On a Four Parameter Theta Function Identity

Göksal BILGICI[*]

Department of Computer Education and Instructional Technology, Kastamonu University, 37100, Kastamonu, Turkey

Abstract

In this paper, we give a four parameter theta function identity and prove it by using some properties of Jacobi's theta functions and Jacobi's fundamental formulae.

Key words: Jacobi's theta functions, Jacobi's fundamental formulae, infinite products

1. Introduction

The theta functions were first studied by Jacobi who obtained their properties by algebraic methods. These functions are used to express the Jacobi's elliptic functions.

Let τ be a complex number whose imaginary part is positive and we write $q = e^{\pi i \tau}$, so that $|q| < 1$.

There are four Jacobi's theta functions, namely

$$\theta_1(z,q) = \sum_{n=0}^{\infty} (-1)^n q^{\left(n+\frac{1}{2}\right)^2} \sin(2n+1)\, z,$$

$$\theta_2(z,q) = 2\sum_{n=0}^{\infty} q^{\left(n+\frac{1}{2}\right)^2} \cos(2n+1)\, z,$$

$$\theta_3(z,q) = 1 + 2\sum_{n=1}^{\infty} q^{n^2} \cos 2nz,$$

$$\theta_4(z,q) = 1 + \sum_{n=0}^{\infty} (-1)^n q^{n^2} \cos 2nz.$$

It is obvious that $\theta_1(z,q)$ is an odd function of z and the others are even functions of z. Theta functions are quasi doubly-periodic functions of z, which

[*] Email address: gbilgici@kastamonu.edu.tr
Tel: +903662142041-177

means that they have two periods; but one of them is a quasi-period at least. The quasi-double periodicity can be seen in the following table.

Table 1. *Quasi-double periodicity of theta functions*

θ_i	$\theta_i(z+\pi)/\theta_i(z)$	$\theta_i(z+\pi\tau)/\theta_i(z)$
θ_1	-1	$-q^{-1}e^{-2iz}$
θ_2	-1	$q^{-1}e^{-2iz}$
θ_3	1	$q^{-1}e^{-2iz}$
θ_4	1	$-q^{-1}e^{-2iz}$

The Jacobi's theta functions may be written in terms of each other as follows

$$\theta_1(z,q) = -ie^{iz+\frac{\pi i\tau}{4}}\theta_4\left(z+\frac{1}{2}\pi\tau,q\right),$$

$$\theta_2(z,q) = \theta_1\left(z+\frac{1}{2}\pi,q\right),$$

$$\theta_3(z,q) = \theta_4\left(z+\frac{1}{2}\pi,q\right).$$

We can express the Jacobi's theta functions as infinite products. To do this, we define

$$(a;q)_\infty = \prod_{r=1}^{\infty}(1-zq^{r-1})$$

and

$$[a;q]_\infty = (a;q)_\infty(a^{-1}q;q)_\infty.$$

By these definitions, we have

$$\theta_1(z,q) = iq^{1/4}z^{-1/2}[z;q^2]_\infty(q^2;q^2)_\infty, \qquad (1)$$

$$\theta_2(z,q) = q^{1/4}z^{-1/2}[-z;q^2]_\infty(q^2;q^2)_\infty, \qquad (2)$$

$$\theta_3(z,q) = [-zq;q^2]_\infty(q^2;q^2)_\infty, \qquad (3)$$

$$\theta_4(z,q) = [zq;q^2]_\infty(q^2;q^2)_\infty. \qquad (4)$$

In partition theory, we encounter four parameter theta function identity frequently. For example, Atkin and Swinnerton-Dyer [1] gave

$$P^2(b)P(c+d)P(c-d) - P^2(c)P(b+d)P(b-d) + y^{c-d}P^2(d)P(b+c)P(b-c) = 0 \qquad (5)$$

where $P(a) = [y^a; y^m]_\infty$ and none of $b, c, d, b \pm c, b \pm d, c \pm d$ is divisible by m. The proof of this equation is based on a four parameter theta functions identity. Similarly, in his doctoral thesis [3], O'Brien gave

$$P(b+e)P(b-e)P(c+d)P(c-d) - P(b+d)P(b-d)P(c+e)P(c-e) + y^{c-d}P(b+c)P(b-c)P(d+e)P(d-e) = 0 \quad (6)$$

where none of $b \pm c, b \pm d, b \pm e, c \pm d, c \pm e, d \pm e$ is divisible by m. O'Brien made reference to elliptic function identity Eq.$(\mathrm{LVII})_2$ in [2,p.160].

We note that the infinite product $P(a)$ satisties

$$P(m-a) = P(a) \text{ and } P(-a) = P(m+a) = -y^{-a}P(a).$$

More properties of Jacobi's Theta Functions can be found in [4].

2. The Results

In this section, we write$[a] = [a; q]_\infty$ for abbreviation. We prove the following theorem by using Jacobi's theta functions.

Theorem 1. We have

$$(\alpha\beta\gamma\delta)^{-\frac{1}{2}}[\alpha][\beta][\gamma][\delta] - \left[(\alpha\beta^{-1}\gamma^{-1}\delta^{-1}q)^{\frac{1}{2}}\right]\left[(\alpha^{-1}\beta\gamma^{-1}\delta^{-1}q)^{\frac{1}{2}}\right]\left[(\alpha^{-1}\beta^{-1}\gamma\delta^{-1}q)^{\frac{1}{2}}\right]\left[(\alpha^{-1}\beta^{-1}\gamma^{-1}\delta q)^{\frac{1}{2}}\right] + \left[(\alpha\beta\gamma\delta q^{-1})^{-\frac{1}{2}}\right]\left[(\alpha\beta\gamma^{-1}\delta^{-1}q)^{\frac{1}{2}}\right]\left[\alpha\beta^{-1}\gamma\delta^{-1}q)^{\frac{1}{2}}\right]\left[(\alpha\beta^{-1}\gamma^{-1}\delta q)^{\frac{1}{2}}\right] = 0$$

where α, β, γ and δ are complex numbers in upper half-plane.

Proof. We need Jacobi's fundamental formulae. Let w', x', y', z' be defined in terms of variables w, x, y, z as follows

$$\begin{aligned} 2w' &= -w + x + y + z, \\ 2x' &= w - x + y + z, \\ 2y' &= w + x - y + z, \\ 2z' &= w + x + y - z. \end{aligned}$$

We define

$$[r] = \theta_r(w)\theta_r(x)\theta_r(y)\theta_r(z)$$

and

$$[r]' = \theta_r(w')\theta_r(x')\theta_r(y')\theta_r(z')$$

as in [4,p.468]. By using elliptic functions theory, Whittaker and Watson gave

$$2[1] = [1]' + [2]' - [3]' + [4]', \quad (7)$$

$$2[2] = [1]' + [2]' + [3]' - [4]', \quad (8)$$

$$2[3] = -[1]' + [2]' + [3]' + [4]', \quad (9)$$

$$2[4] = [1]' - [2]' + [3]' + [4]'. \quad (10)$$

In Eq.(10), substituting $w, -x, -y, -z$ for w, x, y, z respectively, we obtain

$$2\theta_4\left(\frac{-w-x-y-z}{2}\right)\theta_4\left(\frac{w+x-y-z}{2}\right)\theta_4\left(\frac{w-x+y-z}{2}\right)\theta_4\left(\frac{w-x-y+z}{2}\right)$$
$$=\theta_1(w)\theta_1(-x)\theta_1(-y)\theta_1(-z)-\theta_2(w)\theta_2(-x)\theta_2(-y)\theta_2(-z) \quad (11)$$
$$+\theta_3(w)\theta_3(-x)\theta_3(-y)\theta_3(-z)+\theta_4(w)\theta_4(-x)\theta_4(-y)\theta_4(-z)$$

and if we substitute $-w, -x, -y, -z$ for w, x, y, z respectively in Eq.(10), we have

$$2\theta_4\left(\frac{w-x-y-z}{2}\right)\theta_4\left(\frac{-w+x-y-z}{2}\right)\theta_4\left(\frac{-w-x+y-z}{2}\right)\theta_4\left(\frac{-w-x-y+z}{2}\right)$$
$$=\theta_1(-w)\theta_1(-x)\theta_1(-y)\theta_1(-z)-\theta_2(-w)\theta_2(-x)\theta_2(-y)\theta_2(-z) \quad (12)$$
$$+\theta_3(-w)\theta_3(-x)\theta_3(-y)\theta_3(-z)+\theta_4(-w)\theta_4(-x)\theta_4(-y)\theta_4(-z).$$

After subtracting Eq.(12) from Eq.(11), since θ_1 is an odd function and the others are even function, we get

$$+\theta_4\left(\frac{w-x-y-z}{2}\right)\theta_4\left(\frac{-w+x-y-z}{2}\right)\theta_4\left(\frac{-w-x+y-z}{2}\right)\theta_4\left(\frac{-w-x-y+z}{2}\right)$$
$$-\theta_4\left(\frac{-w-x-y-z}{2}\right)\theta_4\left(\frac{w+x-y-z}{2}\right)\theta_4\left(\frac{w-x+y-z}{2}\right)\theta_4\left(\frac{w-x-y+z}{2}\right)$$
$$=\theta_1(w)\theta_1(x)\theta_1(y)\theta_1(z). \quad (13)$$

We use Eqs.(1), (4) and obtain Eq.(13) in terms of infinite product. Finally, by substituting $\alpha, \beta, \gamma, \delta, q$ for $e^{2iw}, e^{2ix}, e^{2iy}, e^{2iz}, q^2$, we obtain the theorem. □

If we substitute y^a, y^b, y^c, y^d, y^m for $\alpha, \beta, \gamma, \delta$ and q, respectively, we prove the following.

Corollary 2. We define

$$x=\frac{m-(a+b+c+d)}{2}.$$

where a, b, c, d and m are positive integers and none of a, b, c, d, $a+x$, $b+x$, $c+x$, $d+x$, $a+b+x$, $a+c+x$, $a+d+x$ is divisible by m. We have

$$y^aP(a)P(b)P(c)P(d)-P(a+x)P(b+x)P(c+x)P(d+x)$$
$$+P(x)P(a+b+x)P(a+c+x)P(a+d+x)=0.$$

Corollary 2 gives the same results with the identity given by O'Brien. For example, we have to choose the sum $a+b+c+d$ as an odd integer because of divisibility conditions. Thus, any pair of a, b, c and d cannot be equal. For $m=7$, we may choose $(a,b,c,d)=(1,1,1,2)$ and this selection gives the relation

$$yP^3(1)P(2)-P^3(2)P(3)+P^3(3)P(1)=0$$

which can be found by taking $(b,c,d) = (3,2,1)$ in Eq.(5) and $(b,c,d,e) = (3,2,1,0)$ in Eq.(6). For $m = 11$, Eqs.(5), (6) and Corollary 2 give ten relations. For $m = 13$, whereas Eq.(5) gives twenty relations, Eq.(6) and Corollary 2 give another relation in addition to these twenty relations. If we take $(a,b,c,d) = (1,2,3,5)$ in Corollary 2 and $(b,c,d,e) = (5,3,2,1)$ in Eq.(6), we obtain

$$yP(1)P(2)P(3)P(5) - P(2)P(3)P(4)P(6) + P(1)P(4)P(5)P(6) = 0.$$

References

[1] A.O.L. Atkin and H.P.F. Swinnerton-Dyer, Some Properties of Partition, Pros. London Math. Soc., 4 (3) (1954) 84-106

[2] J. Molk and J. Tannery, Elements de la Theorie des Fonctions Elliptiques Tome II, Gauthier-Villars and Son, Paris (1896)

[3] J.N. O'Brien, Some Properties of Partition With Special reference to primes other than 5,7 and 11, Ph.D.Thesis, Durham University (1965)

[4] E.T. Whittaker and G.N. Watson, A Course of Modern Analysis, Fourth Edition, Cambridge University Press (1965)