THE CLIQUE BEHAVIOR OF CIRCULANTS WITH THREE SMALL JUMPS

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ABSTRACT. The clique graph $K(G)$ of a graph G is the intersection graph of all its (maximal) cliques, and G is said to be *clique divergent* if the order of its n-th iterated clique graph $K^n(G)$ tends to infinity with n. In general, deciding whether a graph is clique divergent is not known to be computable. We characterize the dynamical behavior under the clique operator of circulant graphs of the form $C_n(a,b,c)$ with $0 < a < b < c < \frac{n}{3}$: Such a circulant is clique divergent if and only if it is not clique-Helly. Owing to the Dragan-Szwarcfiter Criterion to decide clique-Hellyness, our result implies that the clique divergence of these circulants can be decided in polynomial time. Our main difficulty was the case $C_n(1,2,4)$, which is clique divergent but no previously known technique could be used to prove it.

1. INTRODUCTION

We identify vertex sets with their induced subgraphs; in particular, we usually write $x \in G$ instead of $x \in V(G)$, although we may use the latter for the added emphasis. A *clique* of a graph is a maximal complete subgraph. The *clique graph* $K(G)$ of a graph G is the intersection graph of its cliques. *Iterated clique graphs* are defined inductively by $K^0(G) = G$ and $K^{n+1}(G) = K(K^n(G))$. The graph G is *clique divergent* if the orders of its iterated clique graphs are unbounded, otherwise we say that it is *clique convergent*. It is easy to see that any graph must be either clique divergent or clique convergent. Determining the *clique behavior* of a graph means determining whether it is clique divergent or clique convergent. In general,

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this is not known to be computable. For extensive bibliography on clique graphs and iterated clique graphs see [1, 36, 40]. More recent work may be found in [2–5, 7, 8, 15, 22, 23, 25–33, 42]. Iterated clique graphs have already found applications to Quantum Gravity [37–39].

There are no known general theorems characterizing the clique behavior of graphs. Several important families and classes of graphs have been studied instead. By dint of their great symmetry, circulants (defined below) are quite amenable to this kind of study and have been investigated already in [17, 23]. Besides their intrinsic interest, circulants and their clique behavior have been applied to characterize the clique behavior of circular arc graphs [30] and also the complements of some graphs admitting an admissible locally bijective coloring [32]. Furthermore clique behavior of circulants was applied to characterize self-clique Helly circular arc graphs [3]. In this work's main result (Thm. 2.9) we shall contribute to this program by characterizing the clique behavior of circulants with three small jumps, which in turn will be used in the investigation of the clique behavior of locally small graphs [24].

A graph G is *clique-Helly* (or simply *Helly*) if the set of cliques of G satisfies the Helly property: every collection of pairwise intersecting cliques has a non-empty total intersection. Equivalently, a graph G is Helly if every extended triangle is a cone [9, 41]. This characterization makes it possible to test for the Helly property in polynomial time. It is known that every Helly graph is clique convergent [10], but the opposite is not true in general [6, 10, 12, 21]. However, there are several families of graphs which are known to be divergent precisely when they are not Helly, such as cographs [16], complements of cycles [23, 34], powers of cycles [23] and chessboard graphs [25]. We shall prove that the circulants studied here also exhibit this property, so their divergence can be decided polynomially.

The circulant $C_n(a_1, a_2, \dots, a_m)$ is the graph defined on $\mathbb{Z}_n$ where two vertices $x, y \in \mathbb{Z}_n$ are adjacent if and only if $x - y \in \{\pm a_1, \pm a_2, \dots, \pm a_m\}$. Here we shall focus on circulants with only three *jumps*: $C_n(a, b, c)$ and we also require the jumps to be short enough, so that no three of them suffice to go all around the graph (i.e. $a, b, c < \frac{n}{3}$). The general case $a, b, c \leq \frac{n}{2}$ will require a more detailed analysis.

If H is a graph we say that the graph G is *locally* H if the neighbors of each vertex of G induce a subgraph isomorphic to H. We will deal with locally C_6 and locally A graphs, where C_6 is the 6-cycle and A is the graph on 6 vertices and 6 edges that looks like A. The following theorem is a restatement of Theorem 0.1 in [27].

Theorem 1.1. [27] *Let* $C = C_n(a, b, c)$ *with* $0 < a < b < c < \frac{n}{3}$. *Then:*

(1) *C is locally C_6 if and only if $a + b = c$ and $b \neq 2a$.*

(2) *C is locally A if and only if $b = 2a$ and $c = 4a$.*

(3) *C is clique Helly if and only if it is not locally C_6 nor locally A.*

In the second case, if $\gcd(n, a) = d$ *and* $a = sd$, $C_n(a, 2a, 4a)$ *is isomorphic to* $C_n(d, 2d, 4d)$ *which in turn is isomorphic to the disjoint union of d copies of* $C_{\frac{n}{d}}(1, 2, 4)$. □

An extended abstract reporting part of this work, without proofs, was published in [27].

2. Coil Graphs

By Theorem 1.1, non-Helly circulants of the kind described can only be either locally C_6 or (a number of disjoint copies of) the circulant $C_n(1, 2, 4)$. It is known that locally C_6 graphs are all clique divergent [18, 19]. Hence, only the case of the circulant $C_n(1, 2, 4)$ remains to be studied for $n \geq 13$. As it turns out, no previously known technique could be used to prove the divergence of these circulants. For instance, triangular coverings [19] can be used to prove that for $n = 11$ and $n \geq 13$, all of the graphs $C_n(1, 2, 4)$ have the same clique behavior (i.e. either they are all clique divergent or they are all clique convergent) but this technique does not allow us to determine whether this behavior is convergent or divergent. $C_n(1, 2, 4)$ does not have any local cutpoints nor dominated vertices, so the corresponding techniques [11, 12] do not work here. Also, $C = C_n(1, 2, 4)$ becomes clique convergent when we remove any vertex (as $K^4(C - \{0\})$ is already a Helly graph), this implies that C does not have any clique divergent retract (other than itself) hence retractions [34, 35] can not be used here. Actually, for even n, it could be the case that C is rank divergent [22] with its antipodal coaffination: the sequence of ranks of the iterated clique graphs of C is $7, 14, 14, 14, 28, \ldots$ (and then computer crashes due to lack of memory). But as in the case of retractions, it is easy to prove that C does not contain any proper subgraph which is also rank divergent, so there is no way to simplify the problem here. Similar considerations apply to expansivity [23, 35], local colorability [32] and other techniques.

The general strategy to prove that $C_n(1, 2, 4)$ is clique divergent is as follows: First we define the class of coil graphs. Then we prove that $K(G)$ is also a coil graph for any coil graph G and that $|K(G)| \geq 2 \cdot |G|$ for these graphs, so every coil graph is clique divergent. The result finally follows by observing that $K^2(C_n(1,2,4))$ belongs to the class of coil graphs.

We describe the notation that we will use in the definition of coil graphs. We use interval notation to denote sets of consecutive integers: $[a,b] = \{a, a+1, \ldots, b\}$. The vertex set of a coil graph is partitioned into orbits, and each of these orbits is indexed by a non negative integer. Given two orbits indexed by a, b, the adjacencies between these two orbits will be determined by a certain set T^{ab}. Any such set T^{ab} must always be one of those which we shall now define, and must also satisfy the requirements of Definition 2.1.

Our first five sets are intervals, and they will describe adjacencies among different orbits: $T_2 = [-2,4]$, $T_1 = [-2,3]$, $T_0 = [-3,3]$, $T_{-1} = [-3,2]$, $T_{-2} = [-4,2]$. See the middle column of Fig. 1, where fat and thin vertices are meant only to aid the visualization of the patterns involved (the right column will be explained later). For adjacencies within orbits, we shall also need $T_{124} = \{\pm 1, \pm 2, \pm 4\}$ and $T_{12} = \{\pm 1, \pm 2\}$, which are not intervals. All these sets will be thought of as subsets of some $\mathbb{Z}_n$ and, as said above, T^{ab} must always be one of these.

Definition 2.1. *A coil graph G (see Fig. 2) is a graph in which $\mathbb{Z}_n$ acts freely ($n \geq 13$), so that it decomposes into $r+1 \geq 2$ orbits $G/\mathbb{Z}_n = \{G^0, G^1, \ldots, G^r\}$ with $G^a = \{x_i^a \mid i \in \mathbb{Z}_n\}$, whose adjacencies are given by $x_i^a \sim x_j^b$ iff $(j-i) \in T^{ab}$ (which implies $T^{ba} = -T^{ab}$) and such that:*

(1) $T^{00} = T_{124}$.

(2) *For $a \neq 0$, $T^{aa} = T_{12}$ and $T^{0a} = T^{a0} = T_0$.*

(3) $T^{ab} \in \{T_2, T_1, T_0, T_{-1}, T_{-2}\}$ *for $a, b \neq 0$, $a \neq b$.*

(4) *For all $x, y \in G$ with $x \sim y$ we have that $|N[x] \cap N[y] \cap G^a| \geq 3$ for $a = 0, 1, \ldots, r$.*

In a coil graph G, we say that G^0 is the *special orbit* and that, for $a \geq 1$, G^a is an *ordinary orbit.* Also, edges of the form $x_i^0 x_{i+4}^0$ are called *long edges.* An example of a coil graph is represented in Fig. 2. There, each row of vertices represents one of the four orbits of G. The number n is not specified and may be any integer $n \geq 13$. Adjacencies between orbits are given by the following *connection matrix*, whose row and column indices

$T_2 = [-2, 4]$

$T_1 = [-2, 3]$

$T_0 = [-3, 3]$

$T_{-1} = [-3, 2]$

$T_{-2} = [-4, 2]$

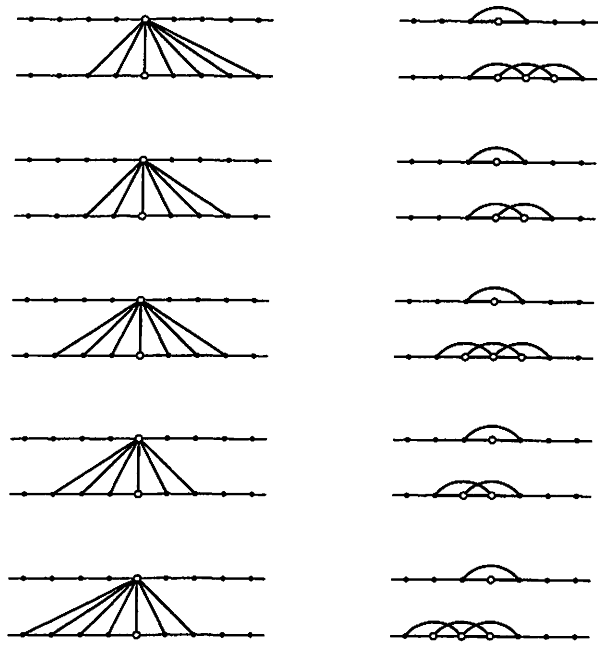

FIGURE 1. The five types of connections between distinct orbits.

start at 0:

$$(T^{ab}) = \begin{pmatrix} T_{124} & T_0 & T_0 & T_0 \\ T_0 & T_{12} & T_1 & T_2 \\ T_0 & T_{-1} & T_{12} & T_1 \\ T_0 & T_{-2} & T_{-1} & T_{12} \end{pmatrix}$$

Most edges are not drawn, but all the edges that are not depicted are obtained from those already drawn by means of translations by elements in $\mathbb{Z}_n$. It can be verified by direct means that the graph G in Fig. 2 is precisely $K^3(C_n(1,2,4))$, but we shall not use this fact in the proof. On the other hand, $K^2(C_n(1,2,4))$ is isomorphic to the subgraph of G induced by the first two rows in the figure: $G^0 \cup G^1$.

Lemma 2.2. *Let k be a natural number. If $\mathcal{F}$ is a finite non-empty family of finite integer intervals such that for any $X, Y \in \mathcal{F}$ we have $|X \cap Y| \geq k$, then $|\bigcap \mathcal{F}| \geq k$.*

Proof. Take $A \in \mathcal{F}$ such that $\max A = \min\{\max X \mid X \in \mathcal{F}\}$, and also take $B \in \mathcal{F}$ such that $\min B = \max\{\min X \mid X \in \mathcal{F}\}$. Then $A \cap B \subseteq \bigcap \mathcal{F}$, and $|A \cap B| \geq k$. □

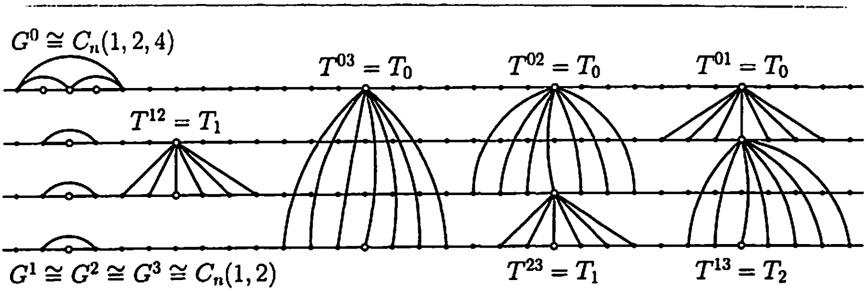

FIGURE 2. An example of a coil graph: $G = K^3(C_n(1,2,4))$.

By the *common neighbourhood* of $Y \subseteq V(G)$ we shall mean the set $N[Y] = \bigcap_{y \in Y} N[y]$.

Lemma 2.3. *Let Q be a clique of a graph G. If X is a set of vertices of G such that $S = X \cap N[Q \setminus X] \neq \varnothing$, then $X \cap Q \neq \varnothing$. Under the same hypothesis, we also have that $X \cap Q \subseteq S$ and that $X \cap Q$ is a clique of the subgraph of G induced by S.*

Proof. $X \cap Q = \varnothing$ implies $N[Q \setminus X] = N[Q] = Q$ and $S = X \cap Q = \varnothing$, a contradiction. Since $Q \subseteq N[Q \setminus X]$, we have $X \cap Q \subseteq S$. Finally, just observe that $S \cap N[X \cap Q] = X \cap N[Q \setminus X] \cap N[X \cap Q] = X \cap N[Q] = X \cap Q$, and therefore $X \cap Q$ is a clique of S. □

Theorem 2.4. *If G is a coil graph, so is $K(G)$.*

Proof. We need to know the cliques of G (see Fig. 3 for an example). As $\mathbb{Z}_n$ acts on G, it suffices to know a set of representatives of those cliques under this action. It will be seen that we can focus on the cliques Q where $Q \cap G^0$ is centered around x_0^0: all other cliques are obtained from these via translations by elements of $\mathbb{Z}_n$. Note that the cliques of the subgraphs G^a are precisely their triangles: up to translations, the cliques of G^0 are either $q_0^0 := \{x_{-1}^0, x_0^0, x_1^0\}$ or $p_0^0 := \{x_{-2}^0, x_0^0, x_2^0\}$ and, for $a > 0$, G^a only has the clique $q_0^a := \{x_{-1}^a, x_0^a, x_1^a\}$ up to translations. It follows that each clique $Q \in K(G)$ has at most 3 vertices in each orbit of G (i.e. $|Q \cap G^a| \leq 3$). Let us show that they also have at least 3.

Claim 1: Every clique Q of G has exactly 3 vertices in each orbit: $|Q \cap G^a| = 3$ for all a.

Assume first that Q contains some long edge of G^0, say $x_{-2}^0 x_2^0 \in Q$. The common neighbors of these two vertices are exactly those in $\{x_0^0\} \cup q_0^1 \cup q_0^2 \cup \cdots \cup q_0^r$. A direct verification shows that all these vertices induce a complete subgraph of G regardless of the particular connections T^{ab} between the orbits of G. Hence, if $x^0_{-2}x^0_2 \in Q$, then Q is univocally determined to be $Q = p^0_0 \cup q^1_0 \cup q^2_0 \cup \cdots \cup q^r_0$. If the long edge were $x^0_{i-2}x^0_{i+2} \in Q$, using translations we would get that $Q = p^0_i \cup q^1_i \cup q^2_i \cup \cdots \cup q^r_i$, which satisfies our claim in this case.

Suppose from now on that Q contains no long edge $x^0_{k-2}x^0_{k+2}$.

By Definition 2.1, whenever $x^a_i \sim x^b_j$ we know that $|j-i| \leq 4$. Since $n \geq 13$, despite the cyclic arrangement of vertices in G it makes sense to speak about a leftmost vertex $x^a_u \in Q$ and a rightmost vertex $x^b_v \in Q$, so that whenever $x^c_i \in Q$ we must have $i \in [u, v]$. We now apply Lemma 2.3 with $X = G^0$. Since $v - u \leq 4$ and the connection T^{0a} from the orbit G^0 to any other orbit G^a is always T_0, we have that $S = N[Q \setminus G^0] \cap G^0 \neq \varnothing$ has at least three vertices and is an interval of integers. Since $Q \cap G^0$ is a clique of $G[S]$, it follows that $|Q \cap G^0| \geq 3$, so $|Q \cap G^0| = 3$ and $Q \cap G^0 = q^0_i$ for some $i \in \mathbb{Z}_n$.

Let $a \neq 0$. For every vertex $x \in G$ the set $N[x] \cap G^a$ is always a set of consecutive vertices and, by Definition 2.1(4), for every pair $x, y \in Q$ the set $N[x] \cap N[y] \cap G^a = (N[x] \cap G^a) \cap (N[y] \cap G^a)$ has at least 3 elements. Then we apply Lemma 2.2 to get that $Q \cap G^a = (\bigcap_{x \in Q} N[x]) \cap G^a = \bigcap_{x \in Q}(N[x] \cap G^a)$ has at least 3 vertices. Hence $|Q \cap G^a| = 3$ for all a as claimed.

Claim 2: If $Q \cap G^0 = p^0_i$ then $Q \cap G^a = q^a_i$ for all $a > 0$. Otherwise $Q \cap G^0 = q^0_i$ and, for all $a > 0$, $Q \cap G^a = q^a_{t_a + i}$ for some $t_a \in \{-1, 0, 1\}$.

When $Q \cap G^0 = p^0_i$, Q contains the long edge $x^0_{i-2}x^0_{i+2}$ and by the previous discussion we have $Q = p^0_i \cup q^1_i \cup q^2_i \cup \cdots \cup q^r_i$. Otherwise, up to translations, $Q \cap G^0 = q^0_0$, and since $|Q \cap G^a| = 3$, it follows that $Q \cap G^a$ is a triangle of the form $q^a_{t_a}$, but $q^a_{t_a}, q^0_0 \subseteq Q$ implies (because of $T^{0a} = T_0$) that $t_a \in \{-1, 0, 1\}$, as claimed.

Note however that some combinations of values of t_i are not valid, for instance $q^a_{-1}, q^b_1 \subseteq Q$ is only possible when $T^{ab} = T_2$. The third column in Fig. 1 shows the valid combinations for each possible type of connection.

The action of $\mathbb{Z}_n$ on G is inherited by $K(G)$ in the obvious way. Let $X^0_0 := p^0_0 \cup q^1_0 \cup q^2_0 \cup \cdots \cup q^r_0$ and $X^\alpha_0 := q^0_0 \cup q^1_{t_1(\alpha)} \cup q^2_{t_2(\alpha)} \cup \cdots \cup q^r_{t_r(\alpha)}$ with $1 \leq \alpha < |K(G)|/n$ and $t_a(\alpha) \in \{-1, 0, 1\}$ be the cliques just described when $Q \cap G^0$ is centered around x^0_0 (see Fig. 3). Also, let $X^\alpha_i := X^\alpha_0 + i$ be the translation of X^α_0 by the element $i \in \mathbb{Z}_n$. It is clear now that $V(K(G)) = \{X^\alpha_i \mid i \in \mathbb{Z}_n, 0 \leq \alpha < |K(G)|/n\}$, and that the action of $\mathbb{Z}_n$

on $K(G)$ is free. We shall use Greek letters to index the orbits $(KG)^\alpha$ of $K(G)$, and $\mathcal{T}^{\alpha\beta}$ to specify the connections between these orbits.

Now we are ready to prove that $K(G)$ satisfies properties (1)-(4) of Definition 2.1.

Proof of (1): Recall that $X_i^0 = p_i^0 \cup q_i^1 \cup q_i^2 \cup \cdots \cup q_i^r$. Now a direct verification shows that for $i \neq j$, $X_i^0 \cap X_j^0 \neq \varnothing$ iff $[p_i^0 \cap p_j^0 \neq \varnothing$ or $q_i^1 \cap q_j^1 \neq \varnothing]$ iff $(j-i) \in T_{124}$.

Proof of (2): Since $X_i^\alpha = q_i^0 \cup q^1_{t_1(\alpha)+i} \cup q^2_{t_2(\alpha)+i} \cup \cdots \cup q^r_{t_r(\alpha)+i}$, it follows that for $i \neq j$, $X_i^\alpha \cap X_j^\alpha \neq \varnothing$ iff $q_i^0 \cap q_j^0 \neq \varnothing$ iff $(j-i) \in T_{12}$. Also, for $\alpha > 0$, $X_i^0 \cap X_j^\alpha \neq \varnothing$ iff $p_i^0 \cap q_j^0 \neq \varnothing$ iff $(j-i) \in T_0$.

Proof of (3): For all $\gamma \in [1, |K(G)|/n]$ define $t_0(\gamma) = 0$, so we can write $X_i^\alpha = q^0_{t_0(\alpha)+i} \cup q^1_{t_1(\alpha)+i} \cup \cdots \cup q^r_{t_r(\alpha)+i}$ and $X_j^\beta = q^0_{t_0(\beta)+j} \cup q^1_{t_1(\beta)+j} \cup \cdots \cup q^r_{t_r(\beta)+j}$. Take $\alpha, \beta \neq 0$, $\alpha \neq \beta$. Then $X_i^\alpha \cap X_j^\beta \neq \varnothing$ iff there is an $a \in [0, r]$ such that $q^a_{t_a(\alpha)+i} \cap q^a_{t_a(\beta)+j} \neq \varnothing$, and this holds iff $\exists a$ $(t_a(\beta)+j) - (t_a(\alpha)+i) \in [-2,2]$ iff $\exists a$ $(j-i) \in [-2,2] + (t_a(\alpha) - t_a(\beta))$. Hence:

$$X_i^\alpha \cap X_j^\beta \neq \varnothing \text{ iff } (j-i) \in \mathcal{T}^{\alpha\beta} = \bigcup_{a=0}^{r} ([-2,2] + (t_a(\alpha) - t_a(\beta)))$$

Now, since $t_0(\alpha) - t_0(\beta) = 0$ we have that $[-2,2] \subseteq \mathcal{T}^{\alpha\beta}$. Furthermore, since $t_a(\alpha), t_a(\beta) \in [-1,1]$, it follows that $t_a(\alpha) - t_a(\beta) \in [-2,2]$, and since $\mathcal{T}^{\alpha\beta}$ is a union as above, it follows that $\mathcal{T}^{\alpha\beta}$ must be one of the following intervals: $[-2,4]$, $[-2,3]$, $[-3,3]$, $[-3,2]$, $[-4,2]$, $[-4,4]$, $[-4,3]$, $[-3,4]$ or $[-2,2]$. The first five of these are precisely the intervals $T_2, T_1, T_0, T_{-1}, T_{-2}$ which satisfy (3) in the definition of coil graphs, the remaining four cases can not occur:

Case $[-4,4]$: In this case we have some a, b such that $t_a(\alpha) - t_a(\beta) = -2$ and $t_b(\alpha) - t_b(\beta) = 2$. This implies that $t_a(\alpha) = -1$, $t_a(\beta) = 1$, $t_b(\alpha) = 1$, $t_b(\beta) = -1$. But then, on the one hand, $t_a(\alpha) = -1$ and $t_b(\alpha) = 1$ imply $q^a_{i-1}, q^b_{i+1} \subseteq X_i^\alpha$, which we know to force the connections from G^a to G^b to be given by $T^{ab} = T_2$ (see Fig. 1, third column) and, on the other hand, $t_a(\beta) = 1$ and $t_b(\beta) = -1$ imply (by the same argument) that $T^{ab} = T_{-2}$, a contradiction.

Case $[-4,3]$: Here we have some a, b such that $t_a(\alpha) - t_a(\beta) = -2$ and $t_b(\alpha) - t_b(\beta) = 1$, which implies that $t_a(\alpha) = -1$, $t_a(\beta) = 1$ and either $t_b(\alpha) = 1$, $t_b(\beta) = 0$ or $t_b(\alpha) = 0$, $t_b(\beta) = -1$. As before, in the first case $t_a(\alpha) = -1$ and $t_b(\alpha) = 1$ imply $T^{ab} = T_2$, incompatible with $T^{ab} \in \{T_0, T_{-1}, T_{-2}\}$ which is implied by $t_a(\beta) = 1$, $t_b(\beta) = 0$. Similarly, in the second case, $t_a(\alpha) = -1$ and $t_b(\alpha) = 0$ imply $T^{ab} \in \{T_0, T_1, T_2\}$, incompatible with $T^{ab} = T_{-2}$ which is implied by $t_a(\beta) = 1$, $t_b(\beta) = -1$.

Case $[-3, 4]$: Dual of the previous case.

Case $[-2, 2]$: This means that $t_a(\alpha) = t_a(\beta)$ for all a, which implies that X_i^α and X_j^β lie in the same orbit and hence $\alpha = \beta$, contradicting our assumption.

Proof of (4): If X, Y are adjacent vertices of $K(G)$, then they are intersecting cliques of G. Let $x_i^a \in X \cap Y$ and let $(KG)^\alpha$ with $\alpha \geq 0$ be any orbit of $K(G)$, if X_j^α is a vertex of $K(G)$ in that orbit, then we know that $|X_j^\alpha \cap G^a| = 3$ by Claim 1, but then there are three elements $k \in \mathbb{Z}_n$ such that $x_i^a \in X_j^\alpha + k = X_{j+k}^\alpha$, hence $|N[X] \cap N[Y] \cap (KG)^\alpha| \geq 3$ as required.
□

We say that the ordinary orbits of the coil graph in Fig. 2 are *nicely ordered*, in the sense that all the entries above the main diagonal of the connection matrix (T^{ab}) belong to the set $\{T_2, T_1, T_0\}$. In that example, any transposition of two of the three ordinary orbits would spoil the niceness of the order. The orbits of the cliques in Fig. 3 will become nicely ordered if one inserts the orbit of X_0^8 immediately before (or after) that of X_0^4.

Lemma 2.5. *The ordinary orbits of any coil graph G can be nicely ordered.*

Proof. Let $\mathcal{O} = \{G^1, G^2, \ldots, G^r\}$ be the set of ordinary orbits of G. We begin by observing that Definition 2.1(4) prohibits some combinations of connections: If $a, b, c \in [1, r]$ are different, $T^{ab} \in \{T_2, T_1, T_0\}$ and $T^{bc} \in \{T_2, T_1\}$, then we cannot have $T^{ac} \in \{T_{-1}, T_{-2}\}$, for otherwise we would get $x_0^a \sim x_3^b$ and $|N[x_0^a] \cap N[x_3^b] \cap G^c| = 2 < 3$.

Next we define a digraph D whose vertex set is $\mathcal{O}$ and in which there is an arrow $G^a \to G^b$ whenever $T^{ab} \in \{T_2, T_1\}$. We claim that D is acyclic. By the above prohibitions, D has no oriented triangle. Suppose that D has an oriented cycle $C = (G^{a_1}, G^{a_2}, \ldots, G^{a_t})$, and suppose that the length t is minimal. Then $t \geq 4$ and C is an induced subdigraph of D, which means that $T^{a_i a_j} = T_0$ whenever a_i and a_j are distinct and not contiguous in C. But then $T^{a_1 a_{t-1}} = T_0$, $T^{a_{t-1} a_t} \in \{T_2, T_1\}$ and $T^{a_1 a_t} \in \{T_{-1}, T_{-2}\}$, which violates our prohibitions.

As D is acyclic, its arrows define a binary relation on $\mathcal{O}$ whose transitive closure is a partial order, and this in turn can be extended to a (strict) linear order. In other words, D can be extended to a transitive tournament on the same vertex set $\mathcal{O}$. But any arrow of this tournament that was not already present in D joins orbits linked by a T_0 connection, and since $-T_0 = T_0$, we see that our linear order is nice. □

Theorem 2.6. *If G is a coil graph, then $|K(G)| \geq 2 \cdot |G|$. In particular, every coil graph is clique divergent.*

Proof. As we know, the special orbit of $K(G)$ is generated by the clique $X_0^0 := p_0^0 \cup q_0^1 \cup q_0^2 \cup \cdots \cup q_0^r$. By Lemma 2.5 we can assume that the orbits $G^0, G^1, G^2, \ldots, G^r$ of G satisfy $T^{ab} \in \{T_2, T_1, T_0\}$ (so $[-2,3] \subseteq T^{ab}$) for $0 \leq a < b \leq r$. Therefore, if $0 \leq a < b \leq r$, every vertex in the triangle q_i^a is adjacent to (at least) every vertex in the triangles q_i^b and q_{i+1}^b. Thus, starting with the clique $X_0^1 := q_0^0 \cup q_1^1 \cup q_1^2 \cup \cdots \cup q_1^r$ of G, we can "pull back" one triangle at a time to obtain always a new clique of G (see Fig. 3), namely:

$$\begin{aligned}
X_0^2 &:= q_0^0 \cup q_0^1 \cup q_1^2 \cup q_1^3 \cup \cdots \cup q_1^{r-1} \cup q_1^r, \\
X_0^3 &:= q_0^0 \cup q_0^1 \cup q_0^2 \cup q_1^3 \cup \cdots \cup q_1^{r-1} \cup q_1^r, \\
X_0^4 &:= q_0^0 \cup q_0^1 \cup q_0^2 \cup q_0^3 \cup \cdots \cup q_1^{r-1} \cup q_1^r, \\
&\cdots \\
X_0^{r+1} &:= q_0^0 \cup q_0^1 \cup q_0^2 \cup q_0^3 \cup \cdots \cup q_0^{r-1} \cup q_0^r.
\end{aligned}$$

Also, as $T^{0a} = T_0 = [-3,3]$, we can keep pulling triangles back to obtain some extra cliques:

$$\begin{aligned}
X_0^{r+2} &:= q_0^0 \cup q_{-1}^1 \cup q_0^2 \cup q_0^3 \cup \cdots \cup q_0^{r-1} \cup q_0^r, \\
X_0^{r+3} &:= q_0^0 \cup q_{-1}^1 \cup q_{-1}^2 \cup q_0^3 \cup \cdots \cup q_0^{r-1} \cup q_0^r, \\
X_0^{r+4} &:= q_0^0 \cup q_{-1}^1 \cup q_{-1}^2 \cup q_{-1}^3 \cup \cdots \cup q_0^{r-1} \cup q_0^r, \\
&\cdots \\
X_0^{2r+1} &:= q_0^0 \cup q_{-1}^1 \cup q_{-1}^2 \cup q_{-1}^3 \cup \cdots \cup q_{-1}^{r-1} \cup q_{-1}^r.
\end{aligned}$$

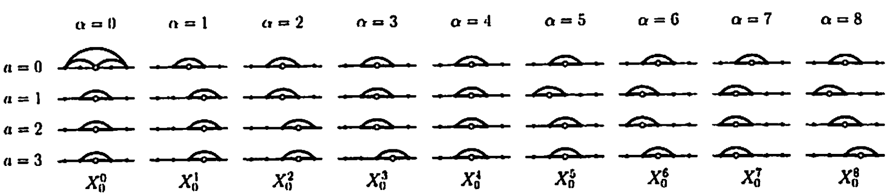

FIGURE 3. The nine types of cliques of $K^3(C_n(1,2,4))$.

The intersection of each of the cliques $X_0^0, X_0^1, \ldots, X_0^{2r+1}$ with the special orbit G^0 of G is centered at x_0^0, so all these cliques generate different orbits. Thus we see that $K(G)$ has at least twice the number of orbits as G, which implies $|K(G)| \geq 2 \cdot |G|$. This, together with Theorem 2.4, proves that G is clique divergent. Fig. 3 shows one orbit, the last, not obtained by the above process. □

Theorem 2.7. *The circulant $C_n(1,2,4)$ with $n \geq 13$ is clique divergent.*

Proof. A direct verification shows that $K^2(C_n(1,2,4))$ is the coil graph with two orbits (which for each n is unique up to isomorphism). Then apply Theorem 2.6. □

Proposition 2.8. *Every locally ᗩ graph is clique divergent.*

Proof. By [13, 4.10] the connected locally ᗩ graphs are precisely $C_n(1,2,4)$ for $n = 11$ or $n \geq 13$. The case $n \geq 13$ is our Theorem 2.7. The natural group quotient $\mathbb{Z}_{22} \to \mathbb{Z}_{11}$ yields a finite (2-to-1) triangular covering map $C_{22}(1,2,4) \to C_{11}(1,2,4)$, so $C_{11}(1,2,4)$ is clique divergent by [19, Cor. 2.3]. □

The Theory of Iterated Clique Graphs, developed in the last 41 years since Hedetniemi and Slater [14], is already rich enough to determine easily the clique behavior of all the instances of $C_n(1,2,4)$: When $n \leq 5$ or $n = 7$, $C_n(1,2,4)$ is a complete graph, so it is clique convergent. $C_6(1,2,4)$ is the octahedron, which is known to be clique divergent [10, 34]. $C_8(1,2,4)$ is clique divergent because its clique graph is the suspension of C_8^2 (i.e. $C_8^2 + I_2$), which is expansive by Neumann-Lara's Connected Summand Theorem [23, Thm. 4.6]. We just saw that $C_n(1,2,4)$ is clique divergent for $n = 11$ and $n \geq 13$ because of Theorem 2.8 and Hall's result 4.10 in [13]. It is also true that $C_n(1,2,4)$ is clique divergent for $n = 9, 10$ and 12, as we shall show in a forthcoming paper in which, for every possible graph H of order at most 6, we will describe (with one exception) the clique behavior of every locally H graph.

Theorem 2.9. *The circulant $C_n(a,b,c)$ with $0 < a < b < c < \frac{n}{3}$ is clique divergent if and only if it is not clique Helly.*

Proof. Immediate from Theorems 1.1 and 2.7, and the known result that every locally C_6 graph is clique divergent [18, 19]. □

It is clear that every circulant $C_n(J)$ with $|J| < 3$ and $\max J < \frac{n}{3}$ is Helly, so the above theorem may be extended to cover also these cases if so desired.

So far, whenever the *clique growth function* $f_G(n) = |K^n(G)|$ of a graph G is sufficiently known, $f_G(n)$ is either non-elementary (grows faster than any elementary function, super-exponential in particular [23, 34]) or it is polynomial (most often linear [17–20], but using strong products we can realize clique growths which are quadratic, cubic and so on [18]). Because of this, Theorem 2.6 is suggestive in this regard and motivates us to ask:

Problem 1. *Is there any graph G whose clique growth function is exponential? More explicitly: is there a graph G such that $|K^n(G)| = \Theta(t^n)$ for some $t > 1$? Is there a graph such that $|K^n(G)| = t^n|G|$ for some $t > 1$? Is this possible for $t = 2$?*

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