Split graphs whose regular endomorphisms form a monoid *

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Abstract

In this paper, the regular endomorphisms of a split graph are investigated. We give a condition under which the regular endomorphisms of a split graph form a monoid.

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1 Introduction and preliminaries

Endomorphism monoids of graphs are generalizations of automorphism groups of graphs. In recent years much attention has been paid to endomorphism monoids of graphs and many interesting results concerning graphs and their endomorphism monoids have been obtained (see [6], [7] and their references). The aim of this research is to develop further relations between graph theory and algebraic theory of semigroups and to apply the theory of semigroups to graph theory. Hou, Luo and Cheng [6] explored the endomorphism monoid of $\overline{P_n}$, the complement of a path P_n with n vertices. It was shown that $End(\overline{P_n})$ is an orthodox monoid. The endomorphism spectrum and the endomorphism type of $\overline{P_n}$ were given. The endomorphism monoids and endomorphism-regularity of split graphs were considered by several authors (see [2], [3], [10] and [13]). Let X be a graph. Denote by $End(X)$ the set of all endomorphisms of X. It is well known that $End(X)$ forms a monoid with respect to composition of mappings. Let $f \in End(X)$. Then f is called a *regular endomorphism*

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of X if it is a regular element in $End(X)$. Denote by $rEnd(X)$ the set of all regular endomorphisms of X. For a monoid S, the composition of two regular elements of S is not regular in general. So it is natural to ask: Under what conditions does the set $rEnd(X)$ form a monoid for a graph X? However, it seems difficult to obtain a general answer to this question. So the strategy for answering this question is to find various kinds of conditions for various kinds of graphs. In this paper we shall give an answer to this question in the range of split graphs.

The graphs considered in this paper are finite undirected graphs without loops and multiple edges. Let X be a graph. The vertex set of X is denoted by $V(X)$ and the edge set of X is denoted by $E(X)$. If two vertices x_1 and x_2 are adjacent in the graph X, the edge connecting x_1 and x_2 is denoted by $\{x_1, x_2\}$ and we write $\{x_1, x_2\} \in E(X)$. For a vertex v of X, denote by $N_X(v)$ (or just by $N(v)$) the set $\{x \in V(X) | \{x, v\} \in E(X)\}$, the cardinality of $N_X(v)$ is called the *degree* or *valency* of v in X and is denoted by $d_X(v)$ (or just $d(v)$). A subgraph H is called an *induced subgraph* of X if for any $a, b \in H$, $\{a, b\} \in E(H)$ if and only if $\{a, b\} \in E(X)$. A graph X is called *complete* if for any $a, b \in V(X)$, $\{a, b\} \in E(X)$. We denote by K_n (or just K) a complete graph with n vertices. A *clique* of a graph X is a maximal complete subgraph of X. A subset $K \subseteq V(X)$ is said to be *complete* if $\{a, b\} \in E(X)$ for any two vertices $a, b \in K$. A subset $S \subseteq V(X)$ is said to be *independent* if $\{a, b\} \notin E(X)$ for any two vertices $a, b \in S$. A graph X is called a *split graph* if its vertex set $V(X)$ can be partitioned into disjoint (non-empty) sets K and S such that K is a complete set and S is an independent set. In this paper, we always assume that a split graph X has a fixed partition $V(X) = K \cup S$, where $K = \{x_1, \cdots, x_n\}$ is a maximum complete set and $S = \{y_1, \cdots, y_m\}$ is an independent set. Since K is a maximum complete set of X, it is easy to see that for any $y \in S$, $0 \leq d_X(y) \leq n - 1$.

Let X and Y be two graphs. A mapping f from $V(X)$ to $V(Y)$ is called a *homomorphism* (from X to Y) if $\{x_1, x_2\} \in E(X)$ implies that $\{f(x_1), f(x_2)\} \in E(Y)$. A homomorphism f from X to itself is called an *endomorphism* of X. A endomorphism f is called *half-strong* if $\{f(a), f(b)\} \in E(X)$ implies that there exist $x_1, x_2 \in V(X)$ with $f(x_1) = f(a)$ and $f(x_2) = f(b)$ such that $\{x_1, x_2\} \in E(X)$. The sets of all endomorphisms and half-strong endomorphisms of X are denoted by $End(X)$ and $hEnd(X)$, respectively.

A *retraction* of a graph X is a homomorphism f from X to a subgraph Y of X such that the restriction $f|_Y$ of f to $V(Y)$ is the identity mapping on $V(Y)$. It is known that the idempotents of $End(X)$ are retractions of X. Denote by $Idpt(X)$ the set of all idempotents of $End(X)$. Let f be an endomorphism of a graph X. A subgraph of X is called the *endomorphic image* of X under f, denoted by I_f, if $V(I_f) = f(V(X))$

and $\{f(a), f(b)\} \in E(I_f)$ if and only if there exist $c \in f^{-1}(f(a))$ and $d \in f^{-1}(f(b))$ such that $\{c, d\} \in E(X)$. By ρ_f we denote the equivalence relation on $V(X)$ induced by f, i.e., for $a, b \in V(X)$, $(a, b) \in \rho_f$ if and only if $f(a) = f(b)$. Denote by $[a]_{\rho_f}$ the equivalence class containing $a \in V(X)$ with respect to ρ_f. Let X be a graph and $f \in End(X)$. For any $a \in I_f$, let

$$A_a^f = \cup_{x \in f^{-1}(a)} N(x).$$

Let G be a semigroup. An element a of S is called *regular* if there exists $x \in S$ such that $axa = a$. A semigroup G is called regular if all its elements are regular.

We shall use the standard terminology and notation of semigroup theory as in [9] and of graph theory as in [1,4,8]. We list some known results which will be used in this paper.

Lemma 1.1([11]) Let X be a graph and $f \in End(X)$. Then
(1) $f \in hEnd(X)$ if and only if I_f is an induced subgraph of X.
(2) If f is regular, then $f \in hEnd(X)$.

Lemma 1.2([12]) Let X be a graph and $f \in End(X)$. Then f is regular if and only if there exists $g, h \in Idpt(X)$ such that $\rho_g = \rho_f$ and $I_h = I_f$.

Lemma 1.3([13]) Let X be a split graph and $f \in End(X)$. Then f is half-strong if and only if $f(A_a^f) = N(a) \cap I_f$ for any $a \in S \cap I_f$ and for any $a \in K \cap I_f$ with $f^{-1}(a) \subset S$.

Lemma 1.4([5]) The regular elements of a semigroup S form a subsemigroup if (and clearly only if) the product of any two idempotents of S is a regular element.

2 Main results

In this section, we shall investigate the regular endomorphisms of a split graph and give the condition under which all regular endomorphisms of a split graph form a monoid.

First we give a characterization of the regular endomorphisms for a split graph X.

Lemma 2.1 Let X be a split graph and $f \in End(X)$. Then the following statements are equivalent.
(1) There exists $h \in Idpt(X)$ such that $I_h = I_f$.
(2) I_f is an induced subgraph of X and $\{x, y\} \notin E(X)$ for any $x \in K \backslash I_f$ and $y \in S \cap I_f$.

Proof (1) $\Rightarrow$ (2). Let $h \in Idpt(X)$. Then h is half-strong and so I_h is an induced subgraph of X. It follows from $I_f = I_h$ that I_f is an induced subgraph of X.

If there exist $x \in K \backslash I_f$ and $y \in S \cap I_f$ such that $\{x, y\} \in E(X)$, then $h(x) = y_1$ for some $y_1 \in S$ with $N(y_1) = K \backslash \{x\}$. Since $h \in Idpt(X)$, $h(y) = y$. Note that $\{x, y\} \in E(X)$ and $\{h(x), h(y)\} = \{y_1, y\} \notin E(X)$. A contradiction. Hence $\{x, y\} \notin E(X)$ for any $x \in K \backslash I_f$ and $y \in S \cap I_f$.

(2) $\Rightarrow$ (1). Let X be a split graph and $f \in End(X)$. Since K is a maximum complete set of X, $f(K)$ is a clique of size n. We have $K \subseteq I_f$ or $|K \cap I_f| = n - 1$.

Assume that $K \subseteq I_f$. For any $x \in S$, there exists a vertex $k_x \in K$ such that x is not adjacent to k_x. Let h be the mapping from $V(X)$ to itself defined by

$$h(x) = \begin{cases} x, & if\ x \in I_f, \\ k_x, & if\ x \in V(X) \setminus I_f. \end{cases}$$

Then $h \in End(X)$. If $x \in I_f$, then $h^2(x) = h(x) = x$; If $x \in V(X) \setminus I_f$, then $h^2(x) = h(k_x) = k_x = h(x)$ since $k_x \in I_f$. Hence $h \in Idpt(X)$. Clearly, I_f and I_h have the same set of vertices. Note that any idempotent endomorphism is half-strong. It follows from Lemmas 1.1 that both I_h and I_f are induced subgraphs of X. Hence $I_h = I_f$.

Assume that $|K \cap I_f| = n - 1$. Then there exists $x_1 \in K \setminus I_f$. Since any endomorphism f maps a clique to a clique of the same size, $f(K)$ is a clique of size n in X. Thus there exists $y_1 \in S \cap I_f$ such that y_1 is adjacent to every vertex of $K \backslash \{x_1\}$. For $x \in S$, k_x have the same meaning as in the case $K \subseteq I_f$. Let h be the mapping from $V(X)$ to itself defined by

$$h(x) = \begin{cases} y_1, & if\ x = x_1, \\ x, & if\ x \in I_f, \\ y_1, & if\ x \notin I_f,\ x \in S\ and\ x_1 \notin N(x), \\ k_x, & if\ x \notin I_f,\ x \in S\ and\ x_1 \in N(x). \end{cases}$$

It is easy to see that h is well-defined. Let $\{x, y\} \in E(X)$. We show that $\{h(x), h(y)\} \in E(X)$. If $x, y \in I_f$, then $\{h(x), h(y)\} = \{x, y\} \in E(X)$; If $x = x_1$ and $y \in K \setminus \{x_1\}$, then $\{h(x), h(y)\} = \{y_1, y\} \in E(X)$; If $x = x_1$ and $y \in S$, then $y \notin I_f$ since $\{x, y\} \notin E(X)$ for any $x \in K \backslash I_f$ and $y \in S \cap I_f$. Thus $\{h(x), h(y)\} = \{y_1, k_y\} \in E(X)$; If $x \in K \setminus \{x_1\}$ and $y \in S \setminus I_f$, then $x \in I_f$. If $y \in N(x_1)$, then $\{h(x), h(y)\} = \{x, k_y\} \in E(X)$. If $y \notin N(x_1)$, then $\{h(x), h(y)\} = \{x, y_1\} \in E(X)$. Therefore $h \in End(X)$. Clearly, $h^2 = h$ and $I_h = I_f$. Consequently, $h \in Idpt(X)$.

Lemma 2.2 Let X be a split graph and $f \in End(X)$. Then there exists $g \in Idpt(X)$ such that $\rho_g = \rho_f$ if and only if there exists $b \in [a]_{\rho_f}$ such that $N(b) = \cup_{x \in [a]_{\rho_f}} N(x)$ for any $a \in V(X)$.

Proof Necessity. Let $f \in End(X)$ and $a \in V(X)$. If $[a]_{\rho_f} \cap K \neq \phi$, then there exists $x_i \in K \cap [a]_{\rho_f}$. Note that $\{x_i, y\} \notin E(X)$ for any $y \in [a]_{\rho_f}$. Hence $N(x_i) = \cup_{x\in[a]_{\rho_f}} N(x)$. If $[a]_{\rho_f} \cap K = \phi$, then we can suppose $[a]_{\rho_f} = \{y_1, y_2, \cdots, y_p\}$ for some $1 \leq p \leq m$, where $m = |S|$. Let $g \in Idpt(X)$ be such that $\rho_g = \rho_f$. Then $g([a]_{\rho_f}) = y_j$ for some $1 \leq j \leq p$. Since g is half-strong, we have that $N(y_j) = \cup_{x\in[a]_{\rho_f}} N(x)$.

Sufficiency. Let $f \in End(X)$. For every ρ_f-class $[a]_{\rho_f}$, there exists $a' \in [a]_{\rho_f}$ such that $N(a') = \cup_{x\in[a]_{\rho_f}} N(x)$. Also, according to the last paragraph of this lemma, if $[a]_{\rho_f} \cap K \neq \phi$, then we can choose $a' \in K$. Define a mapping g from $V(X)$ to itself by

$$g(a) = a' \quad for\ all\ a \in V(X).$$

Then it is easy to check that $g \in Idpt(X)$ and $\rho_g = \rho_f$.

Theorem 2.3 Let X be a split graph and $f \in End(X)$. Then $f \in rEnd(X)$ if and only if the following conditions hold:

(1) f is half-strong.

(2) $\{x, y\} \notin E(X)$ for any $x \in K \backslash I_f$ and $y \in S \cap I_f$.

(3) There exists $b \in [a]_{\rho_f}$ such that $N(b) = \cup_{x\in[a]_{\rho_f}} N(x)$ for any $a \in V(X)$.

Proof It follows directly from Lemmas 1.1, 1.2, 2.1 and 2.2.

Next we start to seek the conditions for a split graph X under which $rEnd(X)$ forms a monoid.

Lemma 2.4 Let X be a split graph. If there exist $y_i, y_j \in S$ such that $N(y_i) \subset N(y_j)$, then $rEnd(X)$ does not form a monoid.

Proof Suppose that there exist $y_i, y_j \in S$ such that $N(y_i) \subset N(y_j)$. Since K is a maximum complete set of X, for any $x \in S$, there exists $k_x \in K$ such that $\{x, k_x\} \notin E(X)$. Let

$$f(x) = \begin{cases} y_j, & x = y_i, \\ x, & \text{otherwise.} \end{cases} \quad \text{and} \quad g(x) = \begin{cases} k_x, & x = y_j, \\ x, & \text{otherwise.} \end{cases}$$

Then f and g are idempotent endomorphisms of X and so they are regular. It is easy to see that $y_j = (fg)(y_i) \in I_{fg}$ and $(fg)^{-1}(y_j) = \{y_i\}$. It follows that $(fg)(A^{fg}_{y_j}) = (fg)(N(y_i)) = N(y_i) \neq N(y_j) = N(y_j) \cap I_{fg}$. Clearly, $fg \in End(X)$. Then by Lemma 1.3 fg is not half-strong and so $fg \notin rEnd(X)$. Therefore $rEnd(X)$ does not form a monoid.

Remark 2.5 In view of Lemma 2.4, (1) if X is a non-connected split graph and there exists $y \in S$ such that $N(y) \neq \emptyset$, then $rEnd(X)$ does not form a monoid. It is easy to see that for a non-connected split graph X, $rEnd(X)$ forms a monoid if and only if $N(y) = \emptyset$ for any $y \in S$; (2) if $rEnd(X)$ forms a monoid, then $N(y_i) \not\subset N(y_j)$ for any $y_i, y_j \in S$ with $i \neq j$.

Up to now, we have obtained the following necessary condition for $rEnd(X)$ being a monoid:

(A) $N(y_i) \subseteq N(y_j)$ implies $N(y_i) = N(y_j)$.

To show that (A) is also sufficient for $rEnd(X)$ being a monoid, we need the following characterization of regular endomorphisms of a split graph satisfying (A).

Theorem 2.6 Let X be a split graph satisfying (A) and let $f \in End(X)$. Then $f \in rEnd(X)$ if and only if the following conditions hold:

(1) $\{x, y\} \notin E(X)$ for any $x \in K \backslash I_f$ and $y \in S \cap I_f$.

(2) $f(N(a)) = N(f(a)) \cap I_f$ for any $a \in S$ with $[a]_{\rho_f} \subseteq S$ and $N(c) = N(a)$ for any $c \in [a]_{\rho_f}$.

Proof Necessity. (1) follows directly from Theorem 2.3. Let $f \in rEnd(X)$ and $a \in S$ with $[a]_{\rho_f} \subseteq S$. Then $f \in hEnd(X)$ and there exists $b \in [a]_{\rho_f}$ such that $N(b) = \cup_{x \in [a]_{\rho_f}} N(x)$. By (A) we have $N(c) = N(a)$ for any $c \in [a]_{\rho_f}$. Thus $A^f_{f(a)} = \cup_{x \in [a]_{\rho_f}} N(x) = N(a)$. Since f is half-strong, by Lemma 1.3, $f(A^f_a) = N(a) \cap I_f$ for any $a \in I_f$. Hence $f(N(a)) = f(A^f_{f(a)}) = N(f(a)) \cap I_f$. Hence (2) holds.

Sufficiency. Let $f \in End(X)$ and $a \in I_f \cap S$. If $f^{-1}(a) \cap K \neq \emptyset$, then $f^{-1}(a) \cap K = \{x\}$ for some $x \in K$ and $a = f(x)$. Now $|N(a)| = n - 1$. Let $N(a) = K \setminus \{x_1\}$ for some $x_1 \in K$. Then $f(K \setminus \{x\}) = K \setminus \{x_1\}$. If $f^{-1}(a) \cap S = \emptyset$, then $A^f_a = N(x)$. If $f^{-1}(a) \cap S \neq \emptyset$, then $\{x, b\} \notin E(X)$ and $N(b) \subseteq N(x)$ for any $b \in f^{-1}(a) \cap S$. Hence $A^f_a = N(x)$. Since $f(x) = a$ and $f(K \setminus \{x\}) = K \setminus \{x_1\}$, $f(N(x)) \subseteq N(f(x)) = N(a) = K \setminus \{x_1\}$ and so $f(N(x)) = K \setminus \{x_1\} = N(a) \cap I_f$. Therefore $f(A^f_a) = f(N(x)) = N(a) \cap I_f$. If $f^{-1}(a) \subseteq S$, then by hypothesis $A^f_a = N(b)$ for some (any) $b \in f^{-1}(a)$ and $f(A^f_a) = f(N(b)) = N(f(b)) \cap I_f = N(a) \cap I_f$.

Let $a \in K \cap I_f$ with $f^{-1}(a) \subseteq S$. Then $A^f_a = N(b)$ for some (any) $b \in f^{-1}(a)$ and $f(A^f_a) = f(N(b)) = N(f(b)) \cap I_f = N(a) \cap I_f$. By Lemma 1.3, f is half-strong.

Let $a \in V(X)$. If $[a]_{\rho_f} \subseteq S$, then $N(c) = N(a)$ for any $c \in [a]_{\rho_f}$. Thus $N(a) = \cup_{x \in [a]_{\rho_f}} N(x)$; If $[a]_{\rho_f} \not\subseteq S$, then there exists $x_0 \in [a]_{\rho_f} \cap K$. Note that $\{b, x_0\} \notin E$ for any $b \in [a]_{\rho_f}$. Hence $N(x_0) = \cup_{x \in [a]_{\rho_f}} N(x)$. By Theorem 2.3, f is regular.

For a split graph satisfying (A), we have

Corollary 2.7 Let X be a split graph satisfying (A) and $f \in Idpt(X)$. If there exists $a \in S$ such that $f(a) \in S$, then $N(a) = N(f(a))$.

Proof If there exists $a \in S$ such that $f(a) \in S$, then $N(a) \subseteq N(f(a))$. It follows from (A) that $N(a) = N(f(a))$.

Now we prove that $rEnd(X)$ forms a monoid for any split graph X satisfying (A).

Theorem 2.8 Let X be a split graph satisfying (A). Then $rEnd(X)$ forms a monoid.

Proof Let $f, g \in Idpt(X)$. We only need to show that $fg \in rEnd(X)$. To prove that $fg \in rEnd(X)$, let $a \in S$ with $[a]_{\rho_{fg}} \subseteq S$. We will show that $(fg)(N(a)) = N((fg)(a)) \cap I_{fg}$ and $N(c) = N(a)$ for any $c \in [a]_{\rho_{fg}}$. Since $[b]_{\rho_g} \subseteq [a]_{\rho_{fg}}$ for any $b \in [a]_{\rho_{fg}}$, $[b]_{\rho_g} \subseteq S$ for any $b \in [a]_{\rho_{fg}}$. Obviously, $g(b) \in [g(a)]_{\rho_f}$ for any $b \in [a]_{\rho_{fg}}$.

If $g(c) \in K$ for some $c \in [a]_{\rho_{fg}}$, then $g^2(c) = g(c)$. Thus $g(c) \in [c]_{\rho_g}$. It contradicts $[c]_{\rho_g} \subseteq S$. Therefore $g(b) \in S$ for any $b \in [a]_{\rho_{fg}}$. By Corollary 2.7, $N(b) = N(g(b))$ for any $b \in [a]_{\rho_{fg}}$.

If $[g(a)]_{\rho_f} \subseteq S$, by Corollary 2.7, $N(g(b)) = N(g(a)) = N((fg)(a))$ for any $b \in [a]_{\rho_{fg}}$. Hence $N(c) = N(a) = N((fg)(a))$ for any $c \in [a]_{\rho_{fg}}$ and $A^{fg}_{fg(a)} = N(a)$. By Lemma 1.3, we have

$$\begin{aligned} (fg)(A^{fg}_{fg(a)}) &= (fg)(N(a)) = f(g(N(a))) = f(N(g(a)) \cap I_g) \\ &= f(N(g(a))) \cap f(I_g)N((fg)(a)) \cap I_f \cap I_{fg} \\ &= N((fg)(a)) \cap I_{fg}. \end{aligned}$$

If $[g(a)]_{\rho_f} \not\subseteq S$, without loss of generality, we may suppose that there exists $r_1 \in [g(a)]_{\rho_f}$ for some $r_1 \in K$. If there exists $k \in K$ such that $g(k) = r_1$, then $k \in [a]_{\rho_{fg}}$. It contradicts $[a]_{\rho_{fg}} \subseteq S$. Hence $r_1 \notin I_g$. Note that $g(K)$ is a clique of size n and $r_1 \notin g(K)$, then there exist $y \in S$ such that $g(r_1) = y$, where r_1 is the unique vertex in K such that $\{r_1, y\} \notin E$. It is easy to see $\{r_1, t\} \notin E$ for any $t \in [r_1]_{\rho_f}$. Hence $N(t) \subseteq N(y)$ for any $t \in [r_1]_{\rho_f}$. By (A), we have $N(t) = N(y) = K \setminus \{r_1\}$. In particular, $N(g(b)) = N(g(a)) = K \setminus \{r_1\}$ for any $b \in [a]_{\rho_{fg}}$. Hence $N(c) = N(g(c)) = N(g(a)) = N(a)$ for any $c \in [a]_{\rho_{fg}}$ and $A^{fg}_{fg(a)} = N(a) = K \setminus \{r_1\}$. Clearly, $N(g(a)) \cap I_g = N(r_1) \cap I_g$. By Lemma 1.3, we have

$$\begin{aligned} (fg)(A^{fg}_{fg(a)}) &= (fg)(N(a)) = f(g(N(a))) = f(N(g(a)) \cap I_g) \\ &= f(N(r_1) \cap I_g) = f(N(r_1)) \cap f(I_g) \\ &= N((fg)(a)) \cap I_f \cap I_{fg} = N((fg)(a)) \cap I_{fg}. \end{aligned}$$

Now $\{x, y\} \notin E(X)$ for any $x \in K \backslash I_{fg}$ and $y \in S \cap I_{fg}$. Otherwise, there exist $x \in K \backslash I_{fg}$ and $y \in S \cap I_{fg}$ such that $\{x, y\} \in E(X)$. Since $I_{fg} \subseteq I_f$,

$y \in I_f$, which implies that $x \in I_f$(if $x \notin I_f$, then we have a contradiction with Lemma 2.1 applied for f). Thus $x \notin I_{fg}$, but $x \in I_f$, so $x \notin I_g$, thus $y \notin I_g$, because otherwise Lemma 2.1 would fail for g. However, $y \in I_{fg}$, so there exists a vertex $z \in I_g$ such that $y = f(z)$. Clearly $z \neq x$. We claim that $z \notin K$. Otherwise, since $z \in I_g$, $fg(z) = f(z) = y \notin K$. Note that f is idempotent and $f(z) = y$. Then $z \notin I_f$, so $z \notin I_{fg}$. Thus $x, z \in K \setminus I_{fg}$ and so the clique number of I_{fg} less than n. A contradiction. However, z cannot be connected to x, because $x \in K \backslash I_g$ and $z \in S \cap I_g$. But this shows that $N(z) \subseteq K \setminus \{x\}$. We have that $x \in K \backslash I_g$, so $g(x) \in S$ such that $N(g(x)) = K \setminus \{x\}$. By property (A) we have $N(z) = N(g(x))$. Since $y, z \in S$ and $f(z) = y$, by Corollary 2.7, $N(y) = N(z) = K \setminus \{x\}$. Thus $g(y) \neq t$ for any $t \in K \setminus \{x\}$. It follows from $x \notin I_g$ that $g(y) \neq x$. Hence $g(y) \in S$. Thus we have $g(x), g(y) \in S$ such that $\{g(x), g(y)\} \in E(X)$. A contradiction.

By Theorem 2.6, $fg \in rEnd(X)$.

Up to now we have

Theorem 2.9 Let X be a split graph. Then $rEnd(X)$ forms a monoid if and only if $N(y_i) \subseteq N(y_j)$ implies $N(y_i) = N(y_j)$.

Proof Necessity follows directly from Lemmas 2.4.
Sufficiency follows directly from Theorem 2.8.

Example 2.10 Let X be a split graph with $K = \{x_1, x_2, x_3\}$ and $S = \{y_1, y_2\}$ such that $N(y_1) = \{x_1, x_2\}$, $N(y_2) = \{x_1\}$. It is easy to see that X is a split graph not satisfying (A). Now let

$$f = \begin{pmatrix} x_1 & x_2 & x_3 & y_1 & y_2 \\ x_1 & x_2 & x_3 & y_1 & y_1 \end{pmatrix} \quad \text{and} \quad g = \begin{pmatrix} x_1 & x_2 & x_3 & y_1 & y_2 \\ x_1 & x_2 & x_3 & x_3 & y_2 \end{pmatrix}.$$

Then $f, g \in rEnd(X)$. Now

$$fg = \begin{pmatrix} x_1 & x_2 & x_3 & y_1 & y_2 \\ x_1 & x_2 & x_3 & x_3 & y_1 \end{pmatrix}$$

It is easy to see that $(fg)^{-1}(y_1) = \{y_2\}$, $(fg)^{-1}(x_2) = \{x_2\}$ and $\{x_2, y_2\} \notin E$. Thus, fg is not half-strong and so $fg \notin rEnd(X)$.

Example 2.11 Let Y be a split graph with $K = \{x_1, x_2, x_3\}$ and $S = \{y_1, y_2\}$ such that $N(y_1) = \{x_1\}$, $N(y_2) = \{x_2, x_3\}$. It is easy to see that Y is a split graph satisfying (A). By Theorem 2.9, $rEnd(Y)$ forms a monoid.

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