l-1)\sum_{x\in V_2} d_{G_1}(x) > 0.
\end{aligned}
$$

For any vertex $x \in V_1 - \{u_1, u_k\}$, we have

$$
\begin{aligned}
& d_{G_1}(x)D_{G_1}(x) - d_{G_2}(x)D_{G_2}(x) \\
= \; & 2D_{G_1}(x) - 2D_{G_2}(x) \\
= \; & 2\sum_{u\in V_2}[d_{G_1}(u_k,x) - d_{G_2}(u_k,x)] + 2[\sum_{u\in V_1} d_{G_1}(u,x) - \sum_{u\in V_1} d_{G_2}(u,x)] \\
= \; & 2(n-k)[d_{G_1}(u_k,x) - d_{G_2}(u_k,x)] + 2[\sum_{u\in V_1} d_{G_1}(u,x) - \sum_{u\in V_1} d_{G_2}(u,x)]
\end{aligned}
$$

and it is easy to see that $d_{G_1}(u_k,x) - d_{G_2}(u_k,x) \geq 0$. Further by Lemma

2.3 (iv), we have

$$A_4 = \sum_{x\in V_1-\{u_1,u_k\}} d_{G_1}(x)D_{G_1}(x) - \sum_{x\in V_1-\{u_1,u_k\}} d_{G_2}(x)D_{G_2}(x)$$
$$= \sum_{x\in V_1-\{u_1,u_k\}} \{2(n-k)[d_{G_1}(u_k,x) - d_{G_2}(u_k,x)] + 2[\sum_{u\in V_1} d_{G_1}(u,x) -$$
$$\sum_{u\in V_1} d_{G_2}(u,x)]\} \geq 0.$$

If $k = 2l+1$, similarly, we have

$$\begin{aligned} A_1 + A_2 &= n-k+(l-1)+(l-1)d_{G_1}(u_k) > 0; \\ A_3 &= (l-1)\sum_{x\in V_2} d_{G_1}(x) > 0; \\ A_4 &= \sum_{x\in V_1-\{u_1,u_k\}} \{2(n-k)[d_{G_1}(u_k,x) - d_{G_2}(u_k,x)] + \\ & \quad 2[\sum_{u\in V_1} d_{G_1}(u,x) - \sum_{u\in V_1} d_{G_2}(u,x)]\} \geq 0. \end{aligned}$$

Hence

$$D'(G_1) - D'(G_2) = A_1 + A_2 + A_3 + A_4 > 0.$$

□

Theorem 2.5. *Let $G \in \mathscr{G}(n,r)$, then $D'(G) \geq 3n^2 + 4nr - 7n - 10r + 4$. The equality holds if and only if $G \cong G^0(n,r)$.*

Proof. By induction on $n+r$. If $r = 0$ or 1, then the theorem holds clearly by lemmas 1.1-1.2. Now, we assume that $r \geq 2$ and $n \geq 5$. If $n = 5$, then the theorem holds clearly by the facts that there is only one graph in $\mathscr{G}(5,2)$. Let $G \in \mathscr{G}(n,r)$, $n \geq 6$ and $r \geq 2$ in what follows.

Case 1. $\delta(G) = 1$.

Let $v \in V(G)$ with $d_G(v) = 1$ and $uv \in E(G)$. Note that $G - v \in \mathscr{G}(n-1,r)$. Then

$$D_{G-v}(u) = \sum_{x\in V(G)-v} d_G(x,u) \geq n-2 \quad (2.8)$$

$$\begin{aligned} D_G(v) &= \sum_{x\in V(G)} d_G(x,v) = \sum_{x\in V(G)-v} [d_G(x,u)+1] \\ &= n-1+\sum_{x\in V(G)-v} d_G(x,u) \geq 2n-3 \quad (2.9) \end{aligned}$$

By lemmas 1.3, 2.2, the inductive assumption, (2.8) and (2.9), we have

$$\begin{aligned} D'(G) &= D'(G-v)+D_{G-v}(u)+D_G(v)+D^*(v) \\ &\geq [3(n-1)^2+4(n-1)r-7(n-1)-10r+4]+(n-2)+ \\ &\quad (2n-3)+(3n+4r-5)=3n^2+4nr-7n-10r+4. \end{aligned}$$

The equality holds if and only if $G \cong G^0(n,r)$.

Case 2. $\delta(G) \geq 2$.

By the definition of cactus, $\delta(G) \geq 2$ and $r \geq 2$, we can choose a cycle $C_k = u_1u_2\ldots u_ku_1$ of G such that $d_G(u_1) = \cdots = d_G(u_{k-1}) = 2$ and $d_G(u_k) \geq 3$.

Subcase 2.1. If $k = 3$, let $G_1 = G - \{u_1, u_2\}$ and $G_2 = C_3$, then $G_1 \in \mathscr{G}(n-2, r-1)$. By Lemma 1.4, we have

$$\begin{aligned} D'(G) &= D'(G_1)+D'(G_2)+2m_1D_{G_2}(u_3)+2m_2D_{G_1}(u_3)+ \\ &\quad (n_1-1)D^*_{G_2}(u_3)+(n_2-1)D^*_{G_1}(u_3). \end{aligned}$$

Note that

$$\begin{aligned} &D'(G_2)=12, \qquad D_{G_2}(u_3)=2, \quad D^*_{G_2}(u_3)=4, \\ &m_1=[(n-1)+r]-3=n+r-4, \qquad m_2=3, \quad n_1=n-2, \quad n_2=3. \end{aligned}$$

Furthermore

$$D_{G_1}(u_3)=\sum_{x\in V(G_1)} d_{G_1}(x,u_3) \geq n_1-1=n-3 \tag{2.10}$$

$$D^*_{G_1}(u_3) \geq \sum_{x\in V(G_1)-u_3} d_{G_1}(x) \geq 2(n-3) \tag{2.11}$$

By the inductive assumption, we have

$$\begin{aligned} &D'(G_1) \\ \geq\ &3(n-2)^2+4(n-2)(r-1)-7(n-2)-10(r-1)+4 \end{aligned} \tag{2.12}$$

Then

$$D'(G) \geq 3n^2+4nr-5n-14r+2 \tag{2.13}$$

The equality holds in (2.10) if and only if $d_{G_1}(u_3) = n_1 - 1$; In (2.11), the first equality holds if and only if $d_{G_1}(x, u_3) = 1$ for any $x \in V(G_1) - u_3$, and since $\delta(G_1) \geq 2$, then the second equality holds if and only if $d_{G_1}(x) = 2$

for any $x \in V(G_1) - u_3$. The equality holds in (2.12) if and only if $G_1 \cong G^0(n-2, r-1)$. Then we have $d_G(u_3) = n-1$ and $2r = n-1$. Note that

$$3n^2 + 4nr - 5n - 14r + 2 - (3n^2 + 4nr - 7n - 10r + 4) = 2(n - 1 - 2r) = 0.$$

Hence $D'(G) \geq 3n^2 + 4nr - 7n - 10r + 4$, the equality holds in (2.13) if and only if $G \cong G^0(n, r)$.

Subcase 2.2. If $k = 5$, let $G_1 = G - \{u_1, u_2, u_3, u_4\}$ and $G_2 = C_5$, then $G_1 \in \mathscr{G}(n-4, r-1)$. By Lemma 1.4, we have

$$\begin{aligned} D'(G) &= D'(G_1) + D'(G_2) + 2m_1 D_{G_2}(u_5) + 2m_2 D_{G_1}(u_5) + \\ &\quad (n_1 - 1)D^*_{G_2}(u_5) + (n_2 - 1)D^*_{G_1}(u_5). \end{aligned}$$

Note that

$$\begin{aligned} &D'(G_2) = 60, \quad D_{G_2}(u_5) = 6, \quad D^*_{G_2}(u_5) = 12, \\ &m_1 = [(n-1) + r] - 5 = n + r - 6, \qquad m_2 = 5, \;\; n_1 = n - 4, \;\; n_2 = 5. \end{aligned}$$

Furthermore

$$D_{G_1}(u_5) = \sum_{x \in V(G_1)} d_{G_1}(x, u_5) \geq n_1 - 1 = n - 5$$

$$D^*_{G_1}(u_5) = \sum_{x \in V(G_1)} d_{G_1}(x) d_{G_1}(x, u_5) \geq \sum_{x \in V(G_1) - u_5} d_{G_1}(x) \geq 2(n-5)$$

By the inductive assumption, we have

$$D'(G_1) \;\; \geq \;\; 3(n-4)^2 + 4(n-4)(r-1) - 7(n-4) - 10(r-1) + 4$$

Then

$$D'(G) \;\; \geq \;\; 3n^2 + 4nr + 7n - 14r - 56.$$

By Lemma 1.5 we have $n \geq 2r + 1$, then

$$\begin{aligned} &3n^2 + 4nr + 7n - 14r - 56 - (3n^2 + 4nr - 7n - 10r + 4) \\ = \;\; &14n - 4r - 60 \geq 24r - 46 > 0. \end{aligned}$$

Hence $D'(G) > 3n^2 + 4nr - 7n - 10r + 4$.

Subcase 2.3. If $k = 4$ or $k \geq 6$, let $G' = G - u_1u_2 + u_2u_k$, obviously, $G' \in \mathscr{G}(n, r)$. By Lemma 2.4, we have $D'(G) > D'(G')$. Note that $\delta(G') = 1$, by case 1, we have

$$D'(G') \geq 3n^2 + 4nr - 7n - 10r + 4.$$

This completes the proof. □

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