A note on fractional (g, f, m)-deleted graphs

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Abstract: A graph G is called a fractional (g, f, m)-deleted graph if after deleting any m edges then the resulting graph admits a fractional (g, f)-factor. In this paper, we prove that if G is a graph of order n, and if $1 \le a \le g(x) \le f(x) \le b$ for any $x \in V(G)$, $\delta(G) \ge \frac{b^2(i-1)}{a} + 2m$, $n > \frac{(a+b)(i(a+b)+2m-2)}{a}$, and $|N_G(x_1) \cup N_G(x_2) \cup \cdots \cup N_G(x_i)| \ge \frac{bn}{a+b}$ for any independent set $\{x_1, x_2, \ldots, x_i\}$ of $V(G)$, where $i \ge 2$, then G is a fractional (g, f, m)-deleted graph. The result is tight on the neighborhood union condition.

Key words: graph, fractional (g, f)-factor, fractional (g, f, m)-deleted graph, neighborhood union condition

1 Introduction

All graphs considered in this paper are finite, loopless, and without multiple edges. Let G be a graph with the vertex set $V(G)$ and the edge set $E(G)$. Let $n = |V(G)|$. For a vertex $x \in V(G)$, the degree and the neighborhood of x in G are denoted by $d_G(x)$ and $N_G(x)$, respectively. Let

*Research supported partially by NSFC (No.11071223), ZJNSF (No.Z6090150), and IP-OCNS-ZJNU

$\Delta(G)$ and $\delta(G)$ denote the minimum degree and the maximum degree of G, respectively. For $S \subseteq V(G)$, we denote by $G[S]$ the subgraph of G induced by S, and let $G - S = G[V(G) \backslash S]$. For two disjoint subsets S and T of $V(G)$, we use $e_G(S,T)$ to denote the number of edges with one end in S and the other end in T. Given a vertex $x \in V(G)$, let $E(x) = \{e \in E(G) | e$ is incident to $x\}$. Let $\omega(G)$ denote the number of components of a graph G. The *toughness* $t(G)$ of a graph G is defined as follows: $t(G) = +\infty$ if G is a complete graph; otherwise,

$$t(G) = \min\{\frac{|S|}{\omega(G-S)} \mid S \subseteq V(G), \omega(G-S) \geq 2\}.$$

Suppose that g and f are two integer-valued functions on $V(G)$ such that $0 \leq g(x) \leq f(x)$ for all $x \in V(G)$. A *fractional* (g,f)*-factor* is a function h that assigns to each edge of a graph G a number in [0,1] so that for each vertex x we have $g(x) \leq d_G^h(x) \leq f(x)$, where $d_G^h(x) = \sum_{e \in E(x)} h(e)$ is called the *fractional degree* of x in G. If $g(x) = f(x) = k$ ($k \geq 1$ is an integer) for all $x \in V(G)$, then a fractional (g,f)-factor is just a fractional k-factor.

Yu [4] proved that if G is a connected graph with $\delta(G) \geq k \geq 1$, $n \geq 4k-3$, and $\max\{d_G(x), d_G(y)\} \geq \frac{n}{2}$ for each pair of nonadjacent vertices x and y, then G has a fractional k-factor. Liu and Zhang [3] revealed the relation between the fractional k-factor and the toughness of a graph by showing that for an integer $k \geq 2$, if $n \geq k+1$ and $t(G) \geq k - \frac{1}{k}$, then G admits a fractional k-factor. Anstee [1] gave a necessary and sufficient condition for a graph to have a fractional (g,f)-factor as follows:

Theorem 1 (Anstee [1]) *Suppose that f and g are two integer-valued functions defined on the vertex set of a graph G such that $0 \leq g(x) \leq f(x)$ for each $x \in V(G)$. Then G has a fractional (g,f)-factor if and only if for every subset S of $V(G)$, $g(T) - d_{G-S}(T) \leq f(S)$, where $T = \{x \in V(G) \backslash S \mid d_{G-S}(x) \leq g(x)\}$.*

Given two integers $k, m \geq 1$, a graph G is called a *fractional (k,m)-deleted graph* if removing any m edges from G, then the resulting graph has a fractional k-factor. Zhou [5] first introduced the concept of a fractional (k,m)-deleted graph, and showed that if G is a graph with $n \geq 4k-5+2(2k+1)m$ and $\delta(G) \geq \frac{n}{2}$, then G is a fractional (k,m)-deleted graph. Moreover, Zhou [6] established a neighborhood condition for a fractional (k,m)-deleted graph.

Theorem 2 (Zhou [6]) *Let $k \geq 2$ and $m \geq 0$ be two integers. If G is a connected graph with $n \geq 9k-1-\sqrt{2(k-1)^2+2}+2(2k+1)m$, $\delta(G) \geq k+m+\frac{(m+1)^2-1}{4k}$, and*

$$|N_G(x) \cup N_G(y)| \geq \frac{1}{2}(n+k-2)$$

for each pair of nonadjacent vertices x and y, then G is a fractional (k,m)-deleted graph.

Other results on fractional k-factor and fractional (k,m)-deleted graphs can refer [2, 7, 8].

In this paper, we first extend the concept of a fractional (k,m)-deleted graph to a fractional (g,f,m)-deleted graph. A graph G is called a *fractional (g,f,m)-deleted graph* if for each edge subset $H \subseteq E(G)$ with $|H| = m$, there exists a fractional (g,f)-factor h such that $h(e) = 0$ for all $e \in H$. That is, after removing any m edges, the resulting graph still has a fractional (g,f)-factor. Our main result is stated below, which presents a neighborhood union condition for a graph to be a fractional (g,f,m)-deleted graph:

Theorem 3 *Let G be a graph of order n. Let a, b, i be three integers with $i \geq 2$ and $1 \leq a \leq b$. Let g, f be two integer-valued functions defined on $V(G)$ such that $a \leq g(x) \leq f(x) \leq b$ for each $x \in V(G)$. If $\delta(G) \geq \frac{b^2(i-1)}{a}+2m$, $n > \frac{(a+b)(i(a+b)+2m-2)}{a}$, and*

$$|N_G(x_1) \cup N_G(x_2) \cup \cdots \cup N_G(x_i)| \geq \frac{bn}{a+b}$$

for any independent subset $\{x_1, x_2, \ldots, x_i\}$ *of* $V(G)$*, then* G *is a fractional* (g, f, m)*-deleted graph.*

2 Proof of Theorem 3

Before showing the proof of Theorem 3, we need to verify the following Lemma 4, which provides a necessary and sufficient condition for a graph to be a fractional (g, f, m)-deleted graph.

Lemma 4 *Let* f, g *be two integer-valued functions defined on* $V(G)$ *such that* $0 \le g(x) \le f(x)$ *for each* $x \in V(G)$*. Then* G *is a fractional* (g, f, m)*-deleted graph if and only if for any subset* S *of* $V(G)$ *and a subset* H *of* $E(G)$ *with* $|H| = m$*,*

$$\delta_G(S,T) = f(S) + \sum_{x\in T} d_{G-S}(x) - g(T) \ge \sum_{x\in T} d_H(x) - e_G(T,S),$$

where $T = \{x \in V(G)\backslash S \mid d_{G-S}(x) - d_H(x) + e_G(x,S) \le g(x)\}$.

Proof. Let $G' = G - H$. Then G is a fractional (g, f, m)-deleted graph if and only if G' has a fractional (g, f)-factor. By Theorem 1, this is true if and only if for any subset S of $V(G)$,

$$\delta_{G'}(S,T') = f(S) + d_{G'-S}(T') - g(T') \ge 0,$$

where $T' = \{x \in V(G')\backslash S \mid d_{G'-S}(x) \le g(x)\}$.

It is easy to see that $d_{G'-S}(x) = d_{G-S}(x) - d_H(x) + e_G(x,S)$ for any $x \in T'$. By the definitions of T' and T, we have $T' = T$. Hence, $\delta_{G'}(S,T') = \delta_G(S,T) - \sum_{x\in T} d_H(x) + e_G(T,S)$. Thus, $\delta_{G'}(S,T') \ge 0$ if and only if $\delta_G(S,T) \ge \sum_{x\in T} d_H(x) - e_G(T,S)$. It follows that G is a fractional (g, f, m)-deleted graph if and only if $\delta_G(S,T) = f(S) + \sum_{x\in T} d_{G-S}(x) - g(T) \ge \sum_{x\in T} d_H(x) - e_G(T,S)$. □

The following corollary immediately follows from Lemma 4:

Corollary 5 *Let g and f be two integer-valued functions defined on $V(G)$ such that $0 \leq g(x) \leq f(x)$ for each $x \in V(G)$. Let G be a graph and $H \subseteq E(G)$ with $|H| = m \geq 0$. Then G is a fractional (g, f, m)-deleted graph if and only if*

$$\delta_G(S,T) = f(S) + \sum_{x \in T} d_{G-S}(x) - g(T) \geq \sum_{x \in T} d_H(x) - e_G(T,S)$$

for all disjoint subsets S and T of $V(G)$.

Proof of Theorem 3. Assume to the contrary that G satisfies the conditions of the theorem, but is not a fractional (g, f, m)-deleted graph. By Corollary 5 and noting the fact that $\sum_{x \in T} d_H(x) - e_G(T,S) \leq 2m$, there exist disjoint subsets S and T of $V(G)$ such that

$$\delta_G(S,T) = f(S) + \sum_{x \in T} d_{G-S}(x) - g(T) \leq 2m - 1. \tag{1}$$

We choose subsets S and T such that $|T|$ is minimal. Obviously, $T \neq \emptyset$.

Claim 1 *$d_{G-S}(x) \leq g(x) - 1 \leq b - 1$ for any $x \in T$.*

Proof. If $d_{G-S}(x) \geq g(x)$ for some $x \in T$, then the subsets S and $T \setminus \{x\}$ satisfy (1). This contradicts the choice of S and T. □

Let $d_1 = \min\{d_{G-S}(x) | x \in T\}$ and choose $x_1 \in T$ such that $d_{G-S}(x_1) = d_1$. If $z \geq 2$ and $T \setminus (\cup_{j=1}^{z-1} N_T[x_j]) \neq \emptyset$, let

$$d_z = \min\{d_{G-S}(x) | x \in T \setminus (\cup_{j=1}^{z-1} N_T[x_j])\}$$

and choose $x_z \in T \setminus (\cup_{j=1}^{z-1} N_T[x_j])$ such that $d_{G-S}(x_z) = d_z$. So, we get a sequence such that $0 \leq d_1 \leq d_2 \leq \cdots \leq d_\pi \leq g(x) - 1 \leq b - 1$ and an independent set $\{x_1, x_2, \ldots, x_\pi\} \subseteq T$.

Claim 2 $|T| \geq (i-1)b + 1$.

Proof. Assume that $|T| \leq (i-1)b$. Then $|S| + d_1 \geq d_G(x_1) \geq \delta(G) \geq \frac{b^2(i-1)}{a} + 2m$. By (1) and $0 \leq d_1 \leq b-1$, we have

$$\begin{aligned} 2m-1 &\geq f(S) - g(T) + d_{G-S}(T) \\ &\geq a|S| + d_1|T| - b|T| \\ &= a|S| + (d_1 - b)|T| \\ &\geq a(\frac{b^2(i-1)}{a} - d_1 + 2m) + (d_1 - b)(i-1)b \\ &= b^2(i-1) + d_1(b(i-1) - a) - b^2(i-1) + 2am \\ &\geq 2m. \end{aligned}$$

This produces a contradiction. □

Since $d_{G-S}(x) \leq b-1$ and $|T| \geq (i-1)b+1$, we get $\pi \geq i$. Thus, we can choose an independent set $\{x_1, x_2, \ldots, x_i\} \subseteq T$.

In view of the condition of the theorem, we get

$$\frac{bn}{a+b} \leq |N_G(x_1) \cup N_G(x_2) \cup \cdots \cup N_G(x_i)| \leq |S| + \sum_{j=1}^{i} d_j$$

and

$$|S| \geq \frac{bn}{a+b} - \sum_{j=1}^{i} d_j. \tag{2}$$

Noting that

$$|N_T[x_j]| - |N_T[x_j] \cap (\cup_{z=1}^{j-1} N_T[x_z])| \geq 1, j = 2, 3, \ldots, i-1$$

and

$$\begin{aligned} |\cup_{z=1}^{j} N_T[x_z]| &\leq \sum_{z=1}^{j} |N_T[x_z]| \\ &\leq \sum_{z=1}^{j} (d_{G-S}(x_z) + 1) \\ &= \sum_{z=1}^{j} (d_z + 1), j = 1, 2, \ldots, i, \end{aligned}$$

we obtain

$$\begin{aligned}
& f(S) + d_{G-S}(T) - g(T) \\
\geq\ & a|S| - b|T| + d_1|N_T[x_1]| + d_2(|N_T[x_2]| - |N_T[x_2] \cap N_T[x_1]|) \\
& + \cdots + d_{i-1}(|N_T[x_{i-1}]| - |N_T[x_{i-1}] \cap (\cup_{j=1}^{i-2} N_T[x_j])|) \\
& + d_i(|T| - |\cup_{j=1}^{i-1} N_T[x_j])||) \\
\geq\ & a|S| + (d_1 - d_i)|N_T[x_1]| + \sum_{j=2}^{i-1} d_j + (d_i - b)|T| - d_i \sum_{j=2}^{i-1} |N_T[x_j]| \\
=\ & a|S| + (d_1 - d_i)(d_1 + 1) + \sum_{j=2}^{i-1} d_j + (d_i - b)|T| - d_i \sum_{j=2}^{i-1} (d_j + 1) \\
=\ & a|S| + d_1^2 + \sum_{j=1}^{i-1} d_j + (d_i - b)|T| - d_i \sum_{j=1}^{i-1} (d_j + 1),
\end{aligned}$$

which implies

$$\begin{aligned}
& (n - |S| - |T|)(b - d_i) \\
\geq\ & f(S) + d_{G-S}(T) - g(T) - 2m + 1 \\
\geq\ & a|S| + d_1^2 + \sum_{j=1}^{i-1} d_j + (d_i - b)|T| - d_i \sum_{j=1}^{i-1} (d_j + 1) - 2m + 1.
\end{aligned}$$

Equivalently,

$$0 \leq n(b-d_i) - (a+b-d_i)|S| + d_i \sum_{j=1}^{i-1} d_j - \sum_{j=1}^{i-1} d_j + d_i(i-1) - d_1^2 + 2m - 1. \quad (3)$$

By (2), (3), $d_1 \leq d_2 \leq \cdots \leq d_i \leq b-1$ and $n > \frac{(a+b)(i(a+b)+2m-2)}{a}$, we have the following:

$$\begin{aligned}
0\ \leq\ & n(b - d_i) - (a + b - d_i)\left(\frac{bn}{a+b} - \sum_{j=1}^{i} d_j\right) + d_i \sum_{j=1}^{i-1} d_j - \sum_{j=1}^{i-1} d_j \\
& + d_i(i-1) - d_1^2 + 2m - 1 \\
=\ & -\frac{an}{a+b} d_i + (a+b) \sum_{j=1}^{i} d_j - d_i \sum_{j=1}^{i} d_j + d_i \sum_{j=1}^{i-1} d_j - \sum_{j=1}^{i-1} d_j
\end{aligned}$$

$$
\begin{aligned}
& +d_i(i-1) - d_1^2 + 2m - 1 \\
= \;& -\frac{an}{a+b}d_i + ((a+b-1)d_1 - d_1^2) + (a+b-1)\sum_{j=2}^{i-1} d_j \\
& +d_i(a+b+i-1) - d_i^2 + 2m - 1 \\
\le \;& -\frac{an}{a+b}d_i + (a+b-1)d_i + (a+b-1)\sum_{j=2}^{i-1} d_i \\
& +d_i(a+b+i-1) - d_i^2 + 2m - 1 \\
= \;& -\frac{an}{a+b}d_i + i(a+b)d_i - d_i^2 + 2m - 1.
\end{aligned}
$$

If $d_i > 0$, then $0 < 2d_i - d_i^2 - 1 \le 0$ since $n > \frac{(a+b)(i(a+b)+2m-2)}{a}$ and $2m(1-d_i) \le 0$, a contradiction.

If $d_i = 0$, then $d_1 = \cdots = d_i = 0$. By (2), we have $|S| \ge \frac{bn}{a+b}$ and $|T| \le n - |S| \le \frac{an}{a+b}$. Since $d_{G-S}(T) \ge \sum_{x\in T} d_H(x) - e_G(T,S)$, we have

$$
\begin{aligned}
& f(S) + d_{G-S} - g(T) - (\sum_{x\in T} d_H(x) - e_G(T,S)) \\
\ge \;& a\cdot\frac{bn}{a+b} - b\cdot\frac{an}{a+b} + (d_{G-S}(T) - \sum_{x\in T} d_H(x) + e_G(T,S)) \\
\ge \;& 0,
\end{aligned}
$$

also a contradiction. This completes the proof of the theorem. □

Theorem 3 is best possible, in some extent, on the conditions. Actually, we can construct some graphs such that the neighborhood union condition in Theorem 3 cannot be replaced by $|N_G(x_1) \cup N_G(x_2) \cup \cdots \cup N_G(x_i)| \ge \frac{bn}{a+b} - 1$.

Let $G_1 = K_{bt}$ be a complete graph, $G_2 = (at+1)K_1$ be a graph consisting of $at+1$ isolated vertices, and $G = G_1 \vee G_2$, where t is sufficiently large (i.e., $t > \frac{i(a+b)+2m-2}{a} - \frac{1}{a+b}$ for some i. Thus, $\delta(G) \ge \frac{b^2(i-1)}{a} + 2m$, and $n > \frac{(a+b)(i(a+b)+2m-2)}{a}$). Then $n = |G_1| + |G_2| = (a+b)t + 1$, and for any independent set $\{x_1, x_2, \ldots, x_i\} \subseteq V(G_2)$, we have

$$
\frac{bn}{a+b} > |N_G(x_1) \cup N_G(x_2) \cup \cdots \cup N_G(x_i)| = bt > \frac{bn}{a+b} - 1.
$$

Let $S = V(G_1)$, and $g(x) = f(x) = a$ for any $x \in V(G_1)$; $T = V(G_2)$, and $g(x) = f(x) = b$ for any $x \in V(G_2)$. Then $f(S) - g(T) + d_{G-S}(T) - (\sum_{x \in T} d_H(x) - e_G(T, S)) = a|S| - b|T| = abt - b(at + 1) = -b < 0$. Hence, G is not a fractional (g, f, m)-deleted graph.

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