On the spectral radius of quasi-unicycle graphs

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Abstract A connected graph $G = (V, E)$ is called a *quasi-unicycle graph*, if there exists $v_0 \in V$ such that $G - v_0$ is a unicycle graph. Denote by $\mathscr{C}(n, d_0)$ the set of quasi-unicycle graphs of order n with the vertex v_0 of degree d_0 such that $G - v_0$ is a unicycle graph. In this paper we determine the maximum spectral radii of quasi-unicycle graphs in $\mathscr{C}(n, d_0)$.

Keywords: Spectral radius; Quasi-unicycle graph

1 Introduction

Let $G = (V, E)$ be a simple undirected graph with n vertices. A connected graph G is called a *unicycle graph* if G has exactly one cycle i.e. $|V| = |E| = n$. For $v \in V$, we use $N(v)$ to denote the neighbors of v and set $d(v) = |N(v)|$. For a subgraph H of G, let $N_H(v) = N(v) \cap V(H)$ and $d_H(v) = |N_H(v)|$ for $v \in V(G)$. A pendant vertex of graph is a vertex of degree 1. We will use $G-x$ or $G-xy$ to denote the graph obtained from G by deleting the vertex x or edge xy. Similarly, $G + xy$ is a graph obtained from G by adding an edge $xy \notin E$ where x, $y \in V$. A connected graph G is called a *quasi-unicycle graph* if there exists a vertex $v_0 \in V$ such that $G - v_0$ is a unicycle graph. Denote by $\mathscr{C}(n, d_0)$ the set of quasi-unicycle graphs of order n with the vertex v_0 of degree d_0 such that $G - v_0$ is a unicycle graph. Clearly $d_0 \geq 1$.

Let $A(G)$ be the adjacency matrix of G. The spectral radius, $\rho(G)$, of G is the largest eigenvalues of $A(G)$. When G is connected, $A(G)$ is irreducible and by Perron-Frobenius Theorem, the spectral radius is simple and has a unique positive eigenvector. We will refer to such an eigenvector as Perron

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vector of G. Note that the spectral radius is increasing as we add an edge to G.

The investigation on the spectral radius of graphs is an important topic of theory of graph spectra. The problem concerning graphs with maximal or minimal spectral radius of a given class of graphs has been studied extensively. For example, A. Berman and X.D. Zhang in [2] study the spectral radius of graphs with n vertices and k cut vertices and describe the graph that has the maximal spectral radius in that class. In addition, B.F. Wu etc. (See [3]) determine the tree of order n with k pendant vertices which has maximal spectral radius. Recently, H. Liu and M. Lu (See [4]) determine the quasi-tree with maximal and the second maximal spectral radii of all quasi-tree graphs.

In this short paper we will determine the maximal spectral radii of all quasi-unicycle graphs in $\mathscr{C}(n, d_0)$.

2 Lemmas

Lemma 2.1. *[1] Let $\phi(G;x)$ be the characteristic polynomial of graph G.*

(1) *Let u be a vertices of G and $C(u)$ be the set of all cycles containing u. Then*

$$\phi(G;x) = x\phi(G-u;x) - \sum_{v\in N(u)} \phi(G-v-u;x) - 2\sum_{Z\in C(u)} \phi(G-V(Z);x).$$

(2) *Let uv be an edge of G and $C(uv)$ be the set of all cycles containing uv. Then*

$$\phi(G;x) = \phi(G-uv;x) - \phi(G-v-u;x) - 2\sum_{Z\in C(uv)} \phi(G-V(Z);x).$$

Lemma 2.2. *[3] Let G be a connected graph and $\rho(G)$ be the spectral radius of $A(G)$. Let u and v be two vertices of G. Suppose $v_1, v_2, \ldots, v_s \in N(v)\backslash N(u)(1 \le s \le d_G(v))$ and $x = (x_1, x_2, \ldots, x_n)^T$ be the Perron vector of $A(G)$, where x_i corresponds to the vertex $v_i (1 \le i \le n)$. Let G^* be the graph obtained from G by deleting vv_i and adding uv_i, $i = 1, 2, \ldots, s$. If $x_u \ge x_v$, then $\rho(G) < \rho(G^*)$.*

Let S_{n-1}^+ be a graph of order $n-1$ obtained from a star $K_{1,n-2}$ by adding an edge, and by the center of S_{n-1}^+ we mean the center of $K_{1,n-2}$.

We now label the vertices of S_{n-1}^+ by $v_1, v_2, v_3, v_4, \ldots, v_{n-1}$ with degree sequence $(d(v_1), d(v_2), d(v_3), d(v_4), \ldots, d(v_{n-1})) = (n-1, 2, 2, 1, \ldots, 1)$. Denote by T_{n,d_0} the graph obtained from S_{n-1}^+ by joining d_0 edges from a new

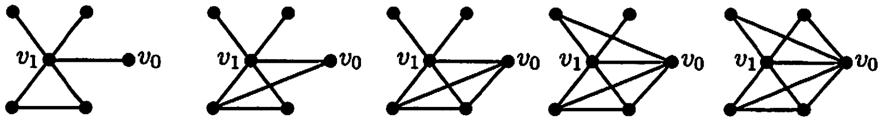

Figure 1.

vertex v_0 to some vertices of S^+_{n-1} such that $N_{T_{n,d_0}}(v_0) = \{v_1, v_2, ..., v_{d_0}\}$. For example, T_{6,d_0}, $d_0 = 1, 2, \ldots, 5$ are shown in Figure 1. By definition, S^+_{n-1} is a quasi-unicycle graph in $\mathscr{C}(n, d_0)$.

Lemma 2.3. *The characteristic polynomial of T_{n,d_0} is as following:*

1. $\phi(T_{n,1}; x) = x^{n-4}[x^4 - nx^2 - 2x + (n-3)]$;
2. $\phi(T_{n,2}; x) = x^{n-4}[x^4 - (n+1)x^2 - 4x + 2(n-4)]$;
3. $\phi(T_{n,3}; x) = x^{n-5}[x^5 - (n+2)x^3 - 8x^2 + 3(n-5)x + 2(n-4)]$;
4. $\phi(T_{n,d_0}; x) = x^{n-6}[x^6 - (n+d_0-1)x^4 - 2(d_0+1)x^3 + (d_0 n + d_0 - d_0^2 - 9)x^2 + 2(n-4)x + (d_0^2 + 3n - 2d_0 - nd_0 - 3)]$ *for* $4 \le d_0 \le n-1$.

Proof. Let v_2, v_3 be the two vertices of degree 2 in S^+_{n-1}. Note that $S^+_{n-1} - v_2v_3 \cong K_{1,n-2}$, hence by Lemma 2.1(2) we have

$$\begin{aligned}\phi(S^+_{n-1}; x) &= \phi(K_{1,n-2}; x) - \phi(K_{1,n-4}; x) - 2x^{n-4}\\ &= x^{n-3}(x^2 - (n-2)) - x^{n-5}(x^2 - (n-4)) - 2x^{n-4}\\ &= x^{n-5}[x^4 - (n-1)x^2 - 2x + (n-4)];\end{aligned}$$

By Lemma 2.1 (1) and simple calculation we have

$$\begin{aligned}\phi(T_{n,1}; x) &= x\phi(S^+_{n-1}; x) - x^{n-4}(x^2 - 1)\\ &= x^{n-4}[x^4 - nx^2 - 2x + (n-3)];\\ \phi(T_{n,2}; x) &= x\phi(S^+_{n-1}; x) - x^{n-4}(x^2 - 1) - \phi(K_{1,n-3}; x) - 2x^{n-3} - 2x^{n-4}\\ &= x^{n-4}[x^4 - (n+1)x^2 - 4x + 2(n-4)];\\ \phi(T_{n,3}; x) &= x\phi(S^+_{n-1}; x) - x^{n-4}(x^2 - 1) - 2\phi(K_{1,n-3}; x) - 4x^{n-3}\\ &\quad - 2\phi(K_{1,n-4}; x) - 6x^{n-4}\\ &= x^{n-5}[x^5 - (n+2)x^3 - 8x^2 + 3(n-5)x + 2(n-4)];\end{aligned}$$

$$
\begin{aligned}
\phi(T_{n,d_0};x) &= x\phi(S_{n-1}^{+};x) - x^{n-4}(x^2-1) - (d_0-3)\phi(S_{n-2}^{+};x) \\
&\quad - 2\phi(K_{1,n-3};x) - 2(d_0-3)x^{n-5}(x^2-1) - 4x^{n-3} \\
&\quad - 2\phi(K_{1,n-4};x) - 4(d_0-3)x^{n-4} - 2\binom{d_0-3}{2}x^{n-6}(x^2-1) \\
&\quad - 6x^{n-4} - 4(d_0-3)x^{n-5} \\
&= x^{n-6}[x^6 - (n+d_0-1)x^4 - 2(d_0+1)x^3 \\
&\quad + (d_0n + d_0 - d_0^2 - 9)x^2 + 2(n-4)x \\
&\quad + (d_0^2 + 3n - 2d_0 - nd_0 - 3)] \text{ for } 4 \le d_0 \le n-1.
\end{aligned}
$$

□

Denote by $\phi_1(x)$, $\phi_2(x)$, $\phi_3(x)$ and $\phi_{d_0}(x)$ the polynomials appeared in $\phi(T_{n,1};x)$, $\phi(T_{n,2};x)$, $\phi(T_{n,3};x)$ and $\phi(T_{n,d_0};x)$ above.

$$
\begin{aligned}
\phi_1(x) &= x^4 - nx^2 - 2x + (n-3); \\
\phi_2(x) &= x^4 - (n+1)x^2 - 4x + 2(n-4); \\
\phi_3(x) &= x^5 - (n+2)x^3 - 8x^2 + 3(n-5)x + 2(n-4); \\
\phi_{d_0}(x) &= x^6 - (n+d_0-1)x^4 - 2(d_0+1)x^3 + (d_0n + d_0 - d_0^2 - 9)x^2 \\
&\qquad + 2(n-4)x + (d_0^2 + 3n - 2d_0 - nd_0 - 3).
\end{aligned}
$$

By Lemma 2.3 we can determine the spectral radii of T_{n,d_0}.

Corollary 2.4. *The spectral radius of T_{n,d_0} is the largest root of $\phi_1(x)$, $\phi_2(x)$, $\phi_3(x)$ and $\phi_{d_0}(x)$ for $d_0 = 1,2,3$ and $4 \le d_0 \le n-1$, respectively.*

3 Maximal quasi-unicycle graphs

Theorem 3.1. *Let $G \in \mathscr{C}(n, d_0)$, $n \ge 5$. Then $\rho(G) \le \rho(T_{n,d_0})$ and equality holds if and only if $G \cong T_{n,d_0}$.*

Proof. We have to prove that if $G \in \mathscr{C}(n, d_0)$, then $\rho(G) \le \rho(T_{n,d_0})$ and equality holds if and only if $G \cong T_{n,d_0}$.

Choose $G \in \mathscr{C}(n, d_0)$ such that $\rho(G)$ is as large as possible. Let $V(G) = \{v_0, v_1, \ldots, v_{n-1}\}$, and $x = \{x_0, x_1, \ldots, x_{n-1}\}^T$ be the Perron vector of $A(G)$, where x_i corresponds to the vertex v_i $(0 \le i \le n-1)$. Suppose that $G - v_0$ is a unicycle graph, denote $G' = G - v_0$. Suppose $v_1 \in V(G')$ such that $d_{G'}(v_1)$ is as large as possible. We first prove the following claims.

Claim 1. v_1 is adjacent to each vertex of $V(G') - v_1$ in G.

Suppose $v_1v_i \notin E(G)$ for some vertex $v_i \in V(G') - v_1$. We have $n \geq 5$ since G' contains at least a 3-cycle. Since G' is connected, there exists a shortest path connecting v_1 and v_i in G', say $P_{v_1v_i} = v_1v_2v_3 \cdots v_i(i \geq 3$ and possibly $v_3 = v_i)$, then $v_1v_j \notin E(G)$ for $3 \leq j \leq i$. On the other hand, $d_{G'}(v_1) \geq d_{G'}(v_2)$ by the choice of v_1, hence there is at least a vertex $v_t \in V(G') - \{v_1, v_2, v_3\}$ such that $v_2v_t \notin E(G)$ and $v_1v_t \in E(G)$. We now set $G^* = G - v_2v_3 + v_1v_3$ if $x_1 \geq x_2$, and $G^* = G - v_1v_t + v_2v_t$ if $x_1 < x_2$. Then, by Lemma 2.2, $\rho(G^*) > \rho(G)$ in either case, but $G^* \in \mathscr{C}(n, d_0)$, a contradiction.

Therefore, $v_1v_i \in E(G)$ for all $v_i \in V(G') - \{v_1\}$, which implies that $G' \cong S_{n-1}^+$, and v_1 is the center of S_{n-1}^+. Let v_2 and v_3 be the vertices with degree two, v_i $(4 \leq i \leq n-1)$ be the pendant vertices in G'.

Claim 2. $N_G(v_0) = \{v_1, v_2, v_3, \ldots, v_{d_0}\}$.

Suppose $v_0v_1 \notin E(G)$. Since $d_0 \geq 1$, without loss of generality, assume that $v_0v_t \in E(G)$. By the choice of v_1, $d_{G'}(v_1) \geq d_{G'}(v_t)$. Since $G' \cong S_{n-1}^+$ $(n \geq 5)$, there is a vertex $v_j \in V(G') - \{v_1, v_t\}$ such that $v_1v_j \in E(G)$ and $v_tv_j \notin E(G)$. If $x_1 \geq x_t$, let $G^* = G - v_0v_t + v_1v_t$; if $x_1 < x_t$, then let $G^* = G - v_1v_j + v_tv_j$. Then, by Lemma 2.2, $\rho(G^*) > \rho(G)$ in either case, but $G^* \in \mathscr{C}(n, d_0)$, a contradiction. Therefore $v_0v_1 \in E(G)$.

Next, suppose $d_0 \geq 2$ and v_0 is adjacent to some pendant vertex v_i $(4 \leq i \leq n-1)$ but not adjacent to v_2 or v_3, say $v_0v_3 \notin E(G)$. We now compare with x_3 and x_i. Let $G^* = G - v_0v_i + v_0v_3$ if $x_3 \geq x_i$, and $G^* = G - v_2v_3 + v_3v_i$ if $x_3 < x_i$. Then, by the same argument, we have $\rho(G^*) > \rho(G)$, a contradiction. Therefore, if v_0 is not adjacent to v_2 and v_3, it cannot be adjacent to any v_i $(4 \leq i \leq n-1)$.

Combining Claim 1 and Claim 2, we have $G \cong T_{n,d_0}$. □

By Corollary 2.4 we can determine the spectral radii of T_{n,d_0} theoretically but it is not easy to work it out. By using Matlab we give the spectral radii of some T_{n,d_0} at the last of this short paper. (See Table 1.)

Note that if we add an edge to a connected graph G, then its spectral radius will increase. So we have the following theorem.

Theorem 3.2. *Let G be a quasi-unicylce graph of order n. Then*

$$\rho(G) \leq \rho(T_{n,n-1})$$

and equality holds if and only if $G \cong T_{n,n-1}$.

Table.1

	$T_{4,1}$	$T_{4,2}$	$T_{4,3}$	$T_{5,1}$	$T_{5,2}$	$T_{5,3}$	$T_{5,4}$	$T_{6,1}$
$\rho(T_{n,d_0})$	2.1701	2.5616	3.0000	2.3429	2.6855	3.0861	3.3234	2.5141
	$T_{6,2}$	$T_{6,3}$	$T_{6,4}$	$T_{6,5}$	$T_{7,1}$	$T_{7,2}$	$T_{7,3}$	$T_{7,4}$
$\rho(T_{n,d_0})$	2.8136	3.1774	3.4037	3.6262	2.6813	2.9439	3.2731	3.4877
	$T_{7,5}$	$T_{7,6}$	$T_{8,1}$	$T_{8,2}$	$T_{8,3}$	$T_{8,4}$	$T_{8,5}$	$T_{8,6}$
$\rho(T_{n,d_0})$	3.7009	3.9095	2.8434	3.0749	3.3723	3.5749	3.7785	3.9793
	$T_{8,7}$	$T_{9,1}$	$T_{9,2}$	$T_{9,3}$	$T_{9,4}$	$T_{9,5}$	$T_{9,6}$	$T_{9,7}$
$\rho(T_{n,d_0})$	4.1755	3.0000	3.2054	3.4742	3.6648	3.8585	4.0514	4.2411

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