# Bounds on locating-total domination number of the Cartesian product of cycles

Huaming Xing[a], Moo Young Sohn[b,*]
[a] Institute of Mathematics, Langfang Normal College, Langfang, 065000, P.R.China
[b] Mathematics, Changwon National University, Changwon, 641-773, Republic of Korea

**Abstract:** A total dominating set $S$ of a graph $G$ with no isolated vertex is a locating-total dominating set of $G$ if for every pair of distinct vertices $u$ and $v$ in $V - S$ are totally dominated by distinct subsets of the total dominating set. The minimum cardinality of a locating-total dominating set is the locating-total domination number. In this paper, we obtain new upper bounds for locating-total domination numbers of the Cartesian product of cycles $C_m$ and $C_n$ and prove that for any positive integer $n \geq 3$, the locating-total domination numbers of the Cartesian product of cycles $C_3$ and $C_n$ is equal to $n$ for $n \equiv 0 \pmod 6$ or $n + 1$ otherwise.
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## 1 Introduction

The location of monitoring devices, such as surveillance camera or fire alarms, to safeguard a system serves as a motivation for this work. The problem of placing monitoring devices in system in such a way that every site in the system (including the monitors themselves) is adjacent to a monitor site can be modeled by total domination in graphs. Applications where it is also important that if there is a problem at a facility, its location can be uniquely identified by the set of monitors, can be modeled by a combination of total-domination and locating sets. Locating-total dominating set in graph was introduced by Haynes and Henning [5] and has been studied in [1, 5–7] and elsewhere.

* Corresponding author, E-mail address: mysohn@changwon.ac.kr
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Graph theory terminology not presented here can be found in [3,4]. All graphs considered in this paper are simple without isolated vertices.

Let $G = (V, E)$ be a graph with vertex set $V$ and edge set $E$. For any vertex $v \in G$, the *open neighborhood* of $v$ is the set $N(v) = \{u \in V | uv \in E\}$, and its *closed neighborhood* is the set $N[v] = N(v) \cup \{v\}$. We denote the degree of a vertex $v$ in $G$ by $d_G(v)$, or simply by $d(v)$ if the graph $G$ is clear from text. The maximum degree of a graph is denoted by $\Delta$. For any $S \subseteq V$, $N(S) = \cup_{v \in S} N(v)$. Let $\langle S \rangle$ denote the subgraph induced by $S$. For $u \in V - S$, if $N(u) \cap S = \{v\}$, then the vertex $u$ is called a *private neighbor* of $v$ (with respect to $S$). If $u \in N(v)$ and $|N(u) \cap S| \geq 2$, then the vertex $u$ is called a *common neighbor* of $v$ (with respect to $S$). For two vertices $u$, $v \in V$, the *distance* between $u$ and $v$ is $d(u, v)$. The distance between a vertex $u$ and a set $S$ of vertices in a graph is defined as $d(u, S) = min\{d(u, v) | v \in S\}$. If $S$ and $T$ are two vertex disjoint subsets of $V$, then we denote the number of all edges of $G$ that join a vertex of $S$ and a vertex of $T$ by $e[S, T]$.

For graphs $G$ and $H$, the *Cartesian product* $G \Box H$ is the graph with vertex set $V(G) \times V(H)$ where two vertices $(u_1, v_1)$ and $(u_2, v_2)$ are adjacent if and only if either $u_1 = u_2$ and $v_1 v_2 \in E(H)$ or $v_1 = v_2$ and $u_1 u_2 \in E(G)$.

Let $\{v_{ij} | (i, j) \in \mathbb{Z}_m \times \mathbb{Z}_n\}$ be the vertex set of $G = C_m \Box C_n$ so that the subgraph induced by $\mathcal{H}_i = \{v_{i0}, v_{i1}, \ldots, v_{i(n-1)}\}$ is isomorphic to the cycle $C_n$ for each $i \in \mathbb{Z}_m$ and that induced by $\mathcal{V}_j = \{v_{0j}, v_{1j}, \ldots, v_{(m-1)j}\}$ is isomorphic to the cycle $C_m$ for each $j \in \mathbb{Z}_n$. The cycles $\langle \mathcal{H}_i \rangle$ and $\langle \mathcal{V}_j \rangle$ are also called *horizontal* and *vertical*, respectively.

A subset $S \subseteq V$ is a *total dominating set* (abbreviated, TDS) if every vertex of $V$ has a neighbor in $S$. The *total domination number* of $G$, denoted by $\gamma_t(G)$, is the minimum cardinality of a total dominating set of $G$. Total domination was introduced by Cockayne et al. [2] and is now well-studied in graph theory [3,4].

A total dominating set $S$ in a graph $G = (V, E)$ is a *locating-total dominating set* (abbreviated, LTDS) of $G$ if for every pair of distinct vertices $u$ and $v$ in $V - S$, $N(u) \cap S \neq N(v) \cap S$. The minimum cardinality of a locating-total dominating set is the *locating-total domination number* $\gamma_t^L(G)$. A locating-total dominating set in $G$ of cardinality $\gamma_t^L(G)$ is referred as a $\gamma_t^L(G)$-set.

A locating-total dominating set $S$ in a graph $G = (V, E)$ is a *locating-paired-dominating set* (abbreviated, LPDS) of $G$ if $S$ contains a perfect matching. The minimum cardinality of an LPDS is the *locating-paird-domination number* $\gamma_{pr}^L(G)$. An LPDS in $G$ of cardinality $\gamma_{pr}^L(G)$ is referred as a $\gamma_{pr}^L(G)$-set. Locating-paired-domination was introduced by McCoy and Henning [6]. In [5], Haynes et al. gave a lower bound on the locating-total domination number of a tree in terms of order and characterized the extremal tree achieving equality in the lower bound. In [1], Chen and Sohn established a lower bound and upper bounds on the locating-total domination number of trees in terms of its order and number of leaves and support vertices. Furthermore they constructively characterized the extremal trees achieving the bounds. In [7], Henning and Rad gave lower bound and upper bounds on the locating-total domination number of a graph, showed that the locating-total domination number and total domination number of a connected cubic graph can differ significantly, and investigated the locating-total domination number of grid graph $P_m \Box P_n$ for small $m$. In [6], Henning and Löwenstein shown that the locating-total domination number of a claw-free cubic graph is at most one-half its order and characterized the graphs achieving this bound. In this paper, we obtain new upper bounds of locating-total domination numbers of the Cartesian product of cycles $C_m$ and $C_n$ and prove that for any positive integer $n \geq 3$, $\gamma_t^L(C_3 \Box C_n)$ is equal to $n$ for $n \equiv 0 \pmod 6$ or $n+1$ otherwise.

## 2 Bounds of locating-total domination number of $C_m \Box C_n$

A lower bound of locating-total domination number of a graph $G$ of order $n \geq 3$ and maximum degree $\Delta \geq 2$ with no isolated vertex is given in [7]. In this section, we present upper bounds on the locating-total domination number of the Cartesian product of cycles $C_m$ and $C_n$.

**Lemma 2.1.** *( [7]) If $G$ is a graph of order $n \geq 3$ and maximum degree $\Delta \geq 2$ with no isolated vertex, then $\gamma_t^L(G) \geq \frac{2n}{\Delta+2}$, and this bound is sharp.*

**Theorem 2.2.** *For any positive integers $m$, $n$ such that $m \equiv 0 \pmod 3$ and $n \geq 3$,*

$$\gamma_t^L(C_m \Box C_n) \leq \begin{cases} \frac{1}{3}mn, & n \equiv 0 \pmod 6; \\ \frac{1}{3}m(n+1), & \textit{otherwise.} \end{cases}$$

Proof: Let $G \cong C_m \Box C_n$, where $m = 3t$ for a positive integer $t$. For any integers $i$, $j$ such that $0 \leq i \leq 2$ and $j \in \mathbb{Z}_n$, let $D_{ij} = \mathcal{V}_j - \cup_{\nu=0}^{t-1}\{v_{(3\nu+i)j}\}$. If $n = 3$, then it is easy to show that $S = D_{00} \cup D_{22}$ is an LPDS of order $4t = \frac{1}{3}m(n+1)$ in $G$. If $n = 4$, then $S = D_{00} \cup D_{12} \cup (\cup_{i=0}^{t-1}\{v_{(3i)3}\})$ is an LPDS of order $4t + t = \frac{1}{3}m(n+1)$ in $G$. If $n = 5$, then $S = D_{00} \cup D_{12} \cup D_{24}$ is an LPDS of order $6t = \frac{1}{3}m(n+1)$ in $G$.
Assume that $n \geq 6$. Let $n = 6k + r$, where $k \geq 1$ and $0 \leq r \leq 5$. Let $S_0 = \cup_{j=0}^{k-1}(D_{0(6j)} \cup D_{1(6j+2)} \cup D_{2(6j+4)})$.
If $r = 0$, then $S = S_0$ is a TDS of order $tn = \frac{1}{3}mn$ in $G$. For any two vertices $u_1, u_2 \in V - S$, if $d(u_1, u_2) \geq 3$ or $d(u_1, u_2) = 1$, then $N(u_1) \cap S \neq$

$N(u_2) \cap S$. Assume that $d(u_1, u_2) = 2$. If $u_1$, $u_2$ in some vertical $\mathcal{V}_j$ with $j \in \mathbb{Z}_n$, then $j$ is odd. Let $u_1 = v_{ij}$, where $i \in \mathbb{Z}_m$. Then, by considering the TDS $S = S_0$, we can check that $N(u_1) \cap S \cap \{v_{i(j-1)}, v_{i(j+1)}\} \neq \emptyset$, $N(u_2) \cap S \cap \{v_{i(j-1)}, v_{i(j+1)}\} = \emptyset$. Thus $N(u_1) \cap S \neq N(u_2) \cap S$. If $u_1$, $u_2$ in some horizontal $\mathcal{H}_i$ with $i \in \mathbb{Z}_m$, then, without loss of generality, we assume that $u_1 = v_{ij}$ and $u_2 = v_{i(j+2)}$, where $\{j, j+2\} \subset \mathbb{Z}_n$. Then $j$ is odd. If $v_{i(j-1)} \in S$, then $v_{i(j-1)} \in N(u_1)$ and $v_{i(j-1)} \notin N(u_2)$. Thus $N(u_1) \cap S \neq N(u_2) \cap S$. If $v_{i(j-1)} \notin S$, then $v_{i(j+3)} \in S$. Since $v_{i(j+3)} \in N(u_2)$ and $v_{i(j+3)} \notin N(u_1)$, we have that $N(u_1) \cap S \neq N(u_2) \cap S$. If $u_1$, $u_2$ are neither in the same horizontal nor in the same vertical, then by symmetry, we may assume that $u_1 = v_{ij}$ and $u_2 = v_{(i+1)(j+1)}$, where $i \in \mathbb{Z}_m$ and $j \in \mathbb{Z}_n$. If $j$ is even, then $v_{(i-1)j} \in N(u_1) \cap S$ and $v_{(i-1)j} \notin N(u_2)$; if $j$ is odd, then $v_{(i+2)(j+1)} \in N(u_2) \cap S$ and $v_{(i+2)(j+1)} \notin N(u_1)$. Thus $N(u_1) \cap S \neq N(u_2) \cap S$. Consequently, $S$ is an LPDS in $G$. Then $\gamma_t^L(G) \leq |S| = \frac{1}{3}mn$.
Similarly, if $r = 1$, then $S = S_0 \cup D_{0(n-1)}$ is an LPDS of order $tn + t = \frac{1}{3}m(n+1)$ in $G$. If $r = 2$, then $S = S_0 \cup \mathcal{V}_{n-2}$ is an LPDS of order $6kt + 3t = \frac{1}{3}m(n+1)$ in $G$. If $r = 3$, then $S = S_0 \cup D_{0(n-3)} \cup D_{1(n-1)}$ is an LPDS of order $6kt+4t = \frac{1}{3}m(n+1)$ in $G$. If $r = 4$, then $S = S_0 \cup D_{0(n-4)} \cup D_{1(n-2)} \cup (\cup_{i=0}^{t-1}\{v_{(3i)(n-1)}\})$ is an LPDS of order $6kt + 5t = \frac{1}{3}m(n+1)$ in $G$. If $r = 5$, then $S = S_0 \cup D_{0(n-5)} \cup D_{1(n-1)} \cup D_{2(n-1)}$ is an LPDS of order $6(k+1)t = \frac{1}{3}m(n+1)$ in $G$.
Therefore, when $n \not\equiv 0 \pmod 6$, $\gamma_t^L(G) \leq |S| = \frac{1}{3}m(n+1)$. This completes the proof. □

By combining Lemma 2.1 and Theorem 2.2, $\gamma_t^L(C_m \square C_n) = \frac{1}{3}mn$ when $m \equiv 0 \pmod 3$ and $n \equiv 0 \pmod 6$. Hence, we have the following.

**Lemma 2.3.** *For any integers $m$ and $n$ such that $m \geq 3, n \geq 3$, $\gamma_t^L(C_m \square C_n) \geq \frac{1}{3}mn$, and this bound is sharp.*

**Theorem 2.4.** *For any integers $m$, $n$ such that $m \equiv 1 \pmod 3$ and $m \geq 4$, $n \geq 4$,*

$$\gamma_t^L(C_m \square C_n) \leq \begin{cases} \frac{1}{3}(m-1)(n+1) + \lceil \frac{n}{2} \rceil, & m = 4,\ n \in \{4, 10\} \\ & \text{or } m > 4,\ n \equiv 4 \pmod 6; \\ \frac{1}{3}(m-1)n + \lceil \frac{n}{2} \rceil, & n \equiv 0 \pmod 6; \\ \frac{1}{3}(m-1)(n+1) + \lceil \frac{n}{2} \rceil - 1, & \text{otherwise.} \end{cases}$$

Proof: Let $G \cong C_m \square C_n$, where $m = 3t+1$ for a positive integer $t$. For any integers $i$, $j$ such that $0 \leq i \leq 3$ and $j \in \mathbb{Z}_n$, let $D_{ij} = \mathcal{V}_j - \cup_{\nu=0}^{t-1}\{v_{(3\nu+i)j}\}$ and let $\lambda(m, n) = \frac{1}{3}(m-1)(n+1) + \lceil \frac{n}{2} \rceil - 1$.

If $n = 4$, then $S = D_{00} \cup (D_{12} - \{v_{(m-2)2}\}) \cup (\cup_{i=0}^{t-1}\{v_{(3i)3}\}) \cup \{v_{(m-3)3}\}$ is an LPDS of order $5t + 2 = \lambda(m, n) + 1$ in $G$. If $n = 5$, then $S = D_{00} \cup D_{12} \cup (D_{24} - \{v_{(m-1)4}\})$ is an LPDS of order $6t + 2 = \lambda(m, n)$ in $G$. Assume that $n \geq 6$. Let $n = 6k + r$, where $k \geq 1$ and $0 \leq r \leq 5$. Let $S_0 = \cup_{j=0}^{k-1}(D_{0(6j)} \cup D_{1(6j+2)} \cup D_{2(6j+4)})$.
By using an identical proof as in the theorem 2.2, we have the following. If $r = 0$, then $S = S_0$ is an LPDS of order $6kt + 3k = \frac{1}{3}(m-1)n + \lceil \frac{n}{2} \rceil$ in $G$. If $r = 1$, then $S = S_0 \cup (D_{0(n-1)} - \{v_{(m-1)(n-1)}\})$ is an LPDS of order $6kt + 3k + 2t = \lambda(m, n)$ in $G$. If $r = 2$, then $S = S_0 \cup (\mathcal{V}_{n-2} - \{v_{(m-1)(n-1)}\})$ is an LPDS of order $6kt + 3k + 3t = \lambda(m, n)$ in $G$. If $r = 3$, then $S = (S_0 \cup D_{0(n-3)} \cup D_{1(n-1)}) - \{v_{(m-1)0}\}$ is an LPDS of order $6kt + 3k + 4t + 1 = \lambda(m, n)$ in $G$. If $m = 4$, $n = 10$ or $m > 4$, $r = 4$, then $S = S_0 \cup D_{0(n-4)} \cup D_{1(n-2)} \cup (\cup_{\ell=0}^{t-1}\{v_{(3\ell)(n-1)}\})$ is a LTDS of order $6kt + 3k + 5t + 2 = \lambda(m, n) + 1$ in $G$. If $m = r = 4$ and $n \neq 10$, then $n \geq 16$. Then $S = \cup_{j=0}^{k-2}(D_{0(6j)} \cup D_{1(6j+2)} \cup D_{2(6j+4)}) \cup D_{3(n-10)} \cup D_{0(n-8)} \cup D_{1(n-6)} \cup D_{2(n-4)} \cup D_{3(n-2)}$ is a LTDS of order $9k + 6 = \lambda(m, n)$ in $G$. If $r = 5$, then $S = S_0 \cup D_{0(n-5)} \cup D_{1(n-3)} \cup (D_{2(n-1)} - \{v_{(m-1)(n-1)}\})$ is an LPDS of order $6kt + 3k + 6t + 2 = \lambda(m, n)$ in $G$. This completes the proof. □

**Theorem 2.5.** *For any integers $m$, $n$ such that $m \equiv 2 \pmod 3$ and $m \geq 5, n \geq 5$,*

$$\gamma_t^L(C_m \Box C_n) \leq \begin{cases} \frac{1}{3}(m+1)n, & n \equiv 0 \pmod 6; \\ \frac{1}{3}(m+1)(n+1) - 2, & \textit{otherwise}. \end{cases}$$

Proof: Let $G \cong C_m \Box C_n$, where $m = 3t+2$ for a positive integer $t$. For any integers $i$, $j$ such that $0 \leq i \leq 4$ and $j \in \mathbb{Z}_n$, let $D_{ij} = \mathcal{V}_j - \cup_{\nu=0}^{t-1}\{v_{(3\nu+i)j}\}$ and let $\mu(m, n) = \frac{1}{3}(m+1)(n+1) - 2$.
If $n = 5$, then $S = D_{00} \cup D_{12} \cup (D_{24} - \{v_{(m-2)4}, v_{(m-1)4}\})$ is an LPDS order $6t + 4 = \mu(m, n)$ in $G$.
Assume that $n \geq 6$. Let $n = 6k + r$, where $k \geq 1$ and $0 \leq r \leq 5$. Let $S_0 = \cup_{j=0}^{k-1}(D_{0(6j)} \cup D_{1(6j+2)} \cup D_{2(6j+4)})$.
By using an identical proof as in the theorem 2.2, we have the following. If $r = 0$, then $S = S_0$ is an LPDS of order $6kt+6k = \frac{1}{3}(m+1)n$ in $G$. If $r = 1$, then $S = S_0 \cup (D_{0(n-1)} - \{v_{(m-2)(n-1)}, v_{(m-1)(n-1)}\})$ is an LPDS of order $6kt+6k+2t = \mu(m, n)$ in $G$. If $r = 2$, then $S = S_0 \cup (\mathcal{V}_{n-2} - \{v_{(m-1)(n-1)}\})$ is an LPDS of order $6kt + 6k + 3t + 1 = \mu(m, n)$ in $G$. If $r = 3$, then $S = (S_0 \cup D_{0(n-3)} \cup D_{1(n-1)}) - \{v_{(m-1)0}, v_{(m-2)(n-1)}\}$ is an LPDS of order $6kt + 6k + 4t + 2 = \mu(m, n)$ in $G$. If $r = 4$, $m = 5$, then $S = S_0 \cup D_{3(n-4)} \cup D_{4(n-2)}$ is an LPDS order $12k + 8 = \mu(m, n)$ in $G$. If $r = 4$,

$m > 5$, then $t \geq 2$. Then $S = S_0 \cup D_{3(n-4)} \cup D_{4(n-2)} \cup (\cup_{\ell=0}^{t-2}\{v_{(3\ell+3)(n-1)}\})$ is a LTDS of order $6kt + 6k + 5t + 3 = \mu(m,n)$ in $G$. If $r = 5$, then $S = S_0 \cup D_{0(n-5)} \cup D_{1(n-3)} \cup (D_{2(n-1)} - \{v_{(m-2)(n-1)}, v_{(m-1)(n-1)}\})$ is an LPDS of order $6kt + 6k + 6t + 4 = \mu(m,n)$ in $G$. This completes the proof. □

# 3 Locating-total domination number of $C_3 \square C_n$

In this section, we investigate the locating-total domination number of $C_3 \square C_n$.

**Lemma 3.1.** *For any integer $n$ with $n \geq 3$, $\gamma_t^L(C_3 \square C_n) \geq n$, with equality if and only if $n \equiv 0 \pmod 6$.*

Proof: Let $G \cong C_3 \square C_n$. By Lemma 2.3, $\gamma_t^L(G) \geq n$. Furthermore, if $n \equiv 0 \pmod 6$, $\gamma_t^L(G) = n$. Now we just need to prove that if $\gamma_t^L(C_3 \square C_n) = n$, then $n \equiv 0 \pmod 6$. Assume that $S$ is a $\gamma_t^L(G)$-set with $|S| = n$. Let $A = \{v \in V - S \mid |N(v) \cap S| = 1\}$ and let $B = (V - S) - A$. Then $|B| = 3n - |S| - |A|$. Since every vertex in $A$ is adjacent to exactly one vertex in $S$, while every vertex in $B$ is adjacent to at least two vertices in $S$, we have

$$e[S, V - S] \geq |A| + 2|B| = |A| + 2(3n - |S| - |A|) = 6n - 2|S| - |A|.$$

Since every vertex $v$ in $S$ is adjacent to at least one other vertex in $S$, $v$ is adjacent to at most 3 vertices in $V - S$. So, $e[S, V - S] \leq 3|S|$. Thus, $3|S| \geq 6n - 2|S| - |A|$, i.e., $5|S| + |A| \geq 6n$. Since $S$ is an LPDS of $G$, no two vertices in $A$ have the same neighbor in $S$. So $|A| \leq |S|$. Thus $|S| \geq n$. Since $|S| = n$, all above inequalities must be the equalities. That is, $3|S| = e[S, V - S] = |A| + 2|B| = 6n - 2|S| - |A|$. We can deduce that $|S| = |A| = |B| = n$; for any $v \in B, |N(v) \cap S| = 2$; and for any $v \in S, |N(v) \cap S| = |N(v) \cap A| = 1, |N(v) \cap B| = 2$. Hence, $S$ is a locating-paired-dominating set of $C_3 \square C_n$. Then $n$ is even.

It suffices to show that $n \equiv 0 \pmod 3$. Let $P_1, P_2, \ldots, P_{\frac{n}{2}}$ be all pairs of the locating-paired-dominating set, and let $\mathcal{P} = \{P_\ell \mid 1 \leq \ell \leq \frac{n}{2}, \ell = \text{integer}\}$. Fix the set $P_\ell \in \mathcal{P}$ with $1 \leq \ell \leq \frac{n}{2}$. Assume that $P_\ell = \{u, v\}$. Since $u, v$ have at most one common neighbor in $B$ and $|N(u) \cap B| = |N(v) \cap B| = 2$, $d(u, S - P_\ell) = d(v, S - P_\ell) = 2$.

We claim that there is no pair of $\mathcal{P}$ in some horizontal of $G$. If it is not the case, then, by symmetry, we may assume that $P_1 = \{v_{00}, v_{01}\}$ is in $\mathcal{H}_0$. Since $d(v_{00}, S - P_1) = d(v_{01}, S - P_1) = 2$, we have that $S \cap \mathcal{V}_0 = \{v_{00}\}$, $S \cap \mathcal{V}_1 = \{v_{01}\}$ and $v_{02} \notin S$. If $v_{02}$ is the private neighbor of $v_{01}$, then

$S \cap \mathcal{V}_2 = \emptyset$ and $d(v_{01}, S - P_1) \geq 3$, a contradiction. Hence, the private neighbor of $v_{01}$ is in $\mathcal{V}_1$. By symmetry, we may assume that $v_{11}$ is the private neighbor of $v_{01}$. Then $v_{21}$ is also a neighbor of $v_{01}$. So, $v_{22} \in S$ and $S \cap \mathcal{V}_2 = \{v_{22}\}$. Then $v_{23} \in S$ in order to totally dominate $v_{22}$. Since $d(v_{23}, S - \{v_{22}, v_{23}\}) = 2$, $S \cap \mathcal{V}_3 = \{v_{23}\}$. Thus, $N[v_{21}] \cap S = N[v_{02}] \cap S = \{v_{01}, v_{22}\}$, which is a contradiction to the fact that $S$ is a $\gamma_t^L(G)$-set. Consequently, for any pair $P_\ell$ in $\mathcal{P}$ with $1 \leq \ell \leq \frac{n}{2}$, $P_\ell$ must be in some vertical of $G$. Furthermore, since $|S| = |A| = |B| = n$, and for any vertex $v \in P_\ell$ with $1 \leq \ell \leq \frac{n}{2}$, $d(v, S - P_\ell) = 2$, then all pairs of $\mathcal{P}$ are either in all odd verticals of $G$ or in all even verticals of $G$.

Without loss of generality, we assume that for any integer $\ell$ with $1 \leq \ell \leq \frac{n}{2}$, $P_\ell$ is in $\mathcal{V}_{2\ell-1}$ and $P_1 = \{v_{01}, v_{11}\}$. Obviously, the private neighbors of $v_{01}$ and $v_{11}$ can not be in the same vertical. This implies that the private neighbors of $v_{01}$ and $v_{11}$ are in two different verticals. By symmetry, we may assume that the private neighbors of $v_{01}$ and $v_{11}$ are $v_{00}$ and $v_{12}$, respectively. Thus, $P_2 = \{v_{03}, v_{23}\}$ and the private neighbors of $v_{03}$ and $v_{23}$ are $v_{04}$ and $v_{22}$, respectively. Similarly, we have that $P_3 = \{v_{15}, v_{25}\}$ and the private neighbors of them are $v_{14}$ and $v_{26}$, respectively. Further, $v_{20} \in A$ and $B \cap (\cup_{i=0}^{5} \mathcal{V}_i) = \{v_{02}, v_{05}, v_{10}, v_{13}, v_{21}, v_{24}\}$. By that analogy, it is easy to see that in any six consecutive verticals, for any integer $i \in \mathbb{Z}_3$, $S \cap \mathcal{H}_i$, $A \cap \mathcal{H}_i$ and $B \cap \mathcal{H}_i$ have the same cardinality 2 and $|S \cap \mathcal{H}_0| = |S \cap \mathcal{H}_1| = |S \cap \mathcal{H}_2|$. This implies that $|S|$ must be a multiple of three. Therefore, $n \equiv 0 \pmod 6$. This completes the proof. □

**Theorem 3.2.** *For any integer $n$ with $n \geq 3$,*

$$\gamma_t^L(C_3 \Box C_n) = \begin{cases} n, & n \equiv 0 \pmod 6; \\ n+1, & \text{otherwise.} \end{cases}$$

Proof: Let $G \cong C_3 \Box C_n$. By Theorem 2.2 and Lemma 2.3, $n \leq \gamma_t^L(G) \leq n+1$. By Lemma 3.1, $\gamma_t^L(C_3 \Box C_n) = n$ if and only if $n \equiv 0 \pmod 6$. Therefore, when $n \not\equiv 0 \pmod 6$, $\gamma_t^L(G) = n+1$. This completes the proof. □

Finally, we pose the following open question that we have yet to settle.

**Question** *Is it true that $\gamma_t^L(C_4 \Box C_n) = \lceil \frac{3}{2} n \rceil$ for any positive integer $n$ such that $n \neq 4, 10$*