Sums of the AB-Generalized Fibonacci Sequence

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Abstract

In this short paper, we introduce the second order linear recurrence relation of the AB-generalized Fibonacci sequence and give the explicit formulas for the sums of the positively and negatively subscripted terms of the AB-generalized Fibonacci sequence by matrix methods. This sum generalizes the one obtained earlier by Kiliç in [2].

Keywords: Fibonacci sequence, generalized Fibonacci sequence, AB-generalized Fibonacci sequence

1 Introduction

For $n > 1$, the Fibonacci sequence is defined by

$$F_{n+1} = F_n + F_{n-1},$$

where $F_0 = 0$ and $F_1 = 1$. The sum of the Fibonacci numbers subscripted from 1 to n can be expressed by the formula

$$\sum_{i=1}^{n} F_i = F_{n+2} - F_1.$$

Recently, Kiliç [2] introduced the generalized Fibonacci sequence and gave the explicit formula for the sum of the terms of this sequence using matrix methods. To obtain this sum, he constructed essential matrices and used the concept of positively and negatively subscripted terms of the generalized

Fibonacci sequence. Kiliç's definition provided a motivation to the construction of the so called AB-generalized Fibonacci sequence.
Let $n > 0$ and let A and B be nonzero integers with $A^2 + 4B \neq 0$ and $A + B \neq 1$. The **AB-generalized Fibonacci sequence** $\{v_n\}$ has the recurrence relation

$$v_{n+1} = Av_n + Bv_{n-1},$$

where $v_0 = 0$ and $v_1 = 1$.
The **Binet's formula** of the n^{th} term of the AB-generalized Fibonacci sequence $\{v_n\}$ has the form

$$v_n = \frac{\alpha^n - \beta^n}{\alpha - \beta},$$

where α and β are the roots of the equation $x^2 - Ax - B = 0$. Using the recurrence relation of the sequence $\{v_n\}$, the n^{th} negatively subscripted term of the sequence $\{v_n\}$ has the form

$$v_{-n} = \frac{\alpha^{-n} - \beta^{-n}}{\alpha - \beta}.$$

Following the methods employed by Kiliç, we show in this paper that

$$\sum_{i=0}^{n} v_{-i} = \frac{1 - Bv_{-(n+1)} - v_{-n}}{A + B - 1} \quad \text{and}$$

$$\sum_{i=1}^{n} v_i = \frac{v_{n+1} + Bv_n - 1}{A + B - 1}.$$

Kiliç proved the results for the special case $B = 1$.

2 Definitions

The following concept is found in [2].

Definition 2.1 [2] Let A be a nonzero integer. The **generalized Fibonacci sequence** $\{u_n\}$ for $n > 1$ is defined by the recurrence relation

$$u_{n+1} = Au_n + u_{n-1},$$

where $u_0 = 0$ and $u_1 = 1$, the sum of the **positively subscripted terms** of the sequence $\{u_n\}$, denoted by S_n^+, is given by

$$S_n^+ = \sum_{i=0}^{n} u_i \,,$$

and the sum of the **negatively subscripted terms** of the sequence $\{u_n\}$, denoted by S_n^-, is given by

$$S_n^- = \sum_{i=0}^{n} u_{-i} \ .$$

Definition 2.2 Let $\{v_n\}$ be an AB-generalized Fibonacci sequence. The sum of the **positively subscripted terms** of the sequence $\{v_n\}$, denoted by S_{+n}, is given by

$$S_{+n} = \sum_{i=0}^{n} v_i,$$

and the sum of the **negatively subscripted terms** of the sequence $\{v_n\}$, denoted by S_{-n}, is given by

$$S_{-n} = \sum_{i=0}^{n} v_{-i}.$$

3 Sum of the Positively Subscripted Terms

To obtain the explicit formula for S_n^+, Kiliç [2] first constructed two key matrices that generate S_n^+. We shall use a similar technique, with one matrix, to generate S_{+n}. Consider the 3×3 matrix M defined by

$$M = \begin{bmatrix} 1 & 0 & 0 \\ 1 & A & 1 \\ 0 & B & 0 \end{bmatrix},$$

where A and B are nonzero integers with $A^2 + 4B \neq 0$ and $A + B \neq 1$.

Lemma 3.1 *For every $n > 0$,*

$$M^n = \begin{bmatrix} 1 & 0 & 0 \\ S_{+n} & v_{n+1} & v_n \\ BS_{+(n-1)} & Bv_n & Bv_{n-1} \end{bmatrix}.$$

Proof: For $n = 1$, we see

$$M^1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & A & 1 \\ 0 & B & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ S_{+1} & v_2 & v_1 \\ BS_{+0} & Bv_1 & Bv_0 \end{bmatrix}.$$

For induction, assume that

$$M^n = \begin{bmatrix} 1 & 0 & 0 \\ S_{+n} & v_{n+1} & v_n \\ BS_{+(n-1)} & Bv_n & Bv_{n-1} \end{bmatrix}.$$

Then

$$M^{n+1} = M^n M = \begin{bmatrix} 1 & 0 & 0 \\ S_{+n} & v_{n+1} & v_n \\ BS_{+(n-1)} & Bv_n & Bv_{n-1} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & A & 1 \\ 0 & B & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ S_{+n} + v_{n+1} & Av_{n+1} + Bv_n & v_{n+1} \\ B(S_{+(n-1)} + v_n) & B(Av_n + Bv_{n-1}) & Bv_n \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ S_{+(n+1)} & v_{n+2} & v_{n+1} \\ BS_{+n} & Bv_{n+1} & Bv_n \end{bmatrix}.$$

Thus, by the Principle of Mathematical Induction, the assertion must be true. □

We shall derive now the explicit formula for S_{+n}.

Theorem 3.2 *For every* $n > 0$,

$$S_{+n} = \frac{v_{n+1} + Bv_n - 1}{A + B - 1}.$$

Proof: Let $n > 0$ and let $K_M(\lambda)$ be the characteristic polynomial of the matrix M.

Then

$$K_M(\lambda) = det \begin{bmatrix} \lambda - 1 & 0 & 0 \\ -1 & \lambda - A & -1 \\ 0 & -B & \lambda \end{bmatrix} = (\lambda - 1)(\lambda^2 - A\lambda - B).$$

Hence, the eigenvalues of the matrix M are

$$\lambda_1 = \frac{A + \sqrt{A^2 + 4B}}{2}, \lambda_2 = \frac{A - \sqrt{A^2 + 4B}}{2} \text{ and } \lambda_3 = 1.$$

Since $A + B \neq 1$ and $A^2 + 4B \neq 0$, it follows that the eigenvalues of the matrix M are distinct. Now, let H be the matrix defined by

$$H = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{1-A-B} & \lambda_1 & \lambda_2 \\ \frac{B}{1-A-B} & B & B \end{bmatrix}.$$

H is well-defined since $A + B \neq 1$ and $det(H) = B(\lambda_1 - \lambda_2) \neq 0$. Let D_1 be the diagonal matrix defined by

$$D_1 = \begin{bmatrix} \lambda_3 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix}.$$

Then,

$$\begin{aligned} MH &= \begin{bmatrix} 1 & 0 & 0 \\ 1 & A & 1 \\ 0 & B & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{1-A-B} & \lambda_1 & \lambda_2 \\ \frac{B}{1-A-B} & B & B \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{1-A-B} & A\lambda_1 + B & A\lambda_2 + B \\ \frac{B}{1-A-B} & B\lambda_1 & B\lambda_2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{1-A-B} & \lambda_1^2 & \lambda_2^2 \\ \frac{B}{1-A-B} & B\lambda_1 & B\lambda_2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{1-A-B} & \lambda_1 & \lambda_2 \\ \frac{B}{1-A-B} & B & B \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix} \\ &= HD_1. \end{aligned}$$

Since $det(H) \neq 0$, the matrix H is nonsingular, that is, H^{-1} exists. Consequently, $H^{-1}MH = D_1$, that is, M is similar to the diagonal matrix D_1. It follows that

$$\begin{aligned} (H^{-1}MH)^n &= D_1^n \\ \underbrace{(H^{-1}MH)(H^{-1}MH)\cdots(H^{-1}MH)}_{n} &= D_1^n \\ H^{-1}\underbrace{M(HH^{-1})M(HH^{-1})M\cdots M(HH^{-1})M}_{n}H &= D_1^n \\ H^{-1}\underbrace{MIMIMI\cdots MIM}_{n}H &= D_1^n \\ H^{-1}\underbrace{MMM\cdots MM}_{n}H &= D_1^n \\ H^{-1}M^nH &= D_1^n. \end{aligned}$$

This means that $M^nH = HD_1^n$. By substitution,

$$M^nH = \begin{bmatrix} 1 & 0 & 0 \\ S_{+n} & v_{n+1} & v_n \\ BS_{+(n-1)} & Bv_n & Bv_{n-1} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{1-A-B} & \lambda_1 & \lambda_2 \\ \frac{B}{1-A-B} & B & B \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ S_{+n} + \frac{v_{n+1}}{1-A-B} + \frac{Bv_n}{1-A-B} & \lambda_1 v_{n+1} + Bv_n & \lambda_2 v_{n+1} + Bv_n \\ BS_{+(n-1)} + \frac{Bv_n}{1-A-B} + \frac{B^2 v_{n-1}}{1-A-B} & B\lambda_1 v_n + B^2 v_{n-1} & B\lambda_2 v_n + B^2 v_{n-1} \end{bmatrix},$$

and

$$HD_1^n = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{1-A-B} & \lambda_1 & \lambda_2 \\ \frac{B}{1-A-B} & B & B \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \lambda_1^n & 0 \\ 0 & 0 & \lambda_2^n \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{1-A-B} & \lambda_1^{n+1} & \lambda_2^{n+1} \\ \frac{B}{1-A-B} & B\lambda_1^n & B\lambda_2^n \end{bmatrix}.$$

Therefore, by equality of matrices,

$$\begin{aligned} S_{+n} + \frac{v_{n+1}}{1-A-B} + \frac{Bv_n}{1-A-B} &= \frac{1}{1-A-B} \\ S_{+n} &= \frac{-v_{n+1} - Bv_n + 1}{1-A-B} \\ S_{+n} &= \frac{v_{n+1} + Bv_n - 1}{A+B-1}. \end{aligned}$$

□

Corollary 3.3 *For every $n > 0$,*

(a) $\lambda_1^{n+1} = \lambda_1 v_{n+1} + Bv_n$; *and*

(b) $\lambda_2^{n+1} = \lambda_2 v_{n+1} + Bv_n$,

where λ_1 and λ_2 are eigenvalues of the matrix M.

It is worth noting that if $B = 1$, then S_{+n} is equal to S_n^+ for all $n > 0$. The following result was obtained by Emrah Kiliç [2].

Theorem 3.4 [2] *For every $n > 0$,*

$$S_n^+ = \frac{u_{n+1} + u_n - 1}{A}.$$

4 Sum of the Negatively Subscripted Terms

Applying the same technique to generate S_{-n}, consider the 3×3 matrix N defined by

$$N = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -\frac{A}{B} & 1 \\ 0 & \frac{1}{B} & 0 \end{bmatrix},$$

where A and B are nonzero integers with $A^2 + 4B \neq 0$ and $A + B \neq 1$.

Lemma 4.1 *For every $n > 0$,*

$$N^n = \begin{bmatrix} 1 & 0 & 0 \\ BS_{-n} & Bv_{-(n+1)} & Bv_{-n} \\ S_{-(n-1)} & v_{-n} & v_{-(n-1)} \end{bmatrix}.$$

Proof: For $n = 1$, we see

$$N^1 = \begin{bmatrix} 1 & 0 & 0 \\ BS_{-1} & Bv_{-2} & Bv_{-1} \\ S_{-0} & v_{-1} & v_{-0} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -\frac{A}{B} & 1 \\ 0 & \frac{1}{B} & 0 \end{bmatrix}.$$

For induction, assume that

$$N^n = \begin{bmatrix} 1 & 0 & 0 \\ BS_{-n} & Bv_{-(n+1)} & Bv_{-n} \\ S_{-(n-1)} & v_{-n} & v_{-(n-1)} \end{bmatrix}.$$

Then

$$N^{n+1} = N^n N = \begin{bmatrix} 1 & 0 & 0 \\ BS_{-n} & Bv_{-(n+1)} & Bv_{-n} \\ S_{-(n-1)} & v_{-n} & v_{-(n-1)} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & -\frac{A}{B} & 1 \\ 0 & \frac{1}{B} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ B(S_{-n} + v_{-(n+1)}) & -Av_{-(n+1)} + v_{-n} & Bv_{-(n+1)} \\ S_{-(n-1)} + v_{-n} & -\frac{A}{B}v_{-n} + \frac{1}{B}v_{-(n-1)} & v_{-n} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ BS_{-(n+1)} & Bv_{-(n+2)} & Bv_{-(n+1)} \\ S_{-n} & v_{-(n+1)} & v_{-n} \end{bmatrix}.$$

Thus, by the Principle of Mathematical Induction, the assertion must be true. □

We shall derive now the explicit formula for S_{-n}.

Theorem 4.2 *For every $n > 0$,*

$$S_{-n} = \frac{1 - Bv_{-(n+1)} - v_{-n}}{A + B - 1}.$$

Proof: Let $n > 0$ and let $K_N(\lambda)$ be the characteristic polynomial of the matrix N.

Then

$$K_N(\lambda) = det\begin{bmatrix} \lambda - 1 & 0 & 0 \\ -1 & \lambda + \frac{A}{B} & -1 \\ 0 & -\frac{1}{B} & \lambda \end{bmatrix} = (\lambda - 1)(\lambda^2 + \tfrac{A}{B}\lambda - \tfrac{1}{B}).$$

Hence, the eigenvalues of the matrix N are

$$\lambda_1 = \frac{-A + \sqrt{A^2 + 4B}}{2B}, \lambda_2 = \frac{-A - \sqrt{A^2 + 4B}}{2B} \text{ and } \lambda_3 = 1.$$

Since $A + B \neq 1$ and $A^2 + 4B \neq 0$, it follows that the eigenvalues of the matrix N are distinct. Now, let J be the 3×3 matrix defined by

$$J = \begin{bmatrix} 1 & 0 & 0 \\ \frac{B}{A+B-1} & \lambda_1 & \lambda_2 \\ \frac{1}{A+B-1} & \frac{1}{B} & \frac{1}{B} \end{bmatrix}.$$

J is well-defined since $A + B \neq 1$ and $det(J) = \frac{1}{B}(\lambda_1 - \lambda_2) \neq 0$. Let D_2 be the diagonal matrix defined by

$$D_2 = \begin{bmatrix} \lambda_3 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix}.$$

Then

$$NJ = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -\frac{A}{B} & 1 \\ 0 & \frac{1}{B} & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ \frac{B}{A+B-1} & \lambda_1 & \lambda_2 \\ \frac{1}{A+B-1} & \frac{1}{B} & \frac{1}{B} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ \frac{B}{1-A-B} & \frac{-A\lambda_1+1}{B} & \frac{-A\lambda_2+1}{B} \\ \frac{1}{A+B-1} & \frac{1}{B}\lambda_1 & \frac{1}{B}\lambda_2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ \frac{B}{A+B-1} & \lambda_1^2 & \lambda_2^2 \\ \frac{1}{A+B-1} & \frac{1}{B}\lambda_1 & \frac{1}{B}\lambda_2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ \frac{B}{A+B-1} & \lambda_1 & \lambda_2 \\ \frac{1}{A+B-1} & \frac{1}{B} & \frac{1}{B} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix}$$

$$= JD_2.$$

Since $det(J) \neq 0$, the matrix J is nonsingular, that is, J^{-1} exists. Consequently, $J^{-1}NJ = D_2$, that is N is similar to the diagonal matrix D_2. Now, following the proof in Theorem 3.2, $N^nJ = JD_2^n$. By substitution,

$$N^nJ = \begin{bmatrix} 1 & 0 & 0 \\ BS_{-n} & Bv_{-(n+1)} & Bv_{-n} \\ S_{-(n-1)} & v_{-n} & v_{-(n-1)} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ \frac{B}{A+B-1} & \lambda_1 & \lambda_2 \\ \frac{1}{A+B-1} & \frac{1}{B} & \frac{1}{B} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ BS_{-n} + \frac{B^2v_{-(n+1)}}{A+B-1} + \frac{Bv_{-n}}{A+B-1} & B\lambda_1 v_{-(n+1)} + v_{-n} & B\lambda_2 v_{-(n+1)} + v_{-n} \\ S_{-(n-1)} + \frac{Bv_{-n}}{A+B-1} + \frac{v_{-(n-1)}}{A+B-1} & \lambda_1 v_{-n} + \frac{v_{-(n-1)}}{B} & \lambda_2 v_{-n} + \frac{v_{-(n-1)}}{B} \end{bmatrix},$$

and

$$JD_2^n = \begin{bmatrix} 1 & 0 & 0 \\ \frac{B}{A+B-1} & \lambda_1 & \lambda_2 \\ \frac{1}{A+B-1} & \frac{1}{B} & \frac{1}{B} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \lambda_1^n & 0 \\ 0 & 0 & \lambda_2^n \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ \frac{B}{A+B-1} & \lambda_1^{n+1} & \lambda_2^{n+1} \\ \frac{1}{A+B-1} & \frac{\lambda_1^n}{B} & \frac{\lambda_2^n}{B} \end{bmatrix}.$$

Therefore, by equality of matrices,

$$\begin{aligned} BS_{-n} + \frac{B^2v_{-(n+1)}}{A+B-1} + \frac{Bv_{-n}}{A+B-1} &= \frac{B}{A+B-1} \\ S_{-n} &= \frac{1 - Bv_{-(n+1)} - v_{-n}}{A+B-1} \end{aligned}$$

□

Corollary 4.3 *For every $n > 0$,*

$$S_{-n} = \begin{cases} \dfrac{1 - B^{-n}v_{n+1} + B^{-n}v_n}{A+B-1} & , \text{ if } n \text{ is even} \\ \dfrac{1 + B^{-n}v_{n+1} - B^{-n}v_n}{A+B-1} & , \text{ if } n \text{ is odd}. \end{cases}$$

Corollary 4.4 *For every $n > 0$,*

(a) $\lambda_1^{n+1} = B\lambda_1 v_{-(n+1)} + v_{-n}$; *and*

(b) $\lambda_2^{n+1} = B\lambda_2 v_{-(n+1)} + v_{-n}$,

where λ_1 and λ_2 are eigenvalues of the matrix N.

If $B = 1$, then S_{-n} is equal to S_n^- for all $n > 0$. The following results were obtained by Emrah Kiliç [2].

Theorem 4.5 [2] *For every $n > 0$,*

$$S_n^- = \frac{1 - v_{-(n+1)} - v_{-n}}{A}.$$

Corollary 4.6 [2] *For every $n > 0$,*

$$S_n^- = \begin{cases} \dfrac{v_n - v_{n+1} + 1}{A} & , \text{ if } n \text{ is even} \\ \dfrac{v_{n+1} - v_n + 1}{A} & , \text{ if } n \text{ is odd} . \end{cases}$$

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