A Result on (a, b, k)-Critical Graphs *

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Abstract

Let G be a graph, and let a, b, k be integers with $0 \le a \le b$, $k \ge 0$. An $[a, b]$-factor of graph G is defined as a spanning subgraph F of G such that $a \le d_F(x) \le b$ for each $x \in V(F)$. Then a graph G is called an (a, b, k)-critical graph if after deleting any k vertices of G the remaining graph of G has an $[a, b]$-factor. In this paper, it is proved that, if a, b, k be integers with $1 \le a < b$, $k \ge 0$ and $b \ge a(k+1)$ and G is a graph with $\delta(G) \ge a+k$ and binding number $b(G) \ge a - 1 + \frac{a(k+1)}{b}$, then G is an (a, b, k)-critical graph. Furthermore, it is showed that the result in this paper is best possible in some sense.

Keywords: graph, $[a, b]$-factor, binding number, (a, b, k)-critical graph

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1 Introduction

All graphs considered in this paper will be finite and undirected simple graphs. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For $x \in V(G)$, the neighborhood $N_G(x)$ of x is the set vertices of G adjacent to x, and the degree $d_G(x)$ of x is $|N_G(x)|$. The minimum vertex degree

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of $V(G)$ is denoted by $\delta(G)$. For $S \subseteq V(G)$, $N_G(S) = \cup_{x \in S} N_G(x)$ and we denote by $G[S]$ the subgraph of G induced by S, by $G - S$ the subgraph obtained from G by deleting vertices in S together with the edges incident to vertices in S. A vertex set $S \subseteq V(G)$ is called independent if $G[S]$ has no edges. Let S and T be disjoint subsets of $V(G)$. We denote by $e_G(S, T)$ the number of edges joining S and T. We write $i(G)$ for the number of isolated vertices in G. The binding number $b(G)$ of G is the minimum value of $\frac{|N_G(X)|}{|X|}$ taken over all non-empty subsets X of $V(G)$ such that $N_G(X) \neq V(G)$. Let a and b be integers with $0 \leq a \leq b$. An $[a, b]$-factor of graph G is defined as a spanning subgraph F of G such that $a \leq d_F(x) \leq b$ for every vertex x of G (Where of course d_F denotes the degree in F). And if $a = b = k$, then an $[a, b]$-factor is called an k-factor. A graph G is called an (a, b, k)-critical graph if after deleting any k vertices of G the remaining graph of G has an $[a, b]$-factor. If G is an (a, b, k)-critical graph, then we also say that G is (a, b, k)-critical. If $a = b = n$, then an (a, b, k)-critical graph is simply called a (n, k)-critical graph. In particular, a $(1, k)$-critical graph is simply called a k-critical graph. The other terminologies and notations not given in this paper can be found in [1].

Favaron [2] studied the properties of k-critical graphs. Liu and Yu [3] studied the characterization of (n, k)-critical graphs. Enomoto et al [4] gave some sufficient conditions of (n, k)-critical graphs. The characterization of (a, b, k)-critical graph with $a < b$ was given by Liu and Wang [5]. Zhou [6] gave two sufficient conditions for graphs to be (a, b, k)-critical. Li [7] gave some sufficient conditions for graphs to be (a, b, k)-critical graphs. Li and Matsuda [8] gave a necessary and sufficient condition for a graph to be (g, f, k)-critical graph, and studied the properties of (g, f, k)-critical graphs.

Katerinis and Woodall [9] proved the following results for the existence of k-factor.

Theorem 1. Let $k \geq 2$ be an integer and let G be a graph with $p \geq 4k - 6$ vertices and binding number $b(G)$ such that kp is even and $b(G) > \frac{(2k-1)(p-1)}{k(p-2)+3}$. Then G has a k-factor.

Theorem 2. If $p > k \geq 2$ and kp is even and $b(G) > \frac{p-1}{2(\sqrt{kp}-k)}$, then G has a k-factor.

Zhou [10] obtained the following result for a graph to be (a, b, k)-critical.

Theorem 3. Let G be a graph, and let $b > 2$ and $k \geq 0$ be integers, $\delta(G) \geq k + 2$. If $b(G) \geq 1 + \frac{bk+4}{3b}$, then G is an $(2, b, k)$-critical graph.

In this paper, we prove the following result, which is an extension of Theorems 1 and 2 and 3. We extend Theorems 1 and 2 and 3 to (a, b, k)-critical graphs.

Theorem 4. Let a, b, k be integers with $1 \le a < b$, $k \ge 0$, and $b \ge a(k+1)$, and let G be a graph with $|V(G)| \ge a+k+1$. If $\delta(G) \ge a+k$ and binding number $b(G) \ge a - 1 + \frac{a(k+1)}{b}$, then G is an (a, b, k)-critical graph.

In Theorem 4, if $k = 0$, then we get the following Corollary.

Corollary 1. Let a, b be integers with $1 \le a < b$, and let G be a graph with $|V(G)| \ge a + 1$. If $\delta(G) \ge a$ and binding number $b(G) \ge a - 1 + \frac{a}{b}$, then G has an $[a, b]$-factor.

2 The Proof of Theorem 4

The proof of Theorem 4 relies heavily on the following lemmas.

Lemma 2.1.[5] Let a, b, k be nonnegative integers with $1 \le a < b$, and let G be a graph with $|V(G)| \ge a+k+1$. Then G is (a, b, k)-critical if and only if for any $S \subseteq V(G)$ and $|S| \ge k$

$$\sum_{j=0}^{a-1}(a-j)p_j(G-S) \le b|S| - bk,$$

or

$$a|T| - d_{G-S}(T) \le b|S| - bk,$$

where $p_j(G - S) = |\{x : d_{G-S}(x) = j\}|$, $T = \{x : x \in V(G) \setminus S, d_{G-S}(x) \le a - 1\}$.

Lemma 2.2.[11] Let H be a graph and $a \ge 1$ be an integer, and let $T_1, \cdots, T_{a-1}$ be a partition of $V(H)$ such that $d_H(x) \le j$ for $\forall x \in T_j$ (T_j may be empty sets), $j = 1, \cdots, a-1$. Then there exist an independent set I and a covered set C such that

$$\sum_{j=1}^{a-1}(a-j)c_j \le (a-1)\sum_{j=1}^{a-1}(a-j)i_j,$$

where $i_j = |I \cap T_j|$, $c_j = |C \cap T_j|$, $j = 1, \cdots, a-1$.

Proof of Theorem 4. Suppose G satisfies the assumption of theorem, but it is not (a, b, k)-critical. Then, by Lemma 2.1, there exists a subset S of $V(G)$ with $|S| \ge k$ such that

$$a|T| - d_{G-S}(T) > b|S| - bk, \tag{1}$$

where $T = \{x : x \in V(G) \setminus S,\ d_{G-S}(x) \le a - 1\}$.

Here, we prove the following claim.

Claim 1. $|S| \geq k+1$.

Proof. If $|S| = k$, then $d_{G-S}(x) \geq d_G(x) - |S| \geq \delta(G) - k \geq a + k - k = a$. Thus

$$0 \geq a|T| - d_{G-S}(T) > b|S| - bk = bk - bk = 0.$$

This is a contradiction. This completes the proof of Claim 1.

Let $T_j = \{x : x \in T,\ d_{G-S}(x) = j\}$, and $|T_j| = t_j$, $j = 0, 1, \cdots, a-1$. Set $H = G[T_1 \cup T_2 \cup \cdots \cup T_{a-1}]$, we have $d_H(x) \leq j$ for $\forall x \in T_j$. According to Lemma 2.2, there exist an independent set I and a covered set C of H such that

$$\sum_{j=1}^{a-1}(a-j)c_j \leq (a-1)\sum_{j=1}^{a-1}(a-j)i_j, \tag{2}$$

where $i_j = |I \cap T_j|$, $c_j = |C \cap T_j|$, $j = 1, \cdots, a-1$.

Let's assume that I be a maximal independent set of H. Put $W = G - S - T$, $U = S \cup C \cup (N_G(I) \cap V(W))$, then

$$|U| \leq |S| + \sum_{j=1}^{a-1} j i_j \tag{3}$$

and

$$i(G-U) \geq t_0 + \sum_{j=1}^{a-1} i_j, \tag{4}$$

where t_0 is the number of isolated vertices in T.

Here, we prove the following claim.

Claim 2. $|U| \geq i(G-U)b(G)$.

Proof. The proof splits into two cases.

Case 1. $i(G-U) = 0$.

Clearly, $|U| \geq i(G-U)b(G)$.

Case 2. $i(G-U) \geq 1$.

We write X for the set of isolated vertices of $G-U$, it is easily seen that

$$|X| = i(G-U) \tag{5}$$

and

$$|N_G(X)| \leq |U|. \tag{6}$$

Since $|X| = i(G-U) \geq 1$, thus we have

$$U \subset V(G). \tag{7}$$

By (6) and (7), we get that

$$N_G(X) \neq V(G).$$

According to the definition of $b(G)$, we obtain

$$\frac{|N_G(X)|}{|X|} \geq b(G). \tag{8}$$

In view of (5), (6) and (8), we have

$$|U| \geq |N_G(X)| \geq |X|b(G) = i(G-U)b(G).$$

This completes the proof of Claim 2.

According to (3), (4) and Claim 2, we have

$$|S| + \sum_{j=1}^{a-1} j i_j \geq b(G)(t_0 + \sum_{j=1}^{a-1} i_j). \tag{9}$$

By (1) and Claim 1, we get that

$$\begin{aligned} & at_0 + \sum_{j=1}^{a-1}(a-j)i_j + \sum_{j=1}^{a-1}(a-j)c_j > b|S| - bk \\ = \ & \frac{b}{k+1}|S| + (b - \frac{b}{k+1})|S| - bk \\ \geq \ & \frac{b}{k+1}|S| + (b - \frac{b}{k+1})(k+1) - bk \\ = \ & \frac{b}{k+1}|S|. \end{aligned}$$

In view of (9), we get

$$\begin{aligned} & at_0 + \sum_{j=1}^{a-1}(a-j)i_j + \sum_{j=1}^{a-1}(a-j)c_j > \frac{b}{k+1}|S| \\ \geq \ & \frac{b}{k+1}(b(G)(t_0 + \sum_{j=1}^{a-1} i_j) - \sum_{j=1}^{a-1} j i_j) \\ = \ & \frac{b}{k+1}b(G)t_0 + \sum_{j=1}^{a-1}(\frac{b}{k+1}b(G) - \frac{b}{k+1}j)i_j, \end{aligned}$$

that is

$$at_0+\sum_{j=1}^{a-1}(a-j)i_j+\sum_{j=1}^{a-1}(a-j)c_j > \frac{b}{k+1}b(G)t_0+\sum_{j=1}^{a-1}(\frac{b}{k+1}b(G)-\frac{b}{k+1}j)i_j. \quad (10)$$

If $a=1$, then $T=\{x : x \in V(G)\setminus S,\ d_{G-S}(x)=0\}$. Thus, we get

$$i(G-S)=|T|=a|T|-d_{G-S}(T)>b|S|-bk\geq \frac{b}{k+1}|S|.$$

We write Y for the set of isolated vertices of $G-S$. By Claim 1, it is easily seen that

$$|Y|=i(G-S)>\frac{b}{k+1}|S|\geq b>1$$

and

$$|N_G(Y)|\leq |S|$$

and

$$N_G(Y)\neq V(G).$$

Thus, by the definition of $b(G)$, we have

$$b(G)\leq \frac{|N_G(Y)|}{|Y|}\leq \frac{|S|}{i(G-S)}<\frac{|S|}{\frac{b}{k+1}|S|}=\frac{k+1}{b}.$$

This contradicts the assumption that $b(G)\geq \frac{k+1}{b}$. In the following, we consider $a\geq 2$. Obviously, $\frac{b}{k+1}b(G)\geq \frac{b}{k+1}(a-1+\frac{a}{b}(k+1))>a$. By (10) and (2), we obtain

$$a\sum_{j=1}^{a-1}(a-j)i_j>\sum_{j=1}^{a-1}(\frac{b}{k+1}b(G)-\frac{b}{k+1}j)i_j, \quad (11)$$

i.e.

$$\sum_{j=1}^{a-1}(a(a-j)-\frac{b}{k+1}b(G)+\frac{b}{k+1}j)i_j>0.$$

Let $\Phi(j)=a(a-j)-\frac{b}{k+1}b(G)+\frac{b}{k+1}j$. By $\frac{b}{k+1}\geq a$, we have $\Phi'(j)\geq 0$. Moreover, $\Phi(a-1)=a-\frac{b}{k+1}b(G)+\frac{b}{k+1}(a-1)\leq a-\frac{b}{k+1}(a-1+\frac{a}{b}(k+1))+\frac{b}{k+1}(a-1)=0$. Thus, we have

$$\Phi(j)\leq 0,\qquad j=1,\cdots,a-1.$$

Thus, we get that

$$a\sum_{j=1}^{a-1}(a-j)i_j \le \sum_{j=1}^{a-1}(\frac{b}{k+1}b(G)-\frac{b}{k+1}j)i_j.$$

Which contradicts (11).

From the argument above, we deduce the contradiction. Hence, G is (a,b,k)-critical.
Completing the proof of Theorem 4.

Remark. Let us show that the condition $b(G) \ge a-1+\frac{a}{b}(k+1)$ in Theorem 4 can not be replaced by $b(G) \ge a-1+\frac{a}{b}(k+1)-\varepsilon$, where ε is any positive real number. In Theorem 4, if $a=1$, $k=0$, then $b(G) \ge \frac{1}{b}$. Let $H = K_n \bigvee (nb+1)K_1$. Let $X = V((nb+1)K_1)$, then $|N_H(X)| = n$. By the definition of $b(H)$, $b(H) = \frac{|N_H(X)|}{|X|} = \frac{n}{nb+1} < \frac{1}{b}$, and $b(H) \to \frac{1}{b}$ when $n \to \infty$. Let $S = V(K_n) \subseteq V(H)$, $T = V((nb+1)K_1) \subseteq V(H)$, then $|S| = n$, $|T| = nb+1$. Thus, we get

$$|T| - d_{H-S}(T) = nb+1 = b|S|+1 > b|S| = b|S| - bk.$$

By Lemma 2.1, there are not any $[a,b]$-factors in H. In the above sense, the result of Theorem 4 is best possible.

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