ORDER DIMENSION OF LAYERED GENERALIZED CROWNS

REBECCA E. GARCIA AND DARREL A. SILVA

ABSTRACT. The order dimension is an invariant on partially ordered sets introduced by Dushnik and Miller in 1941 [1]. It is known that the computation of the order dimension of a partially ordered set in general is highly complex, with current algorithms relying on the minimal coloring of an associated hypergraph, see [5]. The aim of this work is to extend the family of posets whose order dimension is easily determined by a formula. We introduce an operation called layering. Finally, we provide the precise formulas for determining the order dimension of any given number of layers of Trotter's generalized crowns.

1. INTRODUCTION

This paper extends the family of posets whose order dimension is easily determined by a formula. We focus on one fairly distinguished family of partially ordered sets known as *generalized crowns* and create new partially ordered sets with them through an operation we define as *layering*. The main results in this article are found in Section 4. The theorems in this section give the order dimension of *layers of generalized crowns*.

In Section 2, we give all the necessary definitions and notation used throughout our work. In Section 3, we briefly discuss Trotter's *generalized crowns* and a theorem which provides the formula for computing the order dimension of a generalized crown. Finally, we define the notion of *layering* and give new results on the order dimension of layers of a generalized crown in Section 4.

2. NOTATION AND BACKGROUND

This section includes notation and definitions used throughout this paper. The notation and definitions in this section are consistent with those found in [4].

Recall that a **partially ordered set** $\mathbb{P}$ is a pair (X, P), where X is a set, called the **ground set**, and $P \subset X \times X$ is a reflexive, antisymmetric, and transitive binary relation on X. Further recall that a **subposet** $\mathbb{P}(Y) \subset \mathbb{P}$ is a pair $(Y, P(Y))$ such that $Y \subset X$ and $P(Y) = P \cap (Y \times Y)$.

At times, instead of using the formal notation $(x, y) \in P$, we will also denote this relation by writing $x \leq_P y$, or when it is clear that we are working within the partial ordering defined by P, we shall also write $x \leq y$. Whenever $x \neq y$ and

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$x \leq y$, we will denote this by writing $x < y$. We sometimes write $x \in \mathbb{P}$ to denote that the element x is in the ground set of $\mathbb{P}$. Throughout this paper, the word **poset** will be used interchangeably with *partially ordered set.* We say $\mathbb{P}$ is **finite** if the ground set X is finite. In this paper, the posets discussed are finite posets.

To help illustrate the partial ordering P on a given set, one generally refers to the **Hasse diagram** of that poset. Every poset gives rise to a Hasse diagram and vice versa. In this paper, we will not distinguish between the two.

We denote an incomparable pair x, y by $x \parallel y$. In the Hasse diagram of the poset, one always finds a path of edges between comparable vertices. The **incomparable set** of $x \in \mathbb{P}$ is defined by $\mathrm{Inc}_{\mathbb{P}}(x) = \{y \in X : y \parallel x\}$. When the order $\mathbb{P}$ is clear, we will drop the subscript $\mathbb{P}$ in the notation and simply use $\mathrm{Inc}(x)$. The set of all incomparable pairs in $\mathbb{P}$ is denoted by $\mathrm{Inc}(\mathbb{P})$.

The **strict downset** of $x \in \mathbb{P}$ is denoted by $D_{\mathbb{P}}(x)$ and is defined as $\{y \in X : y <_P x\}$. Similarly, the **strict upset** is defined as $U_{\mathbb{P}}(x) = \{y \in X : x <_P y\}$. When the partial order $\mathbb{P}$ is clear, we will drop the subscript $\mathbb{P}$ in the notation and simply use $D(x)$ and $U(x)$ to denote the strict downset of x and strict upset of x, respectively. A poset with the property that any two elements are comparable is said to be **linearly** (or **totally**) **ordered**. An element $x \in \mathbb{P}$ is **minimal** if there is no element $y \in X$ with $y <_P x$. We denote the set of all minimal elements of $\mathbb{P}$ by $\min(\mathbb{P})$. Similarly, an element $x \in \mathbb{P}$ is **maximal** if there is no element $y \in X$ with $x <_P y$. We denote the set of all maximal elements of $\mathbb{P}$ by $\max(\mathbb{P})$.

Definition 2.1. An **extension** of $\mathbb{P} = (X, P)$ is a poset $\mathbb{E} = (X, E)$ over the same ground set X of $\mathbb{P}$ with the property that $P \subseteq E$. A **linear extension** $\mathbb{L}$ of $\mathbb{P}$ is an extension of $\mathbb{P}$ which is also a *total* ordering on the ground set X.

Let $\mathcal{R} = \{\mathbb{L}_1, \ldots, \mathbb{L}_t\}$ be a set of linear extensions of a poset $\mathbb{P}$. We can construct a new poset $\mathbb{Q} := \bigcap_{i=1}^{t} \mathbb{L}_i$, which is an extension of $\mathbb{P}$ in the following way: $x \leq_Q y$ if and only if $x \leq_{L_i} y$, for all i. Note that x and y are incomparable with respect to Q if there are linear extensions $\mathbb{L}_i$ and $\mathbb{L}_j$ such that $(x, y) \in L_i$ and $(y, x) \in L_j$.

Definition 2.2. The set $\mathcal{R}$ of linear extensions is said to **realize** $\mathbb{P}$ if $\mathbb{P} = \bigcap_{i=1}^{t} \mathbb{L}_i$ and we say that $\mathcal{R}$ is a **realizer** of $\mathbb{P}$.

Definition 2.3. The **order dimension** of $\mathbb{P}$ is the minimal possible cardinality of a realizer of $\mathbb{P}$.

There is another characterization of order dimension [2]. The order dimension of a poset $\mathbb{P}$ is the smallest nonnegative integer t for which $\mathbb{P}$ can be embedded in $\mathbb{R}^t$. Such an embedding is defined by representing each element $x \in X$ by a vector $\overline{x} = (x_1, \ldots, x_t) \in \mathbb{R}^t$ such that for all distinct pairs of elements $x, y \in X$, $x \leq_P y$ if and only if $x_i < y_i$ for all i.

The next proposition states one important and very useful characteristic of the order dimension: monotonicity.

Proposition 2.1. [4] *Let $\mathbb{P} = (X, P)$ be a poset and let $Y \subseteq X$ be a nonempty subset. Then,*

$$\dim(\mathbb{P}|_Y) \leq \dim(\mathbb{P}).$$

For any realizer $\mathcal{R}$ of $\mathbb{P}$ and for every incomparable pair x, y in $\mathbb{P}$, there are two linear extensions $\mathbb{L}$ and $\mathbb{L}'$ in $\mathcal{R}$ with $(x, y) \in L$ and $(y, x) \in L'$. Thus the order dimension is at most twice the number of incomparable pairs, which gives a very large upper bound. In fact, most classical work in dimension theory provides nicer upper bounds on the dimension of a poset. Proposition 2.2 shows that one need not check that $(x, y) \in L$ and $(y, x) \in L'$ for all incomparable pairs in $x||y \in \mathbb{P}$. There is a special set of ordered incomparable pairs, known as the set of **critical pairs**, over which one can verify whether a collection of linear extensions is in fact a realizer for $\mathbb{P}$.

Definition 2.4. The *ordered* pair (x, y) is **critical** if

(1) $x \parallel y$.
(2) $D(x) \subset D(y)$.
(3) $U(y) \subset U(x)$.

The set of all critical pairs of $\mathbb{P}$ is denoted $\text{crit}(\mathbb{P})$.

Definition 2.5. We say that an extension $\mathbb{Q} = (X, Q)$ of $\mathbb{P}$ **reverses** a critical pair (x, y) if $y \leq_Q x$.

We shall make frequent use of monotonicity and of the following proposition which was proven by I. Rabinovitch and I. Rival.

Proposition 2.2. [3] *Let $\mathbb{P} = (X, P)$ be a poset and let $\mathcal{R}$ be a family of linear extensions of $\mathbb{P}$. Then the following statements are equivalent:*

(1) *$\mathcal{R}$ is a realizer of $\mathbb{P}$.*
(2) *For every $(x, y) \in \text{crit}(\mathbb{P})$, there exists a linear extension $\mathbb{L}$ in $\mathcal{R}$ such that $y <_L x$.*

3. Generalized Crowns

In this section, we review the theory on *generalized crowns*, which is a special family of posets first defined by Trotter [4]. We begin with the definition of a *generalized crown* and then proceed immediately to a more detailed discussion of a theorem which gives a formula for the order dimension of a generalized crown. Here, we provide an outline of the proof of this theorem, as we will utilize the algorithm for constructing the minimal realizers that give the order dimension of the crowns. This section does not contain any new results.

Historically, the term *crowns* referred to posets as shown in Figure 1.

Trotter defines a generalized crown in [4] as a generalization of this kind of bipartite poset.

Definition 3.1. The **generalized crown** is a finite poset, denoted $\mathbb{S}_n^k$, where $n \geq 3$ and $k \geq 0$, with the following properties:

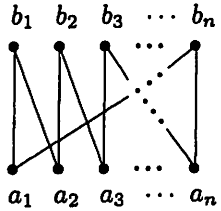

FIGURE 1. Crown

(1) Let $A = \{a_1, \ldots, a_{n+k}\}$ and $B = \{b_1, \ldots, b_{n+k}\}$. The union of these two sets comprises the ground set of $\mathbb{S}_n^k$.
(2) We will identify a_i with $a_{i-(n+k)}$ and b_i with $b_{i-(n+k)}$, whenever $i > n+k$. This indexing scheme is called **cyclic indexing.**
(3) For $i = 1, \ldots, n+k$, we have that $b_i \parallel \{a_i, a_{i+1}, \ldots, a_{i+k}\}$.
(4) For $j = (i+k)+1, \ldots, (i+k)+(n-1)$, we have that $b_i > a_j$.

Note that $\min(\mathbb{S}_n^k) = A$ and $\max(\mathbb{S}_n^k) = B$. For each maximal element $b_i \in B$, there are precisely $k+1$ consecutively indexed elements in A which are incomparable to b_i, with the remaining $n-1$ elements in A belonging to $D(b_i)$. Also note that altogether, there are $2(n+k)$ elements in the ground set of the crown $\mathbb{S}_n^k$. Figure 2 depicts the Hasse diagram for the case when $n = 5$ and $k = 2$.

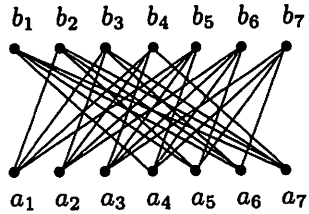

FIGURE 2. $\mathbb{S}_5^2$

We now highlight Trotter's work on generalized crowns and its proof since we will be using the notation and methods for the proofs in our main results. To that end, we begin by presenting two lemmas. The first lemma completely describes $\text{crit}(\mathbb{S}_n^k)$ and the second lemma gives a lower bound to the order dimension of a crown.

Lemma 3.1. [4] *The critical pairs in* $\mathbb{S}_n^k$ *are of the form* (a_i, b_j) *where* $a_i \parallel b_j$ *in* $\mathbb{S}_n^k$.

Lemma 3.2. [4] *Let* $\mathbb{L}$ *be a linear extension of the crown* $\mathbb{S}_n^k$. *Then* $\mathbb{L}$ *reverses at most* $\binom{k+2}{2}$ *critical pairs.*

Let us examine the implications of Lemma 3.2:

(1) Any linear extension $\mathbb{L}$ of a crown $\mathbb{S}_n^k$ reverses at most $\binom{k+2}{2}$ critical pairs.
(2) There are precisely $(n+k)(k+1)$ critical pairs in the crown.
(3) Let $t = \left\lceil \frac{2(n+k)}{k+2} \right\rceil$. Then, $t \cdot \binom{k+2}{2} \geq (n+k)(k+1)$.

By Lemma 3.2 and Proposition 2.2, we need at least $t = \left\lceil \frac{2(n+k)}{k+2} \right\rceil$ many linear extensions in order to realize the crown. In terms of order dimension, this means $\dim(\mathbb{S}_n^k) \geq t$. In fact, the proof gives a recipe for constructing precisely the t linear extensions needed [4].

Theorem 3.1. [4] *For $n \geq 3$ and $k \geq 0$, the order dimension of the crown $\mathbb{S}_n^k$ is given by*

$$\dim(\mathbb{S}_n^k) = \left\lceil \frac{2(n+k)}{k+2} \right\rceil.$$

Proof. Let $t = \left\lceil \frac{2(n+k)}{k+2} \right\rceil$. Write $n + k = (k+2)q + r$, with $r, q \in \mathbb{N}$ and $0 \leq r < k+2$.
Claim: There are t linear extensions which reverse all critical pairs of $\mathbb{S}_n^k$.

The proof of the claim falls into two cases, depending on the parity of t.
Case 1: $t = 2q$ or $t = 2(q+1)$. To prove the claim in this case, it suffices to show that for any subset $B' \subset B$ of $k+2$ consecutively indexed elements, there are two sublinear extensions $\mathbb{L}$ and $\mathbb{L}'$ which reverse all critical pairs of the form $(a_i, b_j) \in A \times B'$. Without loss of generality, we take $B' = \{b_1, \ldots, b_{k+2}\}$. The following two sublinear extensions satisfy this requirement:

$$\mathbb{L} = [b_1, a_1, b_2, a_2, \ldots, b_k, a_k, b_{k+1}, a_{k+1}]$$
$$\mathbb{L}' = [b_{k+2}, a_{2k+2}, b_{k+1}, a_{2k+1}, \ldots, b_3, a_{k+3}, b_2, a_{k+2}].$$

Case 2: $t = 2q + 1$.

(1) Partition the set A of minimal elements into q subsets of $k+2$ consecutively indexed elements: $A^1, A^2, \ldots, A^q$ and let A^{q+1} denote the remaining r elements.
(2) Further refine this partition: For $i = 1, \ldots, q$, let A^{i-} consist of the first $\lfloor \frac{k+2}{2} \rfloor$ elements of A^i and A^{i+} consist of the remaining $\lceil \frac{k+2}{2} \rceil$ elements. Let $A^{(q+1)-} = A^{q+1}$. Thus, for example, $A^{1-} = \{a_1, a_2, \ldots, a_{\lfloor \frac{k+2}{2} \rfloor}\}$.
(3) Let $\mathbb{A}^{i\pm}$ denote the set $A^{i\pm}$ ordered by subscripts and let $\overline{\mathbb{A}^{i\pm}}$ denote the set $A^{i\pm}$ ordered by reverse subscripts. For example,

$$\mathbb{A}^{1-} = \left[a_1 < a_2 < \cdots < a_{\lfloor \frac{k+2}{2} \rfloor}\right] \text{ and}$$

$$\overline{\mathbb{A}^{1-}} = \left[a_{\lfloor \frac{k+2}{2} \rfloor} < \cdots < a_2 < a_1\right].$$

(4) For $i = 1, \ldots, q$, consider the following sublinear extensions:

$$\mathbb{L}_i = \left[\mathbb{A}^{i-} < \overline{\mathbb{A}^{i+}}\right]$$

$$\mathbb{L}'_i = \left[\mathbb{A}^{i+} < \overline{\mathbb{A}^{(i+1)-}}\right]$$

$$\mathbb{L}_{q+1} = \left[\mathbb{A}^{q+1} < \overline{\mathbb{A}^{1-}}\right]$$

(5) For each sublinear extension $\mathbb{L}_i$ constructed in step (4), its maximal element a_m belongs to A. Insert the elements of B which are incomparable to the maximal element a_m, as low as possible in the corresponding sublinear extension $\mathbb{L}_i$. For example, in $\mathbb{L}_{q+1}$, the maximal element is a_1. By construction, $\text{Inc}(a_1) \cap B = \{b_1, b_{n+k}, b_{n+k-1}, \ldots, b_{n+2}, b_{n+1}\}$. We begin inserting these elements b_j in the sublinear extension $\mathbb{L}_{q+1}$. Note that by definition, b_1 is in the upset of every element in A^{q+1}, thus we insert b_1 as low as possible in the sublinear extension $\mathbb{L}_{q+1}$: $\left[\mathbb{A}^{q+1} < b_1 < \overline{\mathbb{A}^{1-}}\right]$. The next element b_{n+k} is incomparable to $a_{n+k} \in A^{q+1}$, so b_{n+k} must be inserted lower than a_{n+k}. Note, however, that b_{n+k} is in the upset of all elements in $A^{q+1} \setminus \{a_{n+k}\}$, thus b_{n+k} cannot be inserted any lower. This yields the sublinear extension

$$\left[\mathbb{A}^{q+1} \setminus \{a_{n+k}\} < b_{n+k} < a_{n+k} < b_1 < \overline{\mathbb{A}^{1-}}\right].$$

Continue the insertion of the elements $b_{n+k-1}, \ldots, b_{n+1}$, as low as possible.

This algorithm produces $2q+1$ sublinear extensions which reverse all critical pairs in the crown. □

4. Layers of Generalized Crowns

Our objective is to determine the order dimension of large posets based on the order dimension of certain subposets. The next definition is the operation we call layering. It produces a larger poset from two compatible posets by gluing one poset above the other in a well-defined way.

Definition 4.1. Let $\mathbb{P}_1 = (X_1, P_1)$ and $\mathbb{P}_2 = (X_2, P_2)$ be two posets with $|\max(\mathbb{P}_1)| = |\min(\mathbb{P}_2)|$. Let $\beta : \max(\mathbb{P}_1) \longrightarrow \min(\mathbb{P}_2)$ be a fixed bijection. The **β-layering of $\mathbb{P}_2$ over $\mathbb{P}_1$** is the poset $\mathbb{P}_1 \rtimes_\beta \mathbb{P}_2 = (X_1 \cup X_2, Q)$, where Q is the transitive closure of

$$P_1 \cup P_2 \cup \{(x, \beta(x)), (\beta(x), x)\}_{x \in \max(\mathbb{P}_1)}.$$

In this construction, one can view the bijection β as a set of instructions for gluing one poset to another compatible poset in such a way as to yield a larger poset. This bijection identifies an element in $\max(\mathbb{P}_1)$ with its image under β, which is an element of $\min(\mathbb{P}_2)$. Figure 3 illustrates this operation.

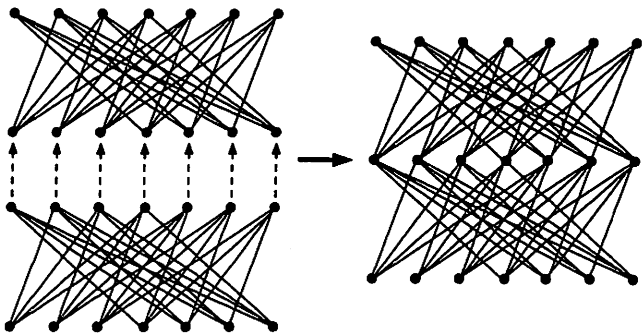

FIGURE 3. Layering $\mathbb{P}_2$ over $\mathbb{P}_1$

In the layered poset $\mathbb{P}_1 \rtimes_\beta \mathbb{P}_2$, we shall call the subposet $\mathbb{P}_2$ the **second layer** or **upper layer** and the subposet $\mathbb{P}_1$ the **first layer** or **lower layer**. We define the **extreme subposet** of $\mathbb{P}_1 \rtimes_\beta \mathbb{P}_2$ to be the subposet generated by the minimal and maximal elements of $\mathbb{P}_1 \rtimes_\beta \mathbb{P}_2$ and denote it by $\mathcal{E}(\mathbb{P}_1 \rtimes_\beta \mathbb{P}_2)$.

We will analyze the dimension of layers of isomorphic copies of the generalized crown $\mathbb{S}_n^k$. For the remainder of this paper, let $\mathbb{P}_1 \cong \mathbb{P}_2 \cong \mathbb{S}_n^k$, where $\mathbb{P}_1 = (A \cup B, P_1)$ and $\mathbb{P}_2 = (B' \cup C, P_2)$ and where $A = \min(\mathbb{P}_1)$, $B = \max(\mathbb{P}_1)$, $B' = \min(\mathbb{P}_2)$, and $C = \max(\mathbb{P}_2)$.

The elements in $\mathbb{P}_1$ and $\mathbb{P}_2$ are indexed as in Definition 3.1. Fix the bijection

$$\beta : B \longrightarrow B'$$

so that $\beta(b_i) = b_i'$, where $b_i \in B$ and $b_i' \in B'$.

We define $\mathbb{P}_1 \rtimes_\beta \mathbb{P}_2$ to be the layered poset

$$(A \cup B \cup B' \cup C, P),$$

where P is the transitive closure of

$$P_1 \cup P_2 \cup \Big\{(b_i, b_i'), (b_i', b_i)\Big\}_{i=1}^{n+k}.$$

This newly defined larger poset identifies the elements in B with B'. For the sake of simplicity and clarity, we employ the following notation:

(1) $\rtimes$ for $\rtimes_\beta$
(2) $A = \{a_1, \ldots, a_{n+k}\}$.
(3) $B = B' = \{b_1, \ldots, b_{n+k}\}$.
(4) $C = \{c_1, \ldots, c_{n+k}\}$.

We will begin our study with an analysis of the case when $n \geq k+3$. We prove a lemma which describes the set of critical pairs in $\mathbb{P}_1 \rtimes \mathbb{P}_2$ as being the disjoint union of the set of critical pairs in the upper layer and the set of critical pairs in the lower layer. Then, we use the results and techniques in the proof of Theorem 3.1 to prove our first main theorem.

Lemma 4.1. *For $n \geq k+3$, the set of critical pairs of $\mathbb{P}_1 \rtimes \mathbb{P}_2$ is the disjoint union of the critical pairs of $\mathbb{P}_1$ and $\mathbb{P}_2$.*

Proof. Define $\mathbb{Q}$ to be the layering $\mathbb{P}_1 \rtimes \mathbb{P}_2$.

Recall that by construction,

$$D_{\mathbb{P}_2}(c_i) = \{b_{i+k+1}, b_{i+k+2}, \ldots, b_{i+k+n-1}\}.$$

Similarly,

$$D_{\mathbb{P}_1}(b_j) = \{a_{j+k+1}, a_{j+k+2}, \ldots, a_{j+k+n-1}\}.$$

Thus,

$$D_{\mathbb{Q}}(c_i) \cap A = \{a_{i+2k+2}, a_{i+2k+3}, \ldots, a_{i+2k+n}, a_{i+2k+n+1}, \ldots, a_{i+2k+2n-2}\}.$$

Observe that the cardinality of this subset of A is $(i+2k+2n-2)-(i+2k+2)+1 = 2n-3 \geq n+k$. From this, we conclude that $D_{\mathbb{Q}}(c_i) \cap A = A$. Therefore, all the elements in A are comparable to all the elements in C and the only possible critical pairs in $\mathbb{Q}$ would arise from the critical pairs of the layers $\mathbb{P}_1$ and $\mathbb{P}_2$.

Finally we wish to show that the set of critical pairs for $\mathbb{P}_1$ and $\mathbb{P}_2$ are critical pairs for $\mathbb{Q}$. We begin by showing that the critical pairs in $\mathbb{P}_1$ are critical pairs in the layered poset $\mathbb{Q}$. A similar argument will yield the result for $\mathbb{P}_2$.

Let $(a_i, b_j) \in \text{crit}(\mathbb{P}_1)$. Since $D_{\mathbb{P}_1}(a_i) = \emptyset$ and $a_i \in \min(\mathbb{Q})$, we have that $D_{\mathbb{Q}}(a_i) = \emptyset$. So, $D_{\mathbb{Q}}(a_i) \subseteq D_{\mathbb{Q}}(b_j)$ holds true vacuously. Finally, note that $U_{\mathbb{Q}}(b_j) \subset C$ and by the previous argument, $C = U_{\mathbb{Q}}(a_i)$. Thus, $U_{\mathbb{Q}}(b_j) \subseteq U_{\mathbb{Q}}(a_i)$. □

Theorem 4.1. *Let $n \geq k+3$. The order dimension of $\mathbb{P}_1 \rtimes \mathbb{P}_2$ is $\dim(\mathbb{S}_n^k)$.*

Proof. Let $t = \dim(\mathbb{S}_n^k)$. Since $\mathbb{S}_n^k$ is a subposet of $\mathbb{P}_1 \rtimes \mathbb{P}_2$, then $t \leq \dim(\mathbb{P}_1 \rtimes \mathbb{P}_2)$. We proceed by constructing t realizers for the layering which yields $t \geq \dim(\mathbb{P}_1 \rtimes \mathbb{P}_2)$.

Observe that by Proposition 2.2 and Lemma 4.1, it is enough to find sublinear extensions of the layering which reverse the critical pairs in each layering. The proof of Theorem 3.1 splits into two cases depending on the parity of t. We do the same here. Let $n+k = (k+2)q + r$, where $0 \leq r < k+2$.

Case 1. When $r = 0$ or $\frac{k+2}{2} < r < k+2$, then t is even. Namely, $t = 2q$ or $t = 2(q+1)$.

We claim that for a subset $\gamma \subset C$ of $k+2$ consecutively indexed elements, we can find a subset $\alpha \subset A$ of $k+2$ consecutively indexed elements and two sublinear extensions which reverse critical pairs of the form

$$(b_i, c_j) \in B \times \gamma \text{ and } (a_u, b_v) \in \alpha \times B.$$

Once we prove this claim, then by partitioning C into subsets γ^i of $k+2$ consecutively indexed elements, the corresponding α^i will also be a partition of

A. These will produce sublinear extensions which will reverse all critical pairs in $\mathbb{P}_1 \rtimes \mathbb{P}_2$.

To prove the claim, let $\gamma = \{c_1, \ldots, c_{k+2}\} \subseteq C$. Set $\alpha = \{a_{k+1}, \ldots, a_{2k+2}\}$. Consider the sublinear extensions which are produced in the proof of Theorem 3.1, which we provide below:

$$\mathbb{L}_1 \quad : [c_1, b_1, c_2, b_2, \ldots, c_{k+1}, b_{k+1}]$$

$$\mathbb{L}_2 \quad : [c_{k+2}, b_{2k+2}, c_{k+1}, b_{2k+1}, \ldots, c_3, b_{k+3}, c_2, b_{k+2}]$$

$$\mathbb{L}_1' \quad : [b_{k+2}, a_{k+2}, b_{k+3}, a_{k+3}, \ldots, b_{2k+2}, a_{2k+2}]$$

$$\mathbb{L}_2' \quad : [b_{k+1}, a_{2k+1}, b_k, a_{2k}, \ldots, b_2, a_{k+2}, b_1, a_{k+1}]$$

Note that $\mathbb{L}_1$ and $\mathbb{L}_2$ reverse all critical pairs of the form $(b_i, c_j) \in B \times \gamma$. Observe that in this case, γ is a fixed set, and we use Trotter's construction to find all the set of elements in B which need to reverse the elements of γ.

Also, $\mathbb{L}_1'$ and $\mathbb{L}_2'$ reverse all critical pairs of the form $(a_u, b_v) \in \alpha \times B$. Here, observe that α is the fixed set. We used Trotter's construction (with a twist) to find all the elements in B which need to reverse the elements of α.

Now consider the sublinear extensions of $\mathbb{P}_1 \rtimes \mathbb{P}_2$:

$$(1) \qquad \begin{aligned} \mathbb{L}_1' < \mathbb{L}_1 \quad & : [b_{k+2}, a_{k+2}, b_{k+3}, a_{k+3}, \ldots, b_{2k+2}, a_{2k+2}, \\ & \quad c_1, b_1, c_2, b_2, \ldots, c_{k+1}, b_{k+1}] \\ \mathbb{L}_2' < \mathbb{L}_2 \quad & : [b_{k+1}, a_{2k+1}, b_k, a_{2k}, \ldots, b_2, a_{k+2}, b_1, a_{k+1}, \\ & \quad c_{k+2}, b_{2k+2}, c_{k+1}, b_{2k+1}, \ldots, c_3, b_{k+3}, c_2, b_{k+2}] \end{aligned}$$

The sublinear extensions in (1) simultaneously reverse critical pairs of the form

$$(b_i, c_j) \in B \times \gamma \text{ and } (a_s, b_t) \in \alpha \times B.$$

Case 2. When $0 < r \leq \frac{(k+2)}{2}$, then $t = 2q + 1$.

For the upper layer $\mathbb{P}_2$, partition the set B into q subsets of $k + 2$ consecutively indexed elements of B.

$$B = \bigcup_{i=1}^{q+1} B^i.$$

Here, B^{q+1} consists of the remaining r elements of B.

As in the proof of Theorem 3.1, for $i = 1, \ldots, q$,

$$B^i = B^{i-} \cup B^{i+},$$

where B^{i-} consist of the first $\lfloor \frac{k+2}{2} \rfloor$ elements of B^i and where B^{i+} consist of the remaining $\lceil \frac{k+2}{2} \rceil$ elements. Let $B^{(q+1)-} = B^{q+1}$. To further illustrate this partition, $B^{1-} = \{b_1, \ldots, b_h\}$, where $h = \lfloor \frac{k+2}{2} \rfloor$.

As in the algorithm described in the proof of Theorem 3.1, we produce sublinear extensions $\mathbb{L}_2^j$, where $j = 1, \ldots, t$.

(1) Let $\mathbb{B}^{i\pm}$ denote the set $B^{i\pm}$ ordered by subscripts and let $\overline{\mathbb{B}^{i\pm}}$ denote the set $B^{i\pm}$ ordered by reverse subscripts.

(2) For $i = 1, \ldots, q$, consider the following sublinear extensions:

$$\begin{aligned} \mathbb{L}_2^{2i-1} &= \left[\mathbb{B}^{i-} < \overline{\mathbb{B}^{i+}}\right] \\ \mathbb{L}_2^{2i} &= \left[\mathbb{B}^{i+} < \overline{\mathbb{B}^{(i+1)-}}\right] \\ \mathbb{L}_2^{t} &= \left[\mathbb{B}^{(q+1)-} < \overline{\mathbb{B}^{1-}}\right] \end{aligned}$$

(3) At this stage, for $j = 1, \ldots, t$ the maximal element of the sublinear extension $\mathbb{L}_2^j$ is $b_{m_j} \in B$. Insert elements of $\mathrm{Inc}(b_{m_j}) \cap C$ as low as possible in $\mathbb{L}_2^j$, as described earlier in the proof of Theorem 3.1, Case 2, point (5).

The next step is to repeat this construction for the lower layer $\mathbb{P}_1$.

(1) For $i = 1, \ldots, q$, consider the following sublinear extensions:

$$\begin{aligned} \mathbb{L}_1^{2i-1} &= \left[\mathbb{A}^{i-} < \overline{\mathbb{A}^{i+}}\right] \\ \mathbb{L}_1^{2i} &= \left[\mathbb{A}^{i+} < \overline{\mathbb{A}^{(i+1)-}}\right] \\ \mathbb{L}_1^{t} &= \left[\mathbb{A}^{(q+1)-} < \overline{\mathbb{A}^{1-}}\right] \end{aligned}$$

(2) For $j = 1, \ldots, t$ the maximal element of the sublinear extension $\mathbb{L}_1^j$ is $a_{m_j} \in A$. Insert elements of $\mathrm{Inc}(a_{m_j}) \cap B$ as low as possible in $\mathbb{L}_1^j$, again as described earlier in the proof of Theorem 3.1, Case 2, point (5).

In the algorithm just described, a_{m_j} is the maximal element of the sublinear extension $\mathbb{L}_1^j$. Let f be the first index such that $\mathrm{Inc}(a_{m_f}) \cap B^1 = \emptyset$.

Finally, we use both sublinear extensions $\mathbb{L}_1^j$ and $\mathbb{L}_2^j$ to construct sublinear extensions for the layered crowns $\mathbb{P}_1 \rtimes \mathbb{P}_2$ in the following way.

$$\begin{array}{lcl} \mathbb{L}_1^{f} & < & \mathbb{L}_2^{1} \\ \mathbb{L}_1^{f+1} & < & \mathbb{L}_2^{2} \\ \mathbb{L}_1^{f+2} & < & \mathbb{L}_2^{3} \\ \vdots & & \vdots \\ \mathbb{L}_1^{t} & < & \mathbb{L}_2^{t-f} \\ \mathbb{L}_1^{1} & < & \mathbb{L}_2^{(t-f)+1} \\ \vdots & & \vdots \\ \mathbb{L}_1^{f-1} & < & \mathbb{L}_2^{t} \end{array}$$

□

In the example that follows, we explicitly construct sublinear extensions for the poset $\mathbb{S}_6^1 \rtimes \mathbb{S}_6^1$.

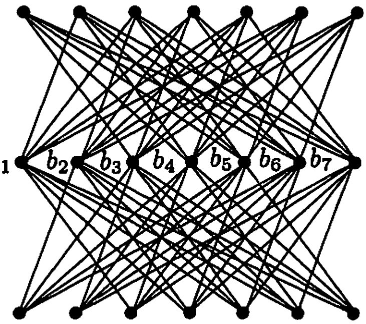

FIGURE 4. $\mathbb{S}_6^1 \rtimes \mathbb{S}_6^1$

Example 4.1. Consider the poset $\mathbb{S}_6^1 \rtimes \mathbb{S}_6^1$, whose HasseNote that $n = 6$ and $k = 1$. Since, $r = 1 < \lceil \frac{k+2}{2} \rceil$, this example falls under Case 2 of the preceeding proof. The two sublinear extensions from the preceeding proof are listed below.

$$\begin{array}{lll|lll}
\mathbb{L}_2^1 & = & [b_1, c_2, b_3, c_1, b_2] & \mathbb{L}_1^1 & = & [a_1, b_2, a_3, b_1, a_2] \\
\mathbb{L}_2^2 & = & [b_2, c_3, b_3, c_4, b_4] & \mathbb{L}_1^2 & = & [a_2, b_3, a_3, b_4, a_4] \\
\mathbb{L}_2^3 & = & [b_4, c_5, b_6, c_4, b_5] & \mathbb{L}_1^3 & = & [a_4, b_5, a_6, b_4, a_5] \\
\mathbb{L}_2^4 & = & [b_5, c_6, b_6, c_7, b_7] & \mathbb{L}_1^4 & = & [a_5, b_6, a_6, b_7, a_7] \\
\mathbb{L}_2^5 & = & [c_7, b_7, c_1, b_1] & \mathbb{L}_1^5 & = & [b_7, a_7, b_1, a_1]
\end{array}$$

In this example, $B^1 = \{b_1, b_2, b_3\}$. The first index f where $\text{Inc}(a_{m_f}) \cap B^1 = \emptyset$ is $f = 3$. Here, $a_{m_3} = a_5$ and $\text{Inc}(a_5) \cap B = \{b_4, b_5\}$ which does not intersect with B^1. So the sublinear extensions which reverse all the critical pairs in $\mathbb{S}_6^1 \rtimes \mathbb{S}_6^1$ are given below.

$$\begin{array}{lll}
\mathbb{L}_1^3 < \mathbb{L}_2^1 & = & [a_4, b_5, a_6, b_4, a_5, b_1, c_2, b_3, c_1, b_2] \\
\mathbb{L}_1^4 < \mathbb{L}_2^2 & = & [a_5, b_6, a_6, b_7, a_7, b_2, c_3, b_3, c_4, b_4] \\
\mathbb{L}_1^5 < \mathbb{L}_2^3 & = & [b_7, a_7, b_1, a_1, b_4, c_5, b_6, c_4, b_5] \\
\mathbb{L}_1^1 < \mathbb{L}_2^4 & = & [a_1, b_2, a_3, b_1, a_2, b_5, c_6, b_6, c_7, b_7] \\
\mathbb{L}_1^2 < \mathbb{L}_2^5 & = & [a_2, b_3, a_3, b_4, a_4, c_7, b_7, c_1, b_1]
\end{array}$$

The preceeding proof actually gives an algorithm for constructing sublinear extensions for the layering of $\mathbb{S}_n^k$ with itself multiple times. By simply repeating the argument for the third layered component, one will have constructed a sublinear extension for the layering of three generalized crowns, and so on. This gives the following theorem:

Theorem 4.2. *Fix $l \in \mathbb{N}$ and for $i = 1, \ldots, l$, let $\mathbb{P}_i \cong \mathbb{S}_k^n$, where $n \geq k + 3$. Then,*

$$\dim(\mathbb{P}_1 \rtimes \cdots \rtimes \mathbb{P}_l) = \dim(\mathbb{S}_n^k).$$

The previous two theorems hinged on the fact that $n \geq k + 3$, since in this case the critical pairs of the layered poset $\mathbb{P}_1 \rtimes \cdots \rtimes \mathbb{P}_l$ come from each layered component $\mathbb{P}_1, \ldots, \mathbb{P}_l$. In the next result, we provide the order dimension of $\mathbb{P}_1 \rtimes \cdots \rtimes \mathbb{P}_l$ when $n < k+3$. It turns out that in this case, the order dimension depends on the number of layers l. A key subposet of this layered poset $\mathbb{P}_1 \rtimes \cdots \rtimes \mathbb{P}_l$ is the extreme subposet $\mathcal{E}(\mathbb{P}_1 \rtimes \cdots \rtimes \mathbb{P}_l)$. This is the subposet generated by the set of minimal elements and maximal elements in $\mathbb{P}_1 \rtimes \cdots \rtimes \mathbb{P}_l$. The extreme subposet plays a pivotal role in finding the critical pairs of the layered poset and consequently in determining the order dimension of the layered poset.

Theorem 4.3. *For $3 \leq n < k + 3$ and for $l \in \mathbb{N}$ with $1 \leq l \leq \lceil \frac{k+1}{n-2} \rceil$, the dimension of l layered crowns*

$$\mathbb{P}_1 \rtimes \cdots \rtimes \mathbb{P}_l,$$

where $\mathbb{P}_i \cong \mathbb{S}_n^k$, is given by

$$\dim(\mathbb{P}_1 \rtimes \cdots \rtimes \mathbb{P}_l) = \left\lceil \frac{2(n+k)}{k+n-l(n-2)} \right\rceil.$$

Proof. There are two parts to this proof. We need to show that

(1) The extreme subposet is a generalized crown, up to a shift in the indices.
(2) The critical pairs in the extreme subposet comprise the set of all critical pairs in the layered poset $\mathbb{P}_1 \rtimes \cdots \rtimes \mathbb{P}_l$.

The first part gives a lower bound to the dimension of the layered poset by the monotonicity of order dimension. The second item gives an upper bound by as we will construct sublinear extensions that will reverse the critical pairs in the extreme subposet and hence in the layered poset.

Part 1: The case $l = 1$ follows by definition. For $l = 2$, we will show that the extreme subposet $\mathcal{E}(\mathbb{P}_1 \rtimes \mathbb{P}_2)$ is $\mathbb{S}_{2n-2}^{k-n+2}$.

Let $c_i \in C = \max(\mathbb{P}_1 \rtimes \mathbb{P}_2)$ for some $i \in \{1, \ldots, n+k\}$. By construction, we have the following:

$$\begin{aligned} c_i &\;||_{\mathbb{P}_2}\; b_i, \ldots, b_{i+k}, \\ c_i &>_{\mathbb{P}_2} b_{i+k+1}, \ldots, b_{i+k+n-1}, \\ D_{\mathbb{P}_1 \rtimes \mathbb{P}_2}(c_i) \cap A &= \{a_{i+k+2-n}, \ldots, a_{i+k}, a_{i+k+1}, \ldots, a_{i+k+(n-2)}\}. \end{aligned}$$

Thus, each c_i is comparable to a distinct set of $2n-3$ consecutively indexed elements in A. This also shows that there are precisely $k - n + 3$ consecutively indexed elements in A which are incomparable to c_i. Thus, each c_i is incomparable to a distinct set of $k - n + 3$ consecutively indexed elements in A. This shows that the extreme subposet is isomorphic to the generalized crown $\mathbb{S}_{2n-2}^{k-n+2}$.

For $2 < l \leq \lceil \frac{k+1}{n-2} \rceil$, denote the layered poset

$$\mathbb{P}_1 \rtimes \cdots \rtimes \mathbb{P}_l = (X^1 \cup \cdots \cup X^{l+1}, P),$$

where $X^i = \min(\mathbb{P}_i)$, for $i = 1, \ldots, l$ and $X^{l+1} = \max(\mathbb{P}_l)$. Also, since $\mathbb{P}_i = \mathbb{S}_n^k$ for each i, observe that for $i = 1, \ldots, l$, $X^{i+1} = \max(\mathbb{P}_i)$.

Let $x_i^{l+1} \in X^{l+1}$. Note that

Downsets	Number of consecutively indexed elements
$D(x_i^{l+1}) \cap X^l$	$n-1$
$D(x_i^{l+1}) \cap X^{l-1}$	$n-1+1(n-2)$
$D(x_i^{l+1}) \cap X^{l-2}$	$n-1+2(n-2)$
$\vdots$	$\vdots$
$D(x_i^{l+1}) \cap X^1$	$n-1+(l-1)(n-2)$

Since $l \leq \lceil \frac{k+1}{n-2} \rceil$, we have that $|D(x_i^{l+1}) \cap X^1| < n+k$. Thus, by construction x_i^{l+1} will be comparable to $n-1+(l-1)(n-2)$ consecutively indexed elements in X^1 and will be incomparable to the remaining consecutively indexed elements in X^1. This shows that the extreme subposet of layered generalized crowns is itself a generalized crown.

Part 2: Now we need to show that the extreme subposet contains all the critical pairs of the layered poset $\mathbb{P}_1 \rtimes \cdots \rtimes \mathbb{P}_l$.

Note that any two elements $x, x' \in X^i$ form a non-critical incomparable pair because there is no containment between the sets of their comparable elements in X^{i-1} or in X^{i+1}.

Let $i < j$ and suppose $x_p^i \in X^i$ and $x_p^j \in X^j$ such that $x_p^i || x_q^j$ in $\mathbb{P}_1 \rtimes \cdots \rtimes \mathbb{P}_l$. If $i = 1$ and $j = l$, then the pair (x_p^i, x_q^j) is critical since $D(x_p^1) = \emptyset$ and $U(x_q^l) = \emptyset$.

Suppose that $1 \leq i < j < l$. Let $d = j - i$. We will show that there is no containment in their respective upsets. Let $Y_p = U(x_p^i) \cap X^{j+1}$ and $Z_q = U(x_q^j) \cap X^{j+1}$.

By construction,

$$Y_p = \{x_{p+(d+1)}^{j+1}, \ldots, x_{p+(d+1)(n-1)}^{j+1}\},$$

whereas

$$Z_q = \{x_{q+1}^{j+1}, \ldots, x_{q+(n-1)}^{j+1}\}.$$

Observe that Y_p is a strict subset of X^{j+1}. In particular,

$$X^{j+1} \setminus Y_p = \{x_{p+(d+1)(n-1)+1}^{j+1}, x_{p+(d+1)(n-1)+2}^{j+1}, \ldots, x_{p+d}^{j+1}\}.$$

We will show that in fact $(X^{j+1} \setminus Y_p) \cap Z_q \neq \emptyset$.

In order for $x_q^j || x_p^i$, the index q must be a value in the set of cyclic indexing values:

$$q \in \{p + d(n-1) + 1, p + d(n-1) + 2, \ldots, p + d - 1\}.$$

Observe the range in values of the indices for the elements in the upsets Z_q of x_q^j as q varies:

q	Indices of Elements in Z^q
$p+d(n-1)+1$	$p+d(n-1)+2,\dots,p+(d+1)(n-1)+2$
$p+d(n-1)+2$	$p+d(n-1)+3,\dots,p+(d+1)(n-1)+3$
$\vdots$	$\vdots$
$p+d-1$	$p+d,\dots,p+d+(n-1)$

This shows that if $x_q^j \| x_p^i$, then $(X^{j+1}\setminus Y_p)\cap Z_q \neq \emptyset$. Thus, $U(x_q^j) \not\subseteq U(x_p^i)$ for $1 \le i < j < l$, as desired. A similar argument using downsets will justify the case when $1 < i < j \le l$. Thus, the only possibility for critical pairs comes from incomparable pairs that are in the extreme subposet $\mathcal{E}(\mathbb{P}_1 \rtimes \cdots \rtimes \mathbb{P}_l)$.

Finally, using Theorem 3.1, one constructs sublinear extensions of the extreme subposet of the layered crown. These sublinear extensions will reverse all critical pairs in the extreme subposet and therefore all critical pairs in the layered crown. □

The final result wraps up the analysis on layering identical crowns. There was sufficient evidence that suggested that for the number of layers $l \ge \lceil \frac{k+1}{n-2} \rceil$, the order dimension of the resulting layered poset stabilizes. This is in fact the case.

Theorem 4.4. *For $3 \le n < k+3$ and for $l \in \mathbb{N}$ with $l \ge \lceil \frac{k+1}{n-2} \rceil$, let $\mathbb{P}_i = \mathbb{S}_n^k$. The dimension of l layered generalized crowns*

$$\mathbb{P}_1 \rtimes \cdots \rtimes \mathbb{P}_l,$$

is given by

$$\dim(\mathbb{P}_1 \rtimes \cdots \rtimes \mathbb{P}_l) = \left\lceil \frac{2(n+k)}{k+n-\lceil \frac{k+1}{n-2} \rceil (n-2)} \right\rceil.$$

Proof. We adopt the same notation as in the proof of Theorem 4.3. Let $\mathbb{P} = \mathbb{P}_1 \rtimes \cdots \rtimes \mathbb{P}_l$. Consider the subposets generated by the subsets $X_j \cup X_{j+M}$, where $M = \lceil \frac{k+1}{n-2} \rceil$, where $j = 1, \dots, l-M+1$. Theorem 4.3 reveals that these subposets are generalized crowns that have order dimension equal to

$$t = \left\lceil \frac{2(n+k)}{k+n-M(n-2)} \right\rceil.$$

Monotonicity of dimension forces this as a lower bound on the dimension of $\mathbb{P}$. We now provide an algorithm to produce the t linear extensions which realize $\mathbb{P}$.

(1) For $j = 1, \dots, l-M+1$, the proof of Theorem 4.3 shows that the critical pairs of $\mathbb{P}$ come from incomparable elements in the subposet $\mathcal{E}_j = \mathbb{P}(X_j \cup X_{j+M})$.
(2) For each subposet $\mathcal{E}_j$, $j = 1, \dots, l-M+1$, let $\mathcal{R}_j = \{\mathbb{L}_1^j, \dots, \mathbb{L}_t^j\}$ be its minimal realizer obtained by the algorithm given in Theorem 3.1.

(3) For $i = 1, \ldots, t$, construct the sublinear extension $\mathbb{L}_i$ for $\mathbb{P}$ by appropriately piecing together sublinear extensions, one from each $\mathcal{R}_j$ as described below:
  (a) Let $\mathbb{L}_1$ begin with the sublinear extension $\mathbb{L}_1^1$.
  (b) By construction $\mathbb{L}_1^1$ contains consecutively indexed elements from the set X_{M+1}.
  (c) Among the sublinear extensions in $\mathcal{R}_2$, there is a linear extension $\mathbb{L}_{m_2}^2$ which contains consecutively indexed elements from the set X_2 which are incomparable to the elements from the set X_{M+1} in Step 3b.
  (d) Again, by construction, $\mathbb{L}_m^2$ contains consecutively indexed elements from the set X_{M+2}. Among the sublinear extensions in $\mathcal{R}_3$, there is a linear extension $\mathbb{L}_{m_3}^3$ which contain consecutively indexed elements from the set X_3 which are incomparable to the elements in X_{M+2}.
  (e) Repeat these steps for each $\mathcal{R}_j$ to obtain $\mathbb{L}_{m_j}^j$.
  (f) The sublinear extension $\mathbb{L}_1$ is then pieced together as:
$$\mathbb{L}_1^1 < \mathbb{L}_{m_2}^2 < \cdots < \mathbb{L}_{m_j}^j < \cdots < \mathbb{L}_{m_{l-M+1}}^{l-M+1}.$$
  (g) Repeat this construction for each $\mathbb{L}_i$, $i = 1, \ldots, t$.

□

Finally, we illustrate the construction of the sublinear extensions in the proof of Theorem 4.4 with the following example.

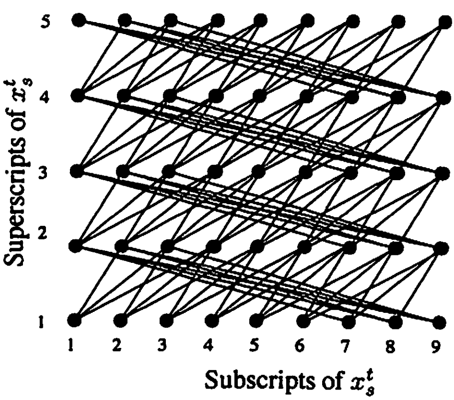

FIGURE 5. Four Layers of $\mathbb{S}_4^5$

Example 4.2. Let $\mathbb{P}$ denote the poset in Figure 5. Here, $\mathbb{P} = \mathbb{S}_4^5 \rtimes \mathbb{S}_4^5 \rtimes \mathbb{S}_4^5 \rtimes \mathbb{S}_4^5$ and $M = 3$. Thus there are two extreme subposets $\mathcal{E}_1 \cong \mathcal{E}_2 \cong \mathbb{S}_8^1$ and the order dimension of $\mathbb{S}_8^1$ is 6. In the table below, we provide all the sublinear extensions that follow from Theorem 3.1.

$\mathcal{E}_1$		$\mathcal{E}_2$	
$\mathbb{L}_1^1$:	$x_1^4 < x_8^1 < x_2^4 < x_9^1$	$\mathbb{L}_1^2$:	$x_1^5 < x_8^2 < x_2^5 < x_9^2$
$\mathbb{L}_2^1$:	$x_3^4 < x_2^1 < x_2^4 < x_1^1$	$\mathbb{L}_2^2$:	$x_3^5 < x_2^2 < x_2^5 < x_1^2$
$\mathbb{L}_3^1$:	$x_4^4 < x_2^1 < x_5^4 < x_3^1$	$\mathbb{L}_3^2$:	$x_4^5 < x_2^2 < x_5^5 < x_3^2$
$\mathbb{L}_4^1$:	$x_6^4 < x_5^1 < x_5^4 < x_4^1$	$\mathbb{L}_4^2$:	$x_6^5 < x_5^2 < x_5^5 < x_4^2$
$\mathbb{L}_5^1$:	$x_7^4 < x_5^1 < x_8^4 < x_6^1$	$\mathbb{L}_5^2$:	$x_7^5 < x_5^2 < x_8^5 < x_6^2$
$\mathbb{L}_6^1$:	$x_9^4 < x_7^1 < x_8^4 < x_6^1$	$\mathbb{L}_6^2$:	$x_9^5 < x_7^2 < x_8^5 < x_6^2$

The sublinear extensions for $\mathbb{P}$ are constructed by piecing together the sublinear extensions above. The table below gives one way of piecing these sublinear extensions.

$$\mathbb{L}_1 = \mathbb{L}_1^1 < \mathbb{L}_2^2: \quad x_1^4 < x_8^1 < x_2^4 < x_9^1 < x_3^5 < x_2^2 < x_2^5 < x_1^2$$

$$\mathbb{L}_2 = \mathbb{L}_2^1 < \mathbb{L}_3^2: \quad x_3^4 < x_2^1 < x_2^4 < x_1^1 < x_4^5 < x_2^2 < x_5^5 < x_3^2$$

$$\mathbb{L}_3 = \mathbb{L}_3^1 < \mathbb{L}_4^2: \quad x_4^4 < x_2^1 < x_5^4 < x_3^1 < x_6^4 < x_5^1 < x_5^4 < x_4^1$$

$$\mathbb{L}_4 = \mathbb{L}_4^1 < \mathbb{L}_5^2: \quad x_6^4 < x_5^1 < x_5^4 < x_4^1 < x_7^5 < x_5^2 < x_8^5 < x_6^2$$

$$\mathbb{L}_5 = \mathbb{L}_5^1 < \mathbb{L}_6^2: \quad x_7^4 < x_5^1 < x_8^4 < x_6^1 < x_9^5 < x_7^2 < x_8^5 < x_6^2$$

$$\mathbb{L}_6 = \mathbb{L}_6^1 < \mathbb{L}_1^2: \quad x_9^4 < x_7^1 < x_8^4 < x_6^1 < x_1^5 < x_8^2 < x_2^5 < x_9^2$$

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DEPARTMENT OF MATHEMATICS AND STATISTICS, SAM HOUSTON STATE UNIVERSITY, HUNTSVILLE, TX 77341, USA

E-mail address: rgarcia@shsu.edu

URL: http://www.shsu.edu/~mth_reg

E-mail address: DarrelSilva@shsu.edu