Multinestings in Octagon Quadrangle Systems

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Abstract

In this paper, we determine the spectrum for *super-perfect* OQSs. OQSs are G-designs in which G is an *octagon quadrangle*, i.e. the graph consisting of an 8-cycle $(x_1, x_2, ..., x_8)$ with two additional chords: the edges $\{x_1, x_4\}$ and $\{x_5, x_8\}$.

1 Introduction and Definitions

In these last years, G-decompositions of λK_v have been examined mainly in the case in which G is a polygon with some chords forming an inside polygon whose sides joining vertices at distance two. Recently hexagon triple systems [13] and dexagon triple systems [14] have been studied. Generally, in these papers, the authors determine the spectrum of the corresponding systems and study problems of embedding. In [6,7] Lucia Gionfriddo studied G-decompositions, in which G is a polygon with chords which determine at least a quadrangle. In particular, in [8] she studied *perfect dodecagon quadrangle systems.* In [2,3,4], the authors introduced and studied *octagon quadrangle systems.* Observe that interesting problems arise when the study is dedicated to colourings in G-designs [1,11]. In this paper the spectrum of *octagon quadrangle systems*, with the condition that *both upper* 4-cycles and *lower* 4-cycles contained in the blocks form two distinct 4-cycle systems, is determined. Further, also the *outside* 8-cycles can form an 8-cycle system.

An *octagon quadrangle* [OQ-graph] is the graph formed by a cycle C_8, $(x_1, x_2, ..., x_8)$, with the two chords $\{x_1, x_4\}, \{x_5, x_8\}$. In what follows, such a graph will denoted by $[(x_1), x_2, x_3, (x_4), (x_5), x_6, x_7, (x_8)]$. The cycle (x_1, x_2, x_3, x_4) will be the *upper* C_4-cycle, the cycle (x_5, x_6, x_7, x_8) will be the *lower* C_4-cycle, while the cycle $(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8)$ will be the *outside* cycle. An *octagon quadrangle system* of order v and index λ, briefly an OQS, is a pair $\Sigma = (X, \mathcal{B})$, where X is a finite set of v vertices and $\mathcal{B}$ is a collection of edge disjoint octagon quadrangles, called *blocks*, which partitions the edge set of λK_v, defined in the vertex set X. An *octagon quadrangle system* $\Sigma = (X, \mathcal{B})$ of order v and index λ is said to be:

i) *upper C_4-perfect*, if all of the *upper* C_4-cycles contained in the octagon quadrangles form a μ-fold 4-cycle system of order v and in this case we say also that Σ is *nesting* an *upper* C_4-system or that a C_4-system is *upper nested* in Σ;

ii) *lower C_4-perfect*, if all of the *lower* C_4-cycles contained in the octagon quadrangles form a μ-fold 4-cycle system of order v and in this case we say also that Σ is *nesting* a *lower* C_4-system or that a C_4-system is *lower nested* in Σ;

iii) *C_8-perfect*, if all of the *outside* C_8-cycles contained in the octagon quadrangles form a ϱ-fold 8-cycle system of order v and in this case we say also that Σ is *nesting* the outside C_8-system or that the outside C_8-system is *nested* in Σ;

iv) *super-perfect*, if Σ is *upper, lower and outside perfect* and in this case we say also that Σ is a *total nesting system.*

In the first two cases, we say that the system has indices (λ, μ), in the third case that it has indices (λ, ϱ), in the last case that it has indices $(\lambda, \varrho, \mu, \mu)$.

In the following sections there are *OQSs* of different types. Observe that, when the vertex set is Z_v, the collection $\mathcal{B}$ of octagon quadrangles is given by a set of *base blocks* so defined: if $\mathcal{B}^* = [(a), b, c, (d), (\alpha), \beta, \gamma, (\delta)]$ is a *base block*, then $\mathcal{B}_i^* = [(a+i), b+i, c+i, (d+i), (\alpha+i), \beta+i, \gamma+i, (\delta+i)]$ is a block of $\mathcal{B}$, for each $i = 1, 2, ..., v \in Z_v$; when the vertex set is $Z_{v-1} \cup \{\infty\}$, the collection $\mathcal{B}$ of blocks is given by a set of *base blocks* defined as above, with the condition that $i = 1, 2, ..., v-1 \in Z_v$ and $\infty + i = \infty$, for every i. The octagon quadrangle $\mathcal{B}_i^*$ is said to be a *translated block* of $\mathcal{B}^*$.

Esempio 1.1 The following blocks define an $OQS(17)$ of indices (5,4,2), which is *upper-C_4 perfect* and *C_8-perfect.* We can see that the upper C_4-cycles form a C_4-system of index $\mu = 2$ *and* the outside C_8-cycles form a C_8-system of index $\varrho = 4$. Observe that the lower C_4-cycles do not form a C_4-system, this OQS is *not strongly perfect.*
Base blocks (mod 17): $[(0), 14, 15, (6), (12), 7, 5, (13)]$, $[(0), 13, 1, (8), (10), 9, 11, (7)]$, $[(0), 13, 1, (2), (11), 4, 16, (6)]$, $[(0), 3, 9, (7), (10), 2, 5, (6)]$.

Esempio 1.2 The following blocks define a *super-perfect* $OQS(13)$ of indices (5,4,2). The upper C_4-cycles form a C_4-system of index $\mu = 2$; the lower C_4-cycles form another C_4-system of index $\mu = 2$; the outside C_8-cycles form a C_8-system of index $\varrho = 4$. There are three cycles-systems *nested* in this $OQS(13)$.
Base blocks (mod 13): $[(0), 3, 10, (1), (7), 9, 4, (2)]$, $[(0), 1, 10, (2), (7), 8, 4, (3)]$, $[(0), 2, 10, (3), (7), 4, 11, (1)]$.

2 Necessary conditions for *super-perfect* OQSs

Theorem 2.1 : *Let Ω be a super-perfect OQS of order v and let Σ_{out}, Σ_{up}, Σ_{low} be the outside C_8 − system, the upper C_4 − system and the lower C_4 − system respectively. If the systems Ω, Σ_{out}, Σ_{up}, Σ_{low} have indices $(\lambda, \varrho, \mu, \mu)$, in the order, then:*
i) $\lambda = 5 \cdot k$, $\varrho = 4 \cdot k$, $\mu = 2 \cdot k$,
for some positive integer k, and
ii) $v \equiv 0$ *or* 1 *mod* 4, $v \geq 8$, *if k is odd,*
iii) $v \equiv 0$ *or* 1 *mod* 2, $v \geq 8$, *if k is even.*

Proof. Let $\Omega = (X, \mathcal{B})$ be a super-perfect OQS of order v and let $\Sigma_{out} = (X, \mathcal{B}_1)$, $\Sigma_{up} = (X, \mathcal{B}_2)$, $\Sigma_{low} = (X, \mathcal{B}_3)$ be the outside C_8 − *system*, the upper C_4 − *system* and the lower C_4 − *system* respectively, nested in Ω. Let $(\lambda, \varrho, \mu, \mu)$ be their indices, in the order.
i) Since $|\mathcal{B}| = |\mathcal{B}_1| = |\mathcal{B}_2| = |\mathcal{B}_3|$, necessarily: $\lambda \cdot v(v-1)/20 = \varrho \cdot v(v-1)/16 = \mu \cdot v(v-1)/8$. It follows: $\lambda/5 = \varrho/4 = \mu/2$, from which *i)* follows.
ii) Immediately from *i)*, if k in an odd number, then $v \equiv 0$ or 1 mod 4, $v \geq 8$, and if k is an even number, then $v \equiv 0$ or 1 mod 2, $v \geq 8$. □

3 Existence of *super-perfect* OQSs of small order

Theorem 3.1 : *There exist super-perfect OQSs of order* $8, 9, 12, 13$ *and indices* $(5, 4, 2, 2)$.

Proof. *i)* Let $\Sigma_8 = (V_8, \mathcal{B})$ be the system defined in $V_8 = Z_7 \cup \{\infty\}$, $\infty \notin Z_7$ whose blocks are all the translated one obtained by the following *base blocks* (mod 7): $[(\infty), 5, 6, (3), (2), 0, 1, (4)]$, $[(1), 0, 2, (4), (6), 3, \infty, (5)]$, where ∞ is a fixed vertex and all the others are obtained cyclically in Z_7. We can verify that Σ_8 is a *super-perfect* $OQS(8)$ of indices $(5, 4, 2, 2)$. The *upper* C_4-system is generated by the two base 4-cycles: $(\infty, 5, 6, 3), (1, 0, 2, 4)$. The *lower* C_4-system is generated by the two base 4-cycles: $(2, 0, 1, 4), (\infty, 5, 6, 3)$.

ii) Let $\Sigma_9 = (Z_9, \mathcal{B})$ be the system defined in Z_9 whose blocks are all the translated one obtained by the following *base blocks* (mod 9): $[(0), 1, 5, (7), (4), 3, 6, (8)]$,$[(3), 0, 5, (2), (4), 8, 6, (7)]$. We can verify that Σ_9 is a *super-perfect* $OQS(9)$ of indices $(5, 4, 2, 2)$. The *upper* C_4-system is generated by the two base 4-cycles: $(0, 1, 5, 7), (3, 0, 5, 2)$. The *lower* C_4-system is generated by the two base 4-cycles: $(4, 3, 6, 8), (4, 8, 6, 7)$.

iii) Let $\Sigma_{12} = (V_{12}, \mathcal{B})$ be the system defined in $V_{12} = Z_{11} \cup \{\infty\}$, $\infty \notin Z_{11}$ whose blocks are all the translated one obtained by the following *base blocks* (mod 11): $[(\infty), 10, 8, (5), (6), 7, 9, (1)]$,$[(0), 3, 8, (1), (6), 7, 9, (2)]$, $[(0), 1, 8, (2), (10), 5, \infty, (7)]$, where ∞ is a fixed vertex and all the others are obtained cyclically in Z_{11}. We can verify that Σ_{12} is a *super-perfect* $OQS(12)$ of indices $(5, 4, 2, 2)$. The *upper* C_4-system is generated by the 4-cycles: $(\infty, 10, 8, 5), (0, 3, 8, 1), (0, 1, 8, 2)$. The *lower* C_4-system is generated by the 4-cycles: $(1, 6, 7, 9), (2, 6, 7, 9), (\infty, 7, 10, 5)$.

iv) Let $\Sigma_{13} = (Z_{13}, \mathcal{B})$ be the system defined in Z_{13} whose blocks are all the translated one obtained by the following *base blocks* (mod 13): $[(0), 3, 10, (1), (7), 9, 4, (2)]$, $[(0), 1, 10, (2), (7), 8, 4, (3)]$, $[(0), 2, 10, (3), (7), 4, 11, (1)]$. We can verify that Σ_{13} is a *super-perfect* $OQS(13)$ of indices $(5, 4, 2, 2)$. The *upper* C_4-system is generated by the 4-cycles: $(0, 3, 10, 1), (0, 1, 10, 2), (0, 2, 10, 3)$. The lower C_4-system is generated by the 4-cycles: $(2, 7, 9, 4), (3, 7, 8, 4), (1, 7, 4, 11)$. □

4 Constructions of *super-perfect* OQSs having minimum index

In this section we construct super-perfect OQSs having indices $(5, 4, 2, 2)$.

Theorem 4.1 : *For every positive integer h, $h \geq 4$, there exist super-perfect OQSs of order $v = 4h + 1$ and indices* $(5, 4, 2, 2)$.

Proof. Let $v = 4h+1$, $h \geq 4$, and let $\Sigma_{4h+1} = (Z_v, \mathcal{B})$ be the system defined in Z_v whose blocks are all the translated ones obtained by the following *base blocks* (mod $v = 4h + 1$):

$[(0), h, 3h + 1, (1), (2h + 1), 3h, h + 1, (2)]$,
$[(0), 1, 3h + 1, (2), (2h + 1), 3h - 1, h + 1, (3)]$,
$[(0), 2, 3h + 1, (3), (2h + 1), 3h - 2, h + 1, (4)]$,
....
....
$[(0), i - 1, 3h + 1, (i), (2h + 1), 3h - i + 1, h + 1, (i + 1)]$, $i < h$,
....

....
$[(0), h-2, 3h+1, (h-1), (2h+1), 2h+2, h+1, (h)]$,
$[(0), h-1, 3h+1, (h), (2h+1), h+1, 3h+2, (1)]$.

We can verify that Σ_{4h+1} is a *super-perfect* $OQS(4h+1)$ of indices $(5,4,2,2)$. Consider that the base blocks

$$[(x_1), x_2, x_3, (x_4), (x_5), x_6, x_7, (x_8)],$$

they are all defined so that: $x_1 = 0$, $x_3 = 3h+1$, $x_5 = 2h+1$ and $x_7 = h+1$, except in the last one, where it is $x_7 = 3h+2$, and they all have fixed values. The other vertices have values which depend on $i = 1, 2, ..., h$ in such a way that every edge describes h consecutive differences. □

Theorem 4.2 : *For every positive integer* h, $h \geq 4$, *there exist super-perfect OQSs of order* $v = 4h$ *and indices* $(5,4,2,2)$.

Proof. Let $v = 4h$, $h \geq 4$, and let $\Sigma_{4h} = (Z_{v-1} \cup \{\infty\}, \mathcal{B})$ be the system defined in $W = Z_{v-1} \cup \{\infty\}$, where $\infty \notin Z_{v-1}$, whose blocks are all the translated ones obtained by the following *base blocks* (mod $v-1 = 4h-1$):

$[(0), h, 3h-1, (1), (2h), 2h+1, 3h, (2)]$,
$[(0), 1, 3h-1, (2), (2h), 2h+2, 3h, (3)]$,
$[(0), 2, 3h-1, (3), (2h), 2h+3, 3h, (4)]$,
....

....
$[(0), i-1, 3h-1, (i), (2h), 2h+i, 3h, (i+1)]$, $i < h-2$,
....

....
$[(0), h-3, 3h-1, (h-2), (2h), 3h-2, 3h, (h-1)]$,
$[(0), h-2, 3h-1, (h-1), (2h), 1, \infty, (h)]$,
$[(\infty), 4h-2, 3h-1, (2h-1), (2h), 2h+1, 3h, (1)]$.

We can verify that Σ_{4h} is a *super-perfect* $OQS(4h)$ of indices $(5,4,2,2)$. Consider that the base blocks

$$[(x_1), x_2, x_3, (x_4), (x_5), x_6, x_7, (x_8)],$$

are all defined so that: $x_1 = 0$ in all the base blocks except in the last block where it is ∞, $x_3 = 3h-1$, $x_5 = 2h$ and $x_7 = 3h$, except in the previous from the last one where it is ∞, and they all have fixed values. The other vertices have values which depend on $i = 1, 2, ..., h$ in such a way every edge describes h consecutive difference. □

5 Conclusive Theorems

Collecting together the results of the previous sections, we have the following conclusive results.

Theorem 5.1 : *There exist super-perfect OQS(v) of indices* $(5, 4, 2, 2)$ *if and only if* $v \equiv 0$ *or* 1 *mod* 4, $v \geq 8$.

Proof. The statement follows from Theorems 3.1, 4.1 and 4.2. □

Theorem 5.2 : *For every* $v \equiv 0$ *or* 1 *mod* 4, $v \geq 8$, *there exist OQSs of order* v *and index* 5 *nesting two* C_4*-systems of index* 2 *and a complete graph* K_v.

Proof. Consider the systems constructed in Theorems 4.1 and 4.2 and observe that every block of these systems

$[(x_1), x_2, x_3, (x_4), (x_5), x_6, x_7, (x_8)]$,

can be partitioned into the two cycles $(x_1, x_2, x_3, x_4), (x_5, x_6, x_7, x_8)$ and the two disjoint edges $\{x_4, x_5\}$,$\{x_1, x_8\}$.

Further, we have seen that the family of all the cycles (x_1, x_2, x_3, x_4) forms a C_4-system of index 2 and the family of all the cycles $(x_5), x_6, x_7, (x_8)$ forms a C_4-system of index 2.

So, we can verify that the family of all the edges $\{x_4, x_5\}$,$\{x_1, x_8\}$ forms a decomposition of K_v into edges. So, the statement follows. □

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